Linear Algebra and its Applications 435 (2011) 842–851

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the Lipschitzian property in linear complementarity problems over symmetric cones

I. Jeyaraman *, V. Vetrivel

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

ARTICLE INFO

Article history:
Received 27 September 2010
Accepted 4 February 2011
Available online 26 February 2011

Submitted by R.A. Brualdi

AMS classification:
90C33
17C55

Keywords:
Euclidean Jordan algebra
Symmetric cone
Complementarity problem
Lipschitzian property
Relaxation transformation

ABSTRACT

Let V be a Euclidean Jordan algebra with symmetric cone K. We show that if a linear transformation L on V has the Lipschitzian property and the linear complementarity problem LCP(L, q) over K has a solution for every invertible $q \in V$, then $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V. We show that the converse holds for Lyapunov-like transformations, Stein transformations and quadratic representations. We also show that the Lipschitzian Q-property of the relaxation transformation R_A on V implies that A is a P-matrix.

1. Introduction

Let $(V, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra, and $K = \{x \circ x : x \in V\}$ be the set of squares in V. Then K is a symmetric cone [7]. Given a linear transformation $L : V \to V$ and $q \in V$, the *linear complementarity problem*, LCP(L, q), is to find a vector $x \in V$ such that

$$x \in K, \quad L(x) + q \in K, \text{ and } \langle x, L(x) + q\rangle = 0.$$

This problem is a particular case of a variational inequality problem [6], and it includes the standard linear complementarity problem [5] and the semidefinite linear complementarity problem [9]. For applications of these problems in optimization, game theory, economics, etc., see [5,6]. A basic problem in LCP(L, q) is to find necessary and sufficient conditions on L so that for all $q \in V$, LCP(L, q) has a unique

* Corresponding author.
E-mail addresses: i_jeyaraman@yahoo.co.in (I. Jeyaraman), vetri@iitm.ac.in (V. Vetrivel).

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.02.009

solution (that is, L has the GUS-property). A related problem is to study the Lipschitzian behavior of the solution map $q \to SOL(L, q)$, where SOL(L, q) is the solution set of LCP(L, q).

When $V = \mathbb{R}^n$ with the usual inner product and Jordan product $x \circ y := x * y$ (the componentwise product of x and y), LCP(L, q) reduces to the *standard linear complementarity problem* LCP(M, q) [5]. In this setting, a well known characterization is that M has the GUS-property if and only if M is a P-matrix (which means that all principal minors of M are positive) [5]. Mangasarian and Shiau [15] showed that if M has the GUS-property, then M is a Lipschitzian matrix. In [8], Gowda gave an alternate proof and at the same time extended their result. Pang (see [8]) conjectured that if M is a Lipschitzian matrix and LCP(M, q) has a solution for all $q \in \mathbb{R}^n$ (that is, M is a Q-matrix), then M is a P-matrix. This conjecture was proved affirmatively by Murthy et al. [16].

If $V = S^n$ (the set of all real symmetric $n \times n$ matrices) with the trace inner product and Jordan product $X \circ Y := \frac{1}{2}(XY + YX)$, then LCP($L$, q) reduces to the *semidefinite linear complementarity problem* SDLCP(L, Q) [9]. Gowda and Song [9] extended many concepts in LCP(M, q) such as P-matrix, Q-matrix and the GUS-property to SDLCP(L, Q). Motivated by the significance of Lipschitzian matrix in LCP(M, q), Balaji et al. [3] studied the Lipschitzian property in SDLCP(L, Q). They showed that, unlike in the standard LCP, the GUS-property need not imply the Lipschitzian property. They also proved that if L on S^n has the Lipschitzian property, then L has the GUS-property under the assumption that L is monotone.

For a real $n \times n$ matrix A, consider the Lyapunov transformation L_A, the Stein transformation S_A, and the multiplication transformation M_A defined on S^n respectively by

$$L_A(X) = AX + XA^T, \quad S_A(X) = X - AXA^T, \quad \text{and} \quad M_A(X) = AXA^T.$$

These transformations have been extensively studied, and are related to dynamical systems, see [9,10]. The following results have been proved in [3].

(i) L_A has the Lipschitzian Q-property if and only if L_A is strongly monotone which implies the GUS-property.
(ii) If S_A has the Lipschitzian Q-property, then $I - A$ is positive definite.
(iii) When A is symmetric, M_A has the Lipschitzian property if and only if M_A is strongly monotone.

When A is normal, S_A has the Lipschitzian Q-property if and only if S_A is strongly monotone, see [10].

Gowda et al. [13] extended several concepts from standard and semidefinite LCPs to the setting of symmetric cone LCPs. They showed that if L has the Lipschitzian GUS-property, then L has the positive principal minor property. As a generalization of this result, Balaji [2] proved that if L on V has the Lipschitzian Q-property, then L has the positive principal minor property. This also extends a result in LCP(M, q) which was proved by Murthy et al. [16, Theorem 4]. However, it is not known if the Lipschitzian Q-property in V implies the GUS-property.

In this paper, we are concerned with the Lipschitzian property of linear transformations on V. Balaji [1] has shown that if L on S^n has the Lipschitzian Q-property, then (i, i)-entry of $L(E_i)$ is positive for all $i = 1, 2, \ldots, n$ where E_i is the diagonal matrix with one in the (i, i)-entry and zero elsewhere. In Section 3, we generalize the above result for a linear transformation on Euclidean Jordan algebras. We show that if L on V has the Lipschitzian property and LCP(L, q) has a solution for every invertible element q in V, then $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V. We show by an example that the converse need not be true. However, we obtain the equivalence for Lyapunov-like transformations, Stein transformations and quadratic representations which generalizes some of the results in [3]. Further, we prove that if the relaxation transformation R_A has the Lipschitzian Q-property in V, then A is a P-matrix. In particular, we study the Lipschitzian Q-property of R_A in the space of S^n and $\mathcal{L}^n$.

2. Preliminaries

2.1. Euclidean Jordan algebras

In this section, we recall some basic concepts and results from Euclidean Jordan algebras. For more details, we refer to [7,13].

A *Euclidean Jordan algebra* is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$, where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional inner product space over $\mathbb{R}$ and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(a) $x \circ y = y \circ x$ for all $x, y \in V$,
(b) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$, where $x^2 := x \circ x$, and
(c) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

In V, the set of squares $K = \{x \circ x : x \in V\}$ is a *symmetric cone* [7]. We write $y \geqslant 0$ if $y \in K$, and $y \leqslant 0$ when $-y \geqslant 0$.

Theorem 2.1 [13]. *For $x, y \in V$, the following conditions are equivalent:*

(i) $x \geqslant 0,\ y \geqslant 0,$ *and* $\langle x, y\rangle = 0.$
(ii) $x \geqslant 0,\ y \geqslant 0,$ *and* $x \circ y = 0.$

The algebra $\mathcal{L}^n$: Consider $\mathbb{R}^n$ $(n > 1)$ with the usual inner product. Let $x = \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} \in \mathbb{R}^n$, where $x_0 \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^{n-1}$. Define the Jordan product $x \circ y$ in $\mathbb{R}^n$ by

$$x \circ y = \begin{pmatrix} x_0 \\ \bar{x} \end{pmatrix} \circ \begin{pmatrix} y_0 \\ \bar{y} \end{pmatrix} := \begin{pmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{pmatrix}.$$

We denote this Euclidean Jordan algebra $(\mathbb{R}^n, \circ, \langle .,. \rangle)$ by $\mathcal{L}^n$, and its cone of squares by $\mathcal{L}^n_+$. $\mathcal{L}^n_+$ is called the *Lorentz cone* (or the *second order cone*) and is given by $\mathcal{L}^n_+ = \{x : x_0 \geqslant \|\bar{x}\|\}$.

For $x \in V$, we define $m(x) := \min\{k > 0 : \{e, x, \ldots, x^k\}$ is linearly dependent$\}$ and *rank* of V by $r = \max\{m(x) : x \in V\}$. An element $c \in V$ is an *idempotent* if $c^2 = c$; it is *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set $\{e_1, e_2, \ldots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if $e_i \circ e_j = 0$ for $i \neq j$, and $\sum_{i=1}^m e_i = e$, where e is the identity element in V satisfies $y \circ e = y$ for all $y \in V$.

Theorem 2.2 (The spectral decomposition theorem [7]). *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$. The numbers λ_i (with their multiplicities) are uniquely determined by x and are called the eigenvalues of x.*

The expression $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r$ is the *spectral decomposition* of x. The set of all eigenvalues of x is called the *spectrum* of x and is denoted by $\sigma(x)$. We say that x is *invertible* if every eigenvalue of x is nonzero.

For $x \in V$, we define the corresponding *Lyapunov transformation* $L_x : V \to V$ by $L_x(z) = x \circ z$. We say that elements x and y *operator commute* if $L_x L_y = L_y L_x$. It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame [7].

Peirce decomposition: Let $\{e_1, e_2, \ldots, e_r\}$ be a fixed Jordan frame in V. For $i, j \in \{1, 2, \ldots, r\}$, consider the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = \mathbb{R}e_i$$

and when $i \neq j$,

$$V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}.$$

Theorem 2.3 [7]. *The space V is the orthogonal direct sum of the spaces* $V_{ij}(i \leqslant j)$. *Furthermore, (i)* $V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj}$, *(ii)* $V_{ij} \circ V_{jk} \subset V_{ik}$ *if* $i \neq k$, *and (iii)* $V_{ij} \circ V_{kl} = \{0\}$ *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

Thus, given any Jordan frame $\{e_1, e_2, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in V_{ij}$.

Quadratic representation: Given any element a in V, the *quadratic representation of a* is the linear map $P_a : V \to V$ defined by $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$.
Principal subtransformations: Given a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ in V, we define $V^{(l)} := V(e_1 + e_2 + \cdots + e_l, l) := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\}$ for $1 \leqslant l \leqslant r$. Then $V^{(l)}$ (called the eigenspace of $e_1 + e_2 + \cdots + e_l$) is a subalgebra of V with rank l [7]. Let $P^{(l)}$ denote the orthogonal projection from V onto $V^{(l)}$. For a linear transformation $L : V \to V$, let $L_{\{e_1,e_2,\ldots,e_l\}} := P^{(l)}L : V^{(l)} \to V^{(l)}$. We call $L_{\{e_1,e_2,\ldots,e_l\}}$ a *principal subtransformation* of L. The determinant of this transformation is called a *principal minor* of L. If all the principal minors of L are positive, then we say that L has the *positive principal minor property*.

2.2. Linear complementarity concepts

Given a linear transformation $L : V \to V$, we say that L is/has

(a) *strongly monotone* if $\langle L(x), x\rangle > 0$ for all $0 \neq x \in V$;
(b) *strictly copositive on K* if $\langle L(x), x\rangle > 0$ for all $0 \neq x \in K$;
(c) the *GUS* (globally uniquely solvable)-*property* if LCP(L, q) has a unique solution for all $q \in V$;
(d) the *GUS-property on K* if LCP(L, q) has a unique solution for all $q \in K$;
(e) the *P-property* if

$$[x \text{ and } L(x) \text{ operator commute and } x \circ L(x) \leqslant 0] \Rightarrow x = 0;$$

(f) the *Q-property* if LCP(L, q) has a solution for all $q \in V$;
(g) the *Lipschitzian property* if there exists a constant $C > 0$ such that

$$SOL(L, q) \subseteq SOL(L, q') + C\|q - q'\|B$$

for all $q, q' \in V$ satisfying SOL(L, q) $\neq \emptyset$ and SOL(L, q') $\neq \emptyset$. Here B is the closed unit ball in V, and SOL(L, q) is the set of all solutions of LCP(L, q).
(h) the *Lipschitzian Q-property* if L has the Lipschitzian and Q-property;
(i) a *Z-transformation* if $x, y \in K$, and $\langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leqslant 0$;
(j) a *Lyapunov-like* transformation if

$$x, y \in K, \quad \text{and } \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Various interconnections between the above properties have been studied in [2,11–14,17–19]. In particular, for the implications (a) ⇒ (c) ⇒ (e) ⇒ (f) and (b) ⇒ (d) ⇒ (f), see [13,18].

3. The Lipschitzian Q-property

Balaji et al. [3, Theorem 3.1] showed that if a linear transformation L on S^n has the Lipschitzian property and SOL(L, I) = {0}, then (i, i)-entry of $L(E_i)$ is positive (equivalently, $\langle L(E_i), E_i\rangle > 0$) for all $i = 1, 2, \ldots, n$. Here I is the identity matrix, and E_i is the symmetric matrix of order n with one in the

(i, i)-entry and zero elsewhere, which is a primitive idempotent in S^n. This result also holds if L has the Lipschitzian Q-property [1, Theorem 3.2.1]. We extend these results to Euclidean Jordan algebras.

Theorem 3.1. *Suppose $L : V \to V$ has the Lipschitzian property and SOL(L, q) is nonempty for every invertible element q in V. Then $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V.*

Proof. Let c be a primitive idempotent in V. When the r (= rank of V) is one, $V = \mathbb{R}c$ and any solution x of LCP$(L, -c)$ is a positive multiple of c, say, $x = \lambda c$ with $\lambda > 0$. From complementarity, $L(x) = c$, and so $c = \lambda L(c)$. This implies $\langle L(c), c\rangle > 0$. Now suppose that $r > 1$. Then by the spectral decomposition of $e - c$, there exists a Jordan frame $\{e_1 = c, e_2, \ldots, e_r\}$ in V. For each $k \in \mathbb{N}$, let $p_k = -e_1 + ke_2 + \cdots + ke_r$ and $q_k = ke_2 + ke_3 + \cdots + ke_r$. Then p_k is invertible in V. Since $q_k \geqslant 0$, we have $0 \in$ SOL(L, q_k). By our assumption, there exists $x_k \in$ SOL(L, p_k) such that $0 \in x_k + C\|q_k - p_k\|B$ for all $k \in \mathbb{N}$, where $C > 0$ and B is the closed unit ball in V. Since $\|q_k - p_k\| = \|e_1\|$, we have $\|x_k\| \leqslant C\|e_1\|$. This means that the sequence $\{x_k\}$ is bounded. Without loss of generality, assume that $x_k \to x$.

Let

$$x_k = \sum_{i=1}^{r} \alpha_i^{(k)} e_i + \sum_{i<j} x_{ij}^{(k)},$$

$$L(x_k) = \sum_{i=1}^{r} \beta_i^{(k)} e_i + \sum_{i<j} y_{ij}^{(k)} \quad \textit{and}$$

$$x = \sum_{i=1}^{r} \alpha_i e_i + \sum_{i<j} x_{ij}$$

be the Peirce decomposition of x_k, $L(x_k)$ and x with respect to the Jordan frame $\{e_1, e_2, \ldots, e_r\}$ respectively. We claim that $\alpha_i = 0$ for $i = 2, \ldots, r$. Since $x_k \in$ SOL(L, p_k), we have $\langle x_k, L(x_k) + p_k\rangle = 0$. This implies that $\left\langle \alpha_1^{(k)} e_1, \left(\beta_1^{(k)} - 1\right) e_1\right\rangle + \sum_{i=2}^{r} \left\langle \alpha_i^{(k)} e_i, \left(\beta_i^{(k)} + k\right) e_i\right\rangle + \sum_{i<j} \left\langle x_{ij}^{(k)}, y_{ij}^{(k)}\right\rangle = 0$, and hence $\sum_{i=2}^{r} \left\langle \alpha_i^{(k)} e_i, e_i\right\rangle = -\frac{1}{k}\left[\left\langle \alpha_1^{(k)} e_1, \left(\beta_1^{(k)} - 1\right) e_1\right\rangle + \sum_{i=2}^{r} \left\langle \alpha_i^{(k)} e_i, \beta_i^{(k)} e_i\right\rangle + \sum_{i<j} \left\langle x_{ij}^{(k)}, y_{ij}^{(k)}\right\rangle\right]$. Since $x_k \to x$, $\alpha_i^{(k)} \to \alpha_i$ and $x_{ij}^{(k)} \to x_{ij}$. Because $\{L(x_k)\}$ converges, $\left\{\beta_i^{(k)} e_i\right\}$ and $\left\{y_{ij}^{(k)}\right\}$ converge. Letting $k \to \infty$ in the above expression, we have $\sum_{i=2}^{r} \langle \alpha_i e_i, e_i\rangle = 0$. Now $x_k \geqslant 0$ implies that $\alpha_i^{(k)} \geqslant 0$ for all k and hence $\alpha_i \geqslant 0$. Therefore $\alpha_i = 0$ for $i = 2, \ldots, r$. From Proposition 3.2 in [11], we have $x = \alpha_1 e_1 \in V_{11} \cap K$. Since K is self-dual and $x_k \in$ SOL(L, p_k), $\langle L(x_k) + p_k, e_1\rangle \geqslant 0$. This implies that $\langle L(x_k), e_1\rangle \geqslant ||e_1||^2$. Taking limits and observing $x = \alpha_1 e_1$, we get $\alpha_1\langle L(e_1), e_1\rangle \geqslant ||e_1||^2 > 0$. Thus, $\langle L(c), c\rangle > 0$. $\square$

Corollary 3.1. *Let $L : V \to V$ be a linear transformation. Under each of the following conditions, the Lipschitzian property of L implies $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V.*

(i) *L has the Q-property.*
(ii) *SOL$(L, q) = \{0\}$ for some $q \in$ int(K).*
(iii) *L is a cone invariant transformation; i.e., $L(K) \subseteq K$.*

Proof. Assume that L has the Lipschitzian property.

If (i) holds, then the result follows from the above theorem.

Suppose that condition (ii) holds. Then by Lemma 5 in [2], L has the GUS property on K. This implies that L has the Q-property and hence condition (i) holds. Thus, we have $\langle L(c), c\rangle > 0$ for all primitive idempotents $c \in V$.

Now, suppose that condition (iii) holds. We claim that SOL$(L, e) = \{0\}$, where $e \in \text{int}(K)$ is the identity element of V. Clearly, $0 \in$ SOL(L, e). Let $x \in$ SOL(L, e). Then $x \geqslant 0$ and $\langle x, L(x) + e\rangle = 0$. Since $L(K) \subseteq K$, $\langle x, L(x)\rangle \geqslant 0$. This implies that $\langle x, e\rangle = 0$. By Theorem 2.1, we have $x \circ e = x = 0$. Hence condition (ii) holds. This completes the proof. □

Remark 3.1. If L has the Lipschitzian Q-property, then L has the positive principal minor property [2, Theorem 5] which implies that $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V [17, Lemma 3.1]. One can ask whether the conditions of Theorem 3.1 imply the Lipschitzian Q-property or the positive principal minor property. We do not have an answer for this question.

As an illustration of Theorem 3.1, we provide the following examples. The proof of these results are modifications of the proof of Example 3.3 in [18] and Corollary 4.1 in [19].

Example 3.1. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. Consider the Lyapunov transformation L_A and the Stein transformation S_A. Then

(i) $\langle L_A(c), c\rangle > 0$ for all primitive idempotents c in S^n if and only if A is positive definite.
(ii) $\langle S_A(c), c\rangle > 0$ for all primitive idempotents c in S^n if and only if $I \pm A$ are positive definite, where I is the identity matrix.

Example 3.2. When $V = \mathcal{L}^n$, $\langle L(c), c\rangle > 0$ for all primitive idempotents c in V if and only if $\langle L(z), z\rangle > 0$ for all nonzero z in the boundary of $\mathcal{L}^n_+$.

The following examples show that both the conditions in the hypothesis of Theorem 3.1 are essential.

Example 3.3. Consider the Euclidean Jordan algebra $\mathbb{R}^2$ with the usual inner product and Jordan product $x \circ y = x * y$. Let $M = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$. Then LCP$(M, q)$ has no solution for all $-q \in \text{int}(\mathbb{R}^2_+)$, which are invertible elements in $\mathbb{R}^2$. We see that every principal minor of M is negative. Since all the entries of M are negative, M has the Lipschitzian property [8, Theorem 14]. But, $\langle Me_1, e_1\rangle = -1 \leqslant 0$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a primitive idempotent in $\mathbb{R}^2$.

Example 3.4. Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 2 \end{bmatrix}$. Consider the Lyapunov transformation L_A on S^2. Since A is positive stable, L_A has the Q-property [9, Theorem 5]. As A is not positive definite, L_A does not have the Lipschitzian property [3, Theorem 3.3]. Also by Example 3.1, we have $\langle L(c), c\rangle \leqslant 0$ for some primitive idempotent c in S^2.

The following example shows that the converse of the Theorem 3.1 is not true even for self-adjoint cone invariant transformation.

Example 3.5. Let $L : \mathcal{L}^2 \to \mathcal{L}^2$ be defined by $L\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}\right) = \begin{pmatrix} 2x_0 \\ -x_1 \end{pmatrix}$. We see that L is induced on $\mathbb{R}^2$ by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and L is self-adjoint. It is easy to show that $L(\mathcal{L}^2_+) \subseteq \mathcal{L}^2_+$. Let $z \neq 0$ belongs to the boundary of $\mathcal{L}^2_+$. Then by the spectral decomposition, there exists a Jordan frame $\{e_1, e_2\}$ such that $z = \lambda e_1$, where $\lambda > 0$ and $e_1 = \frac{1}{2}\begin{pmatrix} 1 \\ u \end{pmatrix}$ with $u \in \mathbb{R}$ and $|u| = 1$ [17, Lemma 4.1]. Since $\langle L(e_1), e_1\rangle = \frac{1}{4}$, we have $\langle L(z), z\rangle > 0$. By Example 3.2, $\langle L(c), c\rangle > 0$ for all idempotents c in $\mathcal{L}^2$. From Proposition 3.1 in [12], L has the Q-property. Since the determinant of L is not positive, L does not have the positive principal minor property [13, Example 2.2]. Hence L does not have the Lipschitzian property [2, Theorem 5].

In spite of the above example, we show below that the converse of Theorem 3.1 holds for Lyapunov-like and Stein transformations on V.

Theorem 3.2. *Let $L : V \to V$ be a Lyapunov-like transformation. Then the following are equivalent:*

(i) *L is strongly monotone.*
(ii) *L has the Lipschitzian Q-property.*
(iii) *$\langle L(c), c\rangle > 0$ for all primitive idempotents c in V.*

Proof. The equivalence of (i) and (ii) follows from Theorem 6 in [2].
$(ii) \Rightarrow (iii)$: This follows from Corollary 3.1.
$(iii) \Rightarrow (i)$: Let $0 \neq x \in V$. By the spectral decomposition, $x = \sum_{i=1}^r \alpha_i e_i$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame. Since L is a Lyapunov-like transformation, $\langle L(e_i), e_j\rangle = 0$ if $i \neq j$. Therefore, $\langle L(x), x\rangle = \sum_{i=1}^r \alpha_i^2 \langle L(e_i), e_i\rangle$. Since $\langle L(e_i), e_i\rangle > 0$ for all i, we have $\langle L(x), x\rangle > 0$. Thus, L is strongly monotone. □

It was shown in [3, Theorem 3.3] that the Lyapunov transformation L_A on S^n is strongly monotone if and only if L_A has the Lipschitzian Q-property (which is equivalent to A is positive definite). Since L_A is a Lyapunov-like transformation on S^n [14], the above result generalizes the Theorem 3.3 in [3].

It has been proved in [3, Theorem 3.2] that the Stein transformation S_A on S^n has the Lipschitzian Q-property implies $I - A$ is positive definite. Further, if $A \in S^n$, then S_A has the Lipschitzian Q-property if and only if $I \pm A$ are positive definite [4, Theorem 5.1.3]. If $A \in S^n$, we have $S_A(X) = X - AXA = (I - P_A)(X)$. We now extend this result to general Euclidean Jordan algebras.

Theorem 3.3. *Let $a \in V$. Consider the Stein transformation S_a defined on V by $S_a = I - P_a$. Then the following statements are equivalent:*

(i) *S_a is strongly monotone.*
(ii) *S_a has the Lipschitzian Q-property.*
(iii) *$\langle S_a(c), c\rangle > 0$ for all primitive idempotents c in V.*
(iv) *$\sigma(\pm a) \subseteq (-1, 1)$.*

Proof. Let $a = \sum_{i=1}^r \lambda_i e_i$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame.

The implication $(i) \Rightarrow (ii)$ follows from Proposition 2.3.11 in [6].
$(ii) \Rightarrow (iii)$: This follows from Corollary 3.1.
$(iii) \Rightarrow (iv)$: Suppose that $\langle S_a(c), c\rangle > 0$ for all primitive idempotents c in V. Now, $\langle S_a(e_i), e_i\rangle = \|e_i\|^2 - \langle P_a(e_i), e_i\rangle$. Since $\langle S_a(e_i), e_i\rangle > 0$ and $P_a(e_i) = \lambda_i^2 e_i$, we have $1 - \lambda_i^2 > 0$ for all i. This implies that $\lambda_i \in (-1, 1)$ for all i. Thus, $\sigma(\pm a) \subseteq (-1, 1)$.
$(iv) \Rightarrow (i)$: Suppose that $\sigma(\pm a) \subseteq (-1, 1)$. Then $1 - \lambda_i^2 > 0$ for all i. Let $x = \sum_{i=1}^r \beta_i e_i$, where $\beta_i > 0$. Then $P_a(x) = \sum_{i=1}^r \lambda_i^2 \beta_i e_i$ and hence $S_a(x) = \sum_{i=1}^r (1 - \lambda_i^2)\beta_i e_i$. Thus, there exists a $x \in \text{int}(K)$ such that $S_a(x) \in \text{int}(K)$. Since S_a is a Z-transformation and self-adjoint, S_a is strongly monotone [14, Corollary 1]. □

Balaji et al. [3] showed that if A is symmetric, then for the multiplication transformation M_A, strong monotonicity property is equivalent to Lipschitzian property. We see that if $A \in S^n$, then the quadratic representation $P_A = M_A$. Thus, the next result generalizes the Theorem 3.5 in [3].

Theorem 3.4. *Let $a \in V$. Then the following are equivalent:*

(i) *P_a is strongly monotone.*
(ii) *P_a has the Lipschitzian property.*
(iii) *$\langle P_a(c), c\rangle > 0$ for all primitive idempotents c in V.*

Proof. It is enough to show that $(iii) \Rightarrow (i)$. Suppose (iii) holds. Since $P_a(K) \subseteq K$ [12, Proposition 6.1], the condition (iii) is equivalent to SOL$(P_a, 0)$ = $\{0\}$ [12, Proposition 3.1]. Therefore, by Theorem 6.5 in [12], we have P_a is strongly monotone. □

The relaxation transformation R_A [19]: Let $\{e_1, e_2, \ldots, e_r\}$ be a Jordan frame in V and $A \in R^{r\times r}$. We define $R_A : V \to V$ as follows.

For any $x \in V$, write the Peirce decomposition $x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$. Then $R_A(x) = \sum_{i=1}^{r} y_i e_i + \sum_{i<j} x_{ij}$, where $(y_1, y_2, \ldots, y_r)^T = A(x_1, x_2, \ldots, x_r)^T$.

In [4], it has been shown that for the relaxation transformation R_A on S^n with respect to the Jordan frame $\{E_1, E_2, \ldots, E_n\}$, the Lipschitzian Q-property implies A is a P-matrix. We now generalize this result to Euclidean Jordan algebras. We need the following lemmas.

Lemma 3.1. *Let V be a Euclidean Jordan algebra of rank r and $A \in \mathbb{R}^{r\times r}$. If R_A has the Q-property in V, then A is a Q-matrix.*

Proof. Let q be a vector in $\mathbb{R}^r$ such that $q^T = (\alpha_1, \alpha_2, \ldots, \alpha_r)$. Take $u = \alpha_1 e_1 + \cdots + \alpha_r e_r \in V$. Since R_A has the Q-property, there exists v in SOL(R_A, u). Let $v = \sum_{i=1}^{r} \beta_i e_i + \sum_{i<j} v_{ij}$ be the Peirce decomposition of v with respect to the Jordan frame $\{e_1, e_2, \ldots, e_r\}$. Then $R_A(v) = \sum_{i=1}^{r} \gamma_i e_i + \sum_{i<j} v_{ij}$, where $(\gamma_1, \gamma_2, \ldots, \gamma_r)^T = A(\beta_1, \beta_2, \ldots, \beta_r)^T$. Since $v \in$ SOL(R_A, u), we have $\beta_i \geqslant 0$, $\gamma_i + \alpha_i \geqslant 0$ and $\beta_i(\gamma_i + \alpha_i) = 0$ for all i. This shows that $x \in$ SOL(A, q), where $x^T = (\beta_1, \beta_2, \ldots, \beta_r)$. Hence the result. □

Lemma 3.2. *If R_A has the Lipschitzian property in V, then A is a Lipschitzian matrix.*

Proof. Let $p, q \in \mathbb{R}^r$ with SOL$(A, p) \neq \emptyset$ and SOL$(A, q) \neq \emptyset$. We now show that if $p^T = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $q^T = (\beta_1, \beta_2, \ldots, \beta_r)$, then SOL$(R_A, u) \neq \emptyset$ and SOL$(R_A, v) \neq \emptyset$, where $u = \alpha_1 e_1 + \cdots + \alpha_r e_r$ and $v = \beta_1 e_1 + \cdots + \beta_r e_r$. Let $x \in$ SOL(A, p) such that $x^T = (x_1, x_2, \ldots, x_r)$ and $(\gamma_1, \gamma_2, \ldots, \gamma_r)^T = A(x_1, x_2, \ldots, x_r)^T$. Then $x_i \geqslant 0$, $\gamma_i + \alpha_i \geqslant 0$ and $x_i(\gamma_i + \alpha_i) = 0$ for all i. Take $w = x_1 e_1 + \cdots + x_r e_r$. Then $R_A(w) = \gamma_1 e_1 + \cdots + \gamma_r e_r$, and hence $w \in$ SOL(R_A, u). Thus, SOL$(R_A, u) \neq \emptyset$ and SOL(R_A, v) $\neq \emptyset$. Since R_A has the Lipschitzian property, there exists a constant $K > 0$ such that SOL$(R_A, u) \subseteq$ SOL(R_A, v) $+K\|u - v\|B$, where B is a closed unit ball in V. Further, there exists a $z \in$ SOL(R_A, v) such that $\|w - z\| \leqslant K\|u - v\|$. Let $z = \sum_{i=1}^{r} y_i e_i + \sum_{i<j} z_{ij}$. Then as in the proof of Lemma 3.1, we have $y \in$ SOL(A, q), where $y^T = (y_1, y_2, \ldots, y_r)$. Now $\left[\sum_{i=1}^{r} (x_i - y_i)^2\right] \min_{1\leqslant j\leqslant r} \|e_j\|^2 \leqslant \sum_{i=1}^{r} (x_i - y_i)^2 \|e_i\|^2 \leqslant \|w - z\|^2$ and hence $\|x - y\|_2 \min_{1\leqslant j\leqslant r} \|e_j\| \leqslant \|w - z\|$, where $\|.\|_2$ is the Euclidean norm in $\mathbb{R}^r$. Also $\|u - v\|^2 = \sum_{i=1}^{r} (\alpha_i - \beta_i)^2 \|e_i\|^2 \leqslant \left[\sum_{i=1}^{r} (\alpha_i - \beta_i)^2\right] \max_{1\leqslant k\leqslant r} \|e_k\|^2$ which implies that $\|u - v\| \leqslant \|p - q\|_2 \max_{1\leqslant k\leqslant r} \|e_k\|$. Since $\|w - z\| \leqslant K\|u - v\|$, we have $\|x - y\|_2 \leqslant C\|p - q\|_2$, where $C = K\dfrac{\max_{1\leqslant k\leqslant r} \|e_k\|}{\min_{1\leqslant k\leqslant r} \|e_k\|}$. Thus, SOL$(A, p) \subseteq$ SOL(A, q) + $C\|p - q\|_2 B_1$, where B_1 is the closed unit ball in $\mathbb{R}^r$. This completes the proof. □

Theorem 3.5. *Suppose that $R_A : V \to V$ has the Lipschitzian Q-property, then A is a P-matrix.*

Proof. Assume that R_A has the Lipschitzian Q-property. Then by Lemmas 3.1 and 3.2, A is a Lipschitzian Q-matrix. This implies that A is a P-matrix [16, Theorem 4]. □

We below specialize our study of Lipschitzian Q-property of R_A on S^n and $\mathcal{L}^n$. Let E be a square matrix with zero diagonal entries and ones elsewhere. Now we have the following result.

Corollary 3.2. *The following statements hold:*

(i) *When $V = \mathcal{L}^2$, R_A has the Lipschitzian Q-property if and only if A is a P-matrix.*
(ii) *If $V = \mathcal{L}^n$ $(n \geqslant 3)$ and R_A has the Lipschitzian Q-property, then A is a P-matrix and $A + E$ is strictly copositive on R^2_+.*
(iii) *If $V = S^n$ and R_A has the Lipschitzian Q-property, then A is a P-matrix and $A + E$ is strictly copositive on R^n_+.*

Proof.

(i) "Only if" part follows from Theorem 3.5.
"If" part: Suppose A is a P-matrix. Then R_A has the P-property [19, Proposition 5.1]. Since $\mathcal{L}^2_+$ is polyhedral, P-property implies Lipschitzian Q-property [13, Theorem 23].
(ii) Suppose that R_A has the Lipschitzian Q-property on $\mathcal{L}^n$, where $n \geqslant 3$. By Theorem 3.5, it is enough to show that $A + E$ is strictly copositive on R^2_+. From Corollary 3.1 and Example 3.2, we have $\langle R_A(z), z\rangle > 0$ for all $z \neq 0$ on the boundary of $\mathcal{L}^n_+$. This implies that $A + E$ is strictly copositive on R^2_+ [17, Proposition 5.2].
(iii) The proof is similar to that of Theorem 5.1 in [19]. □

The following example shows that if A is a Lipschitzian matrix, then R_A need not have the Lipschitzian property. This also shows that the converse of Theorem 3.5 is not true.

Example 3.6. Let $V = S^2$ or $\mathcal{L}^n$ $(n \geqslant 3)$ and $A = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$. Then A is a P-matrix. Hence A is a Lipschitzian matrix [15], and R_A has the P-property [19]. This implies that R_A has the Q-property. But, $A + E = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$ is not strictly copositive on R^2_+, as $\left\langle (A+E)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = -1 \leqslant 0$. Therefore, by above theorem, R_A does not have the Lipschitzian property with respect to any Jordan frame in V.

Acknowledgments

We thank Prof. T. Parthasarathy, ISI Chennai, for his comments and suggestions. We gratefully acknowledge the referee for his suggestions which improved the presentation of the paper, in particular Theorem 3.1. The first author thanks the Council of Scientific and Industrial Research (CSIR), India, for the financial support.

References

[1] R. Balaji, On semidefinite linear complementarity problems, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India, December 2004.
[2] R. Balaji, On an interconnection between the Lipschitz continuity of the solution map and the positive minor property in linear complementarity problems over Euclidean Jordan algebras, Linear Algebra Appl. 426 (2007) 83–95.

[3] R. Balaji, T. Parthasarathy, D. Sampangi Raman, V. Vetrivel, On the Lipschitz continuity of the solution map in semidefinite linear complementarity problems, Math. Oper. Res. 30 (2005) 462–471.
[4] A. Chandrashekaran, Standard and semidefinite linear complementarity problems, Ph.D. Thesis, Indian Institute of Technology Madras, Chennai, India, January 2010.
[5] R.W. Cottle, J.-S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.
[6] F. Facchinei, J.-S. Pang, Finite-dimensional Variational Inequalities and Complementarity Problems, Springer-Verlag, New York, 2003.
[7] J. Faraut, A. Koranyi, Analysis on Symmetric Cones, Oxford University Press, Oxford, 1994.
[8] M.S. Gowda, On the continuity of the solution map in linear complementarity problems, SIAM J. Optim. 2 (1992) 619–634.
[9] M.S. Gowda, Y. Song, On semidefinite linear complementarity problems, Math. Program. 88 (2000) 575–587.
[10] M.S. Gowda, Y. Song, G. Ravindran, On some interconnections between strict monotonicity, globally uniquely solvable, and *P* properties in semidefinite linear complementarity problems, Linear Algebra Appl. 370 (2003) 355–368.
[11] M.S. Gowda, R. Sznajder, Automorphisms invariance of P- and GUS- properties of linear transformations on Euclidean Jordan algebras, Math. Oper. Res. 31 (2006) 109–123.
[12] M.S. Gowda, R. Sznajder, Some global uniqueness and solvability results for linear complementarity problems over symmetric cones, SIAM J. Optim. 18 (2007) 461–481.
[13] M.S. Gowda, R. Sznajder, J. Tao, Some *P*-properties for linear transformations on Euclidean Jordan algebras, Linear Algebra Appl. 393 (2004) 203–232.
[14] M.S. Gowda, J. Tao, *Z*-transformations on proper and symmetric cones, Math. Program. 117 (2009) 195–221.
[15] O.L. Mangasarian, T.-H. Shiau, Lipschitz continuity of solutions of linear inequalities, programs, and complementarity problems, SIAM J. Control Optim. 25 (1987) 583–595.
[16] G.S.R. Murthy, T. Parthasarathy, M. Sabatini, Lipschitzian *Q*-matrices are *P*-matrices, Math. Program. 74 (1996) 55–58.
[17] J. Tao, Positive principal minor property of linear transformations on Euclidean Jordan Algebras, J. Optim. Theory Appl. 140 (2009) 131–152.
[18] J. Tao, Strict semimonotonicity property of linear transformations on Euclidean Jordan algebras, J. Optim. Theory Appl. 144 (2010) 575–596.
[19] J. Tao, M.S. Gowda, Some *P*-properties for nonlinear transformations on Euclidean Jordan algebras, Math. Oper. Res. 30 (2005) 985–1004.