

On Subsystem Codes Beating the Hamming or Singleton Bound

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Subsystem codes are a generalization of noiseless subsystems, decoherence free subspaces, and quantum error-correcting codes. We prove a Singleton bound for \mathbb{F}_q -linear subsystem codes. It follows that no subsystem code over a prime field can beat the Singleton bound. On the other hand, we show the remarkable fact that there exist impure subsystem codes beating the Hamming bound. A number of open problems concern the comparison in performance of stabilizer and subsystem codes. One of the open problems suggested by Poulin's work asks whether a subsystem code can use fewer syndrome measurements than an optimal MDS stabilizer code while encoding the same number of qudits and having the same distance. We prove that linear subsystem codes cannot offer such an improvement under complete decoding.

Keywords: subsystem codes, operator codes, quantum Hamming bound, quantum Singleton bound

1. Introduction

Subsystem codes (sometimes also referred to as operator quantum error-correcting codes) have emerged as an important new discovery in the area of quantum error correcting codes, unifying the classes of stabilizer codes, decoherence free subspaces and noiseless subsystems (Bacon 2006; Knill 2006; Kribs *et al.* 2005, 2006; Kribs 2006; Poulin 2005). From a practical perspective their importance lies in the fact that they seem to offer better error recovery schemes than existing quantum codes. Therefore, it is crucial to know under what circumstances these gains can be attained and how to achieve them.

Recall that a quantum code Q is a subspace in a finite dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^q$. A subsystem code is a quantum code which can be further resolved into a tensor product *i.e.*, $Q = A \otimes B$. Information is stored in system A , while system B , referred to as the gauge subsystem, provides some additional redundancy. By qudit we refer to a quantum bit with q levels. We denote the parameters of a subsystem code by $[[n, k, r, d]]_q$, indicating that it is a q -ary code with length n , encodes k qudits into the subsystem A , and contains r gauge qudits and has distance d .

Our goals in this paper are twofold. After reviewing the necessary background on subsystem codes, we generalize the quantum Singleton bound to \mathbb{F}_q -linear subsystem codes. It follows that no Clifford subsystem code over a prime field can beat the Singleton bound. We use these results to show that if there exists an MDS stabilizer code, then no linear subsystem code can outperform it in the sense of requiring fewer syndrome measurements for error correction.

Bacon & Casaccino (2006) obtain a subsystem code from two classical codes. We show that this method is a special case of the Euclidean construction for subsystem codes proposed in Aly *et al.* (2006) and give a coding theoretic analysis of these codes.

Since the early works on quantum error-correcting codes, it has been suspected that impure codes should somehow perform better than the pure codes. In particular, it was often conjectured that there might exist impure quantum error-correcting codes beating the quantum Hamming bound, but a proof remained elusive. Aly *et al.* (2006) proved a Hamming bound for pure subsystem codes. We show here that there exist impure subsystem codes beating the Hamming bound.

2. Background

Let \mathbb{F}_q be a finite field with q elements and characteristic p . Let $C \subseteq \mathbb{F}_q^n$ be an \mathbb{F}_q -linear classical code denoted by $[n, k, d]_q$, where $k = \dim_{\mathbb{F}_q} C$ and d is the minimum distance of C . We define $\text{wt}(C) = \min\{\text{wt}(c) \mid 0 \neq c \in C\} = d$, where $\text{wt}(c)$ is the Hamming weight of c . Sometimes an alternative notation $(n, K, d)_q$ is also used where $K = |C|$. If C is an \mathbb{F}_p -linear subspace over \mathbb{F}_q , then we say it is an additive code.

If $x, y \in \mathbb{F}_q^n$, then their Euclidean inner product is defined as $x \cdot y = \sum_i x_i y_i$. The Euclidean dual of a code $C \subseteq \mathbb{F}_q^n$ is defined as $C^\perp = \{y \in \mathbb{F}_q^n \mid x \cdot y = 0 \text{ for all } x \in C\}$. We say that a code C is self-orthogonal with respect to the Euclidean inner product if $C \subseteq C^\perp$.

We use the notation $(x|y) = (x_1, \dots, x_n | y_1, \dots, y_n)$ to denote concatenation of $x, y \in \mathbb{F}_q^n$. Let $u = (a|b)$ and $v = (a'|b')$ be in \mathbb{F}_q^{2n} . We define the symplectic weight of u as $\text{swt}(u) = \{(a_i, b_i) \neq (0, 0) \mid 1 \leq i \leq n\}$ and the symplectic weight of a code $C \subseteq \mathbb{F}_q^{2n}$ as $\text{swt}(C) = \min\{\text{swt}(c) \mid 0 \neq c \in C\}$. For codes over \mathbb{F}_q^{2n} another inner product plays a more important role in the context of quantum codes. The trace-symplectic product between u, v is defined as $\langle u|v \rangle_t = \langle (a|b)|(a'|b') \rangle_t = \text{tr}_{q/p}(a' \cdot b - a \cdot b')$. The trace-symplectic dual of $C \subseteq \mathbb{F}_q^{2n}$ is defined as $C^{\perp_t} = \{x \in \mathbb{F}_q^{2n} \mid \langle x|y \rangle_t = 0, \text{ for all } y \in C\}$. If $C \subseteq C^{\perp_t}$, we say that it is self-orthogonal with respect to the trace-symplectic inner product.

(a) Subsystem codes from classical codes

We now briefly review the background on subsystem codes. First we give a group theoretic description and then give an alternate description in terms of classical codes. Further details can be found in Klappenecker & Sarvepalli (2006); Aly *et al.* (2006).

Let q be the power of a prime p and \mathbb{F}_q a finite field with q elements. Let $B = \{|x\rangle \mid x \in \mathbb{F}_q\}$ denote an orthonormal basis for \mathbb{C}^q . Let $X(a)$ and $Z(b)$ be unitary operators on \mathbb{C}^q whose action on any element $|x\rangle$ in B is defined as

$$X(a)|x\rangle = |x+a\rangle \text{ and } Z(b)|x\rangle = \omega^{\text{tr}_{q/p}(bx)}|x\rangle,$$

where $\omega = e^{j2\pi/p}$ is a primitive p^{th} root of unity. These operators are a q -ary generalization of the well-known Pauli matrices X and Z . Their action on an arbitrary element in \mathbb{C}^q is obtained by invoking linearity. Let $\mathcal{H} = \mathbb{C}^q \otimes \dots \otimes \mathbb{C}^q = \mathbb{C}^{q^n}$ and

\mathcal{E} be the error group on \mathcal{H} , defined as the tensor product of n such error operators *i.e.*,

$$\mathcal{E} = \{\omega^c E_1 \otimes \cdots \otimes E_n \mid E_i = X(a_i)Z(b_i); a_i, b_i \in \mathbb{F}_q; c \in \mathbb{F}_p\}.$$

The weight of an error $E = \omega^c E_1 \otimes E_2 \otimes \cdots \otimes E_n$ in \mathcal{E} is defined as the number of E_i which are not equal to identity and it is denoted by $\text{wt}(E)$. We can also associate to E a vector $\overline{E} = (a_1, \dots, a_n | b_1, \dots, b_n) \in \mathbb{F}_q^{2n}$. We define the symplectic weight of \overline{E} as

$$\text{swt}(\overline{E}) = |\{(a_i, b_i) \neq (0, 0) \mid 1 \leq i \leq n\}| = \text{wt}(E).$$

Every nontrivial normal subgroup N in \mathcal{E} defines a subsystem code Q . Let $C_{\mathcal{E}}(N)$ be the centralizer of N in \mathcal{E} and $Z(N)$ the center of N . As a subspace the subsystem code Q defined by N is precisely the same as the stabilizer code defined by $Z(N)$. By Theorem 4 in Klappenecker & Sarvepalli (2006), Q can be decomposed as $A \otimes B$ where $\dim B = |G : Z(N)|^{1/2}$ and

$$\dim A = |Z(\mathcal{E}) \cap G| |\mathcal{E} : Z(\mathcal{E})|^{1/2} |N : Z(N)|^{1/2} / |N|.$$

Since information is stored only on subsystem A , we need only concern errors that affect A . An error E in \mathcal{E} is detectable by subsystem A if and only if E is contained in the set $\mathcal{E} - (NC_{\mathcal{E}}(N) - N)$. The distance of the code is defined as

$$d = \min\{\text{wt}(E) \mid I \neq E \in NC_{\mathcal{E}}(N) - N\} = \text{wt}(NC_{\mathcal{E}}(N) - N).$$

If $NC_{\mathcal{E}}(N) = N$, then we define the distance of the code to be $\text{wt}(N)$. A distance d subsystem code with $\dim A = K$, $\dim B = R$ is often denoted as $((n, K, R, d))_q$ or $[[n, k, r, d]]_q$ if $K = q^k$ and $R = q^r$. We say that N is the *gauge group* of Q and $Z(N)$ its *stabilizer*. The gauge group acts trivially on A .

In Klappenecker & Sarvepalli (2006) we showed that subsystem codes, much like the stabilizer codes, are related to the classical codes over \mathbb{F}_q^{2n} or $\mathbb{F}_{q^2}^n$, but with one important difference. We no longer need the associated classical codes to be self-orthogonal, thereby extending the class of quantum codes. The gauge group N can be mapped to a classical code C over \mathbb{F}_q^{2n} and $C_{\mathcal{E}}(N)$ can be mapped to the trace-symplectic dual of C . The following theorem (Klappenecker & Sarvepalli 2006) shows how subsystem codes are related to classical codes .

Theorem 2.1. *Let C be a classical additive subcode of \mathbb{F}_q^{2n} such that $C \neq \{0\}$ and let D denote its subcode $D = C \cap C^{\perp t}$. If $x = |C|$ and $y = |D|$, then there exists an operator quantum error correcting code $C = A \otimes B$ such that*

- i) $\dim A = q^n / (xy)^{1/2}$,*
- ii) $\dim B = (x/y)^{1/2}$.*

The minimum distance of subsystem A is given by

- (a) $d = \text{swt}((C + C^{\perp t}) - C) = \text{swt}(D^{\perp t} - C)$ if $D^{\perp t} \neq C$;*
- (b) $d = \text{swt}(D^{\perp t})$ if $D^{\perp t} = C$.*

Thus, the subsystem A can detect all errors in \mathcal{E} of weight less than d , and can correct all errors in E of weight $\leq \lfloor (d-1)/2 \rfloor$.

We call codes constructed using theorem 2.1 as *Clifford subsystem codes*. Arguably, these codes cover the most important subsystem codes, including the recently proposed Bacon-Shor codes. In this paper, henceforth by a subsystem code we will mean a Clifford subsystem code.

A further simplification of the above construction is possible which takes any pair of classical codes to give a subsystem code. We will just recall the result here and study its application in the next section.

Corollary 2.2 (Euclidean Construction). *Let $X_i \subseteq \mathbb{F}_q^n$, be $[n, k_i]_q$ linear codes where $i \in \{1, 2\}$. Then there exists an $[[n, k, r, d]]_q$ Clifford subsystem code with*

- $k = n - (k_1 + k_2 + k')/2$,
 - $r = (k_1 + k_2 - k')/2$, and
 - $d = \min\{\text{wt}((X_1^\perp \cap X_2)^\perp \setminus X_1), \text{wt}((X_2^\perp \cap X_1)^\perp \setminus X_2)\}$,
- where $k' = \dim_{\mathbb{F}_q}(X_1 \cap X_2^\perp) \times (X_1^\perp \cap X_2)$.

The result follows from Theorem 2.1 by defining $C = X_1 \times X_2$; it follows that $C^{\perp_s} = X_2^\perp \times X_1^\perp$ and $D = C \cap C^{\perp_s} = (X_1 \cap X_2^\perp) \times (X_2 \cap X_1^\perp)$, and the parameters are easily obtained from these definitions, see Aly *et al.* (2006) for a detailed proof.

(b) Pure and impure subsystem codes

We can extend the notion of purity to subsystem codes also in a straightforward manner. Let N be the gauge group of a subsystem code Q with distance $d = \text{wt}(C_{\mathcal{E}}(Z(N)) - N)$. We say that Q is *pure to d'* if there is no error of weight less than d' in N . The code is said to be *exactly pure to d'* if $\text{wt}(N)$ is d' and it is said to be *pure* if $d' \geq d$. The code is said to be *impure* if it is exactly pure to $d' < d$. This refinement to the notion of purity was made in recognition of certain subtleties that had to be addressed when constructing other subsystem codes from existing subsystem codes, see Aly *et al.* (2006) for details.

In coding theoretic terms this can be translated as follows. Let C be an additive subcode of \mathbb{F}_q^{2n} and $D = C \cap C^{\perp_t}$. By theorem 2.1, we can obtain an $((n, K, R, d))_q$ subsystem code Q from C that has minimum distance $d = \text{swt}(D^{\perp_t} - C)$. If $d' \leq \text{swt}(C)$, then we say that the associated operator quantum error correcting code is *pure to d'* .

Extending these ideas of purity to subsystem codes is useful because it facilitates the analysis of the parameters of the subsystem codes, as will become clear when we derive bounds in the next section. If the codes are pure, then it will be very easy to see that the subsystem code with the parameters $[[n, k, r, d]]_q$ satisfies $k + r \leq n - 2d + 2$. This is because then the subsystem code can also be viewed as an $[[n, k + r, d]]_q$ stabilizer code, see theorem 11 in Aly *et al.* (2006) for further details.

3. Singleton upper bound for \mathbb{F}_q -linear subsystem codes

(a) An upper bound for subsystem codes

We prove that the \mathbb{F}_q -linear subsystem codes with the parameters $[[n, k, r, d]]_q$ satisfy a quantum Singleton like bound *viz.*, $k + r \leq n - 2d + 2$. It will be seen that this reduces to the quantum Singleton bound if $r = 0$. More interestingly, this reveals that there is a trade off in the size of subsystem A and the gauge subsystem. One pays a price for the gains in error recovery. The cost is the reduction in the information to be stored.

Our proof for this result is quite straightforward, though the intermediate details are a little involved. First we show that a linear $[[n, k, r > 0, d]]_q$ subsystem code

that is exactly pure to 1 can be punctured to an $[[n-1, k, r-1, d]]_q$ code which retains the relationship between n, k, r, d . If $d = 2$ by repeated puncturing we either arrive at a pure code or a stabilizer code, both of which have upper bounds. For $d > 2$, two cases can arise, if the code is exactly pure to 1, we simply puncture it to get a smaller code as in $d = 2$ case. Otherwise, we puncture it to get an $[[n-1, k, r+1, d-1]]_q$ code. By repeatedly shortening we either get a stabilizer code or a distance 2 code both of which have an upper bound. Keeping track of the change in the parameters will give us an upper bound on the parameters of the original code.

Let $w = (a_1, a_2, \dots, a_n | b_1, b_2, \dots, b_n) \in \mathbb{F}_q^{2n}$. We denote by $\rho(w) \in \mathbb{F}_q^{2n-2}$, the vector obtained by deleting the first and the $n+1^{\text{th}}$ coordinates of w . Thus we have

$$\rho(w) = (a_2, \dots, a_n | b_2, \dots, b_n) \in \mathbb{F}_q^{2n-2}.$$

Similarly, given a classical code $C \subseteq \mathbb{F}_q^{2n}$ we denote the puncturing of a codeword or code in the first and $n+1$ coordinates by $\rho(C)$.

For \mathbb{F}_q -linear codes instead of considering the trace symplectic inner product we can consider the relatively simpler symplectic product. The symplectic product of $u = (a|b)$ and $v = (a'|b')$ in \mathbb{F}_q^{2n} is defined as $\langle u|v \rangle_s = \langle (a|b)|(a'|b') \rangle_s = a' \cdot b - a \cdot b'$. The symplectic dual of a code $C \subseteq \mathbb{F}_q^{2n}$ is defined as $C^{\perp_s} = \{x \in \mathbb{F}_q^{2n} \mid \langle x|y \rangle_s = 0, \text{ for all } y \in C\}$. It will be seen that $\langle u|v \rangle_t = \text{tr}_{q/p}(\langle u|v \rangle_s)$.

Lemma 3.1. *Let $C \subseteq \mathbb{F}_q^{2n}$ be an \mathbb{F}_q -linear code with $(a|b) \in C$ and $(a'|b') \in C^{\perp_t}$. Then $\langle (a|b)|(a'|b') \rangle_t = 0$ if and only if $\langle (a|b)|(a'|b') \rangle_s = a \cdot b' - a' \cdot b = 0$. It follows that $C^{\perp_t} = C^{\perp_s}$.*

Proof. If $\langle (a|b)|(a'|b') \rangle_s = 0$, then $\text{tr}_{q/p}(a' \cdot b - a \cdot b') = 0$. Since C is linear $(\alpha a | \alpha b)$ is also orthogonal to $(a'|b')$ for any $\alpha \in \mathbb{F}_q^\times$. Hence, $\text{tr}_{q/p}(\alpha a' \cdot b - \alpha a \cdot b') = 0$. But tr is a nondegenerate function. It follows that $a' \cdot b - a \cdot b' = 0$. The converse is straightforward. The equality of $C^{\perp_t} = C^{\perp_s}$ follows immediately from the first part of the statement. \square

As we shall be concerned with \mathbb{F}_q -linear codes in this paper, we will focus only on the symplectic inner product in the rest of the paper.

Lemma 3.2. *Let $C \subseteq \mathbb{F}_q^{2n}$ be an \mathbb{F}_q -linear code. Then C has an \mathbb{F}_q -linear basis of the form*

$$B = \{z_1, \dots, z_k, z_{k+1}, x_{k+1}, z_{k+2}, x_{k+2}, \dots, z_{k+r}, x_{k+r}\}$$

where $\langle x_i|x_j \rangle_s = 0 = \langle z_i|z_j \rangle_s$ and $\langle x_i|z_j \rangle_s = \delta_{i,j}$.

Proof. First we choose a basis $B = \{z_1, \dots, z_k\}$ for a maximal isotropic subspace C_0 of C . If $C_0 \neq C$, then we can choose a codeword x_1 in C that is orthogonal to all of the z_k except one, say z_1 (renumbering if necessary). We can scale x_1 by an element in \mathbb{F}_q^\times so that $\langle z_1|x_1 \rangle_s = 1$. If $\langle C_0, x_1 \rangle \neq C$, then we repeat the process until we have a basis of the desired form. \square

For the remainder of the section, we fix the following notation. By theorem 2.1, we can associate with an \mathbb{F}_q -linear $[[n, k, r, d]]_q$ subsystem code two classical \mathbb{F}_q -linear codes $C, D \subseteq \mathbb{F}_q^{2n}$ such that $D = C \cap C^{\perp_s}$, $|C| = q^{n-k+r}$, $|D| = q^{n-k-r}$ and

$\text{swt}(D^{\perp s} \setminus C) = d$. By lemma 3.2, we can also assume that C is generated by

$$C = \langle z_1, \dots, z_s, z_{s+1}, x_{s+1}, \dots, z_{s+r}, x_{s+r} \rangle,$$

where $s = n - k - r$ and the vectors x_i, z_i in \mathbb{F}_q^{2n} satisfy the relations $\langle x_i | x_j \rangle_s = 0 = \langle z_i | z_j \rangle_s$ and $\langle x_i | z_j \rangle_s = \delta_{i,j}$. These relations on x_i, z_i imply that

$$\begin{aligned} C^{\perp s} &= \langle z_1, \dots, z_s, z_{s+r+1}, x_{s+r+1}, \dots, z_{s+r+k}, x_{s+r+k} \rangle, \\ D = C \cap C^{\perp s} &= \langle z_1, \dots, z_s \rangle, \\ D^{\perp s} &= \langle z_1, \dots, z_s, z_{s+1}, x_{s+1}, \dots, z_n, x_n \rangle. \end{aligned}$$

Lemma 3.3. *An \mathbb{F}_q -linear $[[n, k, r > 0, d \geq 2]]_q$ Clifford subsystem code exactly pure to 1 can be punctured to an \mathbb{F}_q -linear $[[n-1, k, r-1, \geq d]]_q$ code.*

Proof. As mentioned above, we can associate to the subsystem code two classical codes $C, D \subseteq \mathbb{F}_q^{2n}$. Two cases arise depending on $\text{swt}(D)$.

- a) If $\text{swt}(D) = 1$, then without loss of generality we can assume that $\text{swt}(z_1) = 1$. Further, z_1 can be taken to be of the form $(1, 0, \dots, 0 | a, 0, \dots, 0)$. and for $i \neq 1$, because of \mathbb{F}_q -linearity of the codes we can choose every x_i, z_i to be of the form $(0, a_2, \dots, a_n | b_1, b_2, \dots, b_n)$. Further, as x_i, z_i must satisfy the orthogonality relations with z_1 viz., $\langle z_1 | z_i \rangle_s = 0 = \langle z_1 | x_i \rangle_s$, for $i > 1$ we can choose x_i, z_i to be of the form $(0, a_2, \dots, a_n | 0, b_2, \dots, b_n)$. It follows that because of the form of x_i and z_i puncturing the first and $n+1^{\text{th}}$ coordinate will not alter these orthogonality relations, in particular $\langle \rho(x_i) | \rho(z_i) \rangle_s \neq 0$ for $s+1 \leq i \leq n$.

Letting $\rho(x_i) = x'_i, \rho(z_i) = z'_i$ and observing that $\rho(z_1) = (0, \dots, 0 | 0, \dots, 0)$, we see that the code $\rho(C) = \langle z'_2, \dots, z'_s, z'_{s+1}, x'_{s+1}, \dots, z'_{s+r}, x'_{s+r} \rangle$. Denoting by $D_p = \rho(C) \cap \rho(C)^{\perp s}$ it is immediate that D_p is generated by $\{z'_2, \dots, z'_s\}$ while $D_p^{\perp s} = \langle z'_2, \dots, z'_s, z'_{s+1}, x'_{s+1}, \dots, z'_n, x'_n \rangle$. Hence $\rho(C)$ defines an $[[n-1, k, r, \text{swt}(D_p^{\perp s} \setminus \rho(C))]]_q$ code.

Next we show that $\text{swt}(D_p^{\perp s} \setminus \rho(C)) \geq d$. Let $u = (a_2, \dots, a_n | b_2, \dots, b_n)$ be in $D_p^{\perp s} \setminus \rho(C)$, then we can easily verify that $(0, a_2, \dots, a_n | 0, b_2, \dots, b_n)$ is orthogonal to all $z_i, 1 \leq i \leq s$ and hence it is in $D^{\perp s}$. It cannot be in C as that would imply that u is in $\rho(C)$. But $\text{swt}(D^{\perp s} \setminus C) \geq d$. Therefore $\text{swt}(u) \geq d$. and $\rho(C)$ defines an $[[n-1, k, r, \geq d]]_q$ code. By choosing $C' = \langle z'_2, \dots, z'_s, z'_{s+1}, z'_{s+2}, x'_{s+2}, \dots, z'_{s+r}, x'_{s+r} \rangle$ we can conclude that there exists an $[[n-1, k, r-1, d]]_q$ code. Alternatively, apply theorem 16 in Aly *et al.* (2006).

- b) If $\text{swt}(D) > 1$, then we can assume that $\text{swt}(z_{s+1}) = 1$ and form the code $C' = \langle z_1, \dots, z_s, z_{s+1}, z_{s+2}, x_{s+2}, \dots, z_{s+r}, x_{s+r} \rangle$. It is clear that C' defines an $[[n, k, r-1, d]]_q$ code that is pure to 1 with $\text{swt}(C' \cap C'^{\perp s}) = 1$. But this is just the previous case, from which we can conclude that there exists an $[[n-1, k, r-1, \geq d]]_q$ code. □

Lemma 3.3 allows us to establish a bound for distance 2 codes which can then be used to prove the bound for arbitrary distances.

Lemma 3.4. *An impure \mathbb{F}_q -linear $[[n, k, r, d = 2]]_q$ Clifford subsystem code satisfies*

$$k + r \leq n - 2d + 2.$$

Proof. Suppose that there exists an \mathbb{F}_q -linear $[[n, k, r, d = 2]]_q$ impure subsystem code such that $k + r > n - 2d + 2$; in particular, this code must be pure to 1. By lemma 3.3 it can be punctured to give an $[[n - 1, k, r - 1, \geq d]]_2$ subsystem code. If this code is pure, then $k + r - 1 \leq n - 1 - 2d + 2$ holds, contradicting our assumption $k + r > n - 2d + 2$; hence, the resulting code is once again impure and pure to 1.

Now we repeatedly apply lemma 3.3 to puncture the shortened codes until we get an $[[n - r, k, 0, \geq d]]_q$ subsystem code. But this is a stabilizer code which must obey the Singleton bound $k \leq n - r - 2d + 2$, contradicting our initial assumption $k + r > n - 2d + 2$. Therefore, we can conclude that $k + r \leq n - 2d + 2$. \square

If the codes are of distance greater than 2, then we puncture the code until it either has distance 2 or it is a pure code. The following result tells us how the parameters of the subsystem codes vary on puncturing.

Lemma 3.5. *An impure \mathbb{F}_q -linear $[[n, k, r, d \geq 3]]_q$ Clifford subsystem code exactly pure to $d' \geq 2$ implies the existence of an \mathbb{F}_q -linear $[[n - 1, k, r + 1, \geq d - 1]]_q$ subsystem code.*

Proof. Recall that the existence of an $[[n, k, r, d \geq 3]]_q$ subsystem code implies the existence of \mathbb{F}_q -linear codes C and D such that

$$C = \langle z_1, \dots, z_s, z_{s+1}, x_{s+1}, \dots, z_{s+r}, x_{s+r} \rangle,$$

with $s = n - k - r$, and $D = C \cap C^{\perp_s}$, see above.

The stabilizer code defined by D satisfies $k + r = n - s \leq n - 2d + 2$, or equivalently $s \geq 2d - 2$; it follows that $s \geq 2$, since $d \geq d' \geq 2$. Without loss of generality, we can take z_1 to be of the form $(1, a_2, \dots, a_n | b_1, b_2, \dots, b_n)$ for if no such codeword exists in D , then $(0, 0, \dots, 0 | 1, 0, \dots, 0)$ is contained in D^{\perp_s} , contradicting the fact that $\text{swt}(D^{\perp_s}) \geq 2$. Consequently, we can choose z_2 in D to be of the form $(0, c_2, \dots, c_n | 1, d_2, \dots, d_n)$, and we may further assume that $b_1 = 0$ in z_1 . The form of z_1 and z_2 allows us to assume that any remaining generator of C is of the form $(0, u_2, \dots, u_n | 0, v_2, \dots, v_n)$.

Let ρ be the map defined by puncturing the first and $(n + 1)^{\text{th}}$ coordinate of a vector in C . Define for all i the punctured vectors $x'_i = \rho(x_i)$ and $z'_i = \rho(z_i)$. Then one easily checks that $\langle \rho(x_i) | \rho(x_j) \rangle_s = 0 = \langle \rho(z_i) | \rho(z_j) \rangle_s$ for all indices i and j , and $\langle \rho(x_i) | \rho(z_j) \rangle_s = \delta_{i,j}$ if $i \geq s + 1$ or $j \geq 3$, and that $\langle \rho(z_1) | \rho(z_2) \rangle_s = -1$.

Let us look at the punctured code $\rho(C)$,

$$\rho(C) = \langle z'_3, \dots, z'_s, z'_{s+1}, x'_{s+1}, \dots, z'_{s+r}, x'_{s+r}, z'_1, z'_2 \rangle.$$

Since $\langle \rho(z_1) | \rho(z_2) \rangle_s = -1$ we have $D_p = \rho(C) \cap \rho(C)^{\perp_s} = \langle z'_3, \dots, z'_s \rangle$, whence $|D_p| = |D|/q^2$. As $\text{swt}(C) \geq 2$, it follows that $|\rho(C)| = |C|$. Thus $\rho(C)$ defines an $[[n - 1, k, r + 1, \text{swt}(D_p^{\perp_s} \setminus \rho(C))]]_q$ subsystem code.

Recall that the code D is generated by $s \geq 2$ vectors; we will show next that our assumptions actually force $s \geq 3$. Indeed, if $s = 2$, then $|D| = q^2$ and $|D^{\perp_s}| = q^{2n-2}$. Under the assumption $\text{swt}(D^{\perp_s}) \geq 2$, it follows that $|\rho(D^{\perp_s})| = |D^{\perp_s}| = q^{2n-2}$. But as $\rho(D^{\perp_s}) \subseteq \mathbb{F}_q^{2n-2}$ this implies that $\rho(D^{\perp_s}) = \mathbb{F}_q^{2n-2}$. Since \mathbb{F}_q^{2n-2} has $2n - 2$ independent codewords of symplectic weight one, D^{\perp_s} must have $2n - 2$ independent codewords of symplectic weight two. However, this contradicts our assumptions on the minimum distance of the subsystem code:

- (a) If C is a proper subspace of D^{\perp_s} , then the minimum distance d is given by $d = \text{swt}(D^{\perp_s} \setminus C) \geq 3$; thus, the weight 2 vectors must all be contained in C , which shows that $|C| = q^{2n-2} = |D|$, contradicting $|C| < |D^{\perp_s}|$.
- (b) If $C = D^{\perp_s}$, then the minimum distance is given by $d = \text{swt}(D^{\perp_s}) = 2$, contradicting our assumption that $d \geq 3$.

Thus, from now on, we can assume that $s \geq 3$.

Before bounding the minimum distance of the punctured subsystem code, we are going to show that $D_p^{\perp_s} = \rho(D^{\perp_s})$. Let $w = (u_1, u_2, \dots, u_n | v_1, v_2, \dots, v_n)$ be a vector in D^{\perp_s} . For $3 \leq i \leq s$, the vectors z_i are of the form $(0, a_2, \dots, a_n | 0, b_2, \dots, b_n)$; thus, it follows from $\langle w | z_i \rangle_s = 0$ that $\langle \rho(w) | z'_i \rangle_s = 0$. Hence $\rho(w)$ is in $D_p^{\perp_s}$, which implies $\rho(D^{\perp_s}) \subseteq D_p^{\perp_s}$. We have $|D_p^{\perp_s}| = q^{2n-2}/|D_p| = q^{2n}/|D| = |D^{\perp_s}|$, and we note that $|D^{\perp_s}| = |\rho(D^{\perp_s})|$, because $\text{swt}(D^{\perp_s}) \geq 2$; hence, $D_p^{\perp_s} = \rho(D^{\perp_s})$.

Let $w' = (u_2, \dots, u_n | v_2, \dots, v_n)$ be an arbitrary vector in $\rho(D^{\perp_s}) \setminus \rho(C)$. It follows that there exist some α, β in \mathbb{F}_q such that $w = (\alpha, u_2, \dots, u_n | \beta, v_2, \dots, v_n)$ is in D^{\perp_s} ; it is clear that w cannot be in C , since then $\rho(w) = w'$ would be in $\rho(C)$; hence, $\text{swt}(w) \geq d$. It immediately follows that $\text{swt}(D_p^{\perp_s} \setminus \rho(C)) \geq d - 1$. Hence $\rho(C)$ defines an $[[n-1, k, r+1, \geq d-1]]_q$ subsystem code. \square

Now we are ready to prove the upper bound for an arbitrary subsystem code. Essentially we reduce it to a pure code or distance two code by repeated puncturing and bound the parameters by carefully tracing the changes.

Theorem 3.6. *An \mathbb{F}_q -linear $[[n, k, r, d \geq 2]]_q$ Clifford subsystem code satisfies*

$$k + r \leq n - 2d + 2. \quad (3.1)$$

Proof. The bound holds for all pure codes, see Aly *et al.* (2006). So assume that the code is impure. If $d = 2$, then the relation holds by lemma 3.4; so let $d \geq 3$. If the code is exactly pure to 1, then it can be punctured using lemma 3.3 to give an $[[n-1, k, r-1, d' = d]]_q$ code, otherwise it can be punctured using lemma 3.5 to obtain an $[[n-1, k, r+1, d' \geq d-1]]_q$ code. If the punctured code is pure, then it follows that either $k+r-1 \leq n-1-2d+2$ or $k+r+1 \leq n-1-2d'+2 \leq n-1-2(d-1)+2$ holds; in both cases, these inequalities imply that $k+r \leq n-2d+2$.

If the resulting code is impure, then if it is exactly pure to 1 we puncture the code again using lemma 3.3, if not we puncture using lemma 3.5, until we get a pure code or a code with distance two. Assume that we punctured i times using lemma 3.3 and j times using lemma 3.5, then the resulting code is an $[[n-i-j, k, r+j-i, d' \geq d-j]]_q$ subsystem code. Since pure subsystem codes and distance 2 subsystem codes satisfy

$$k + r + j - i \leq n - i - j - 2d' + 2 \leq n - i - j - 2(d - j) + 2,$$

it follows that $k + r \leq n - 2d + 2$ holds. \square

When the subsystem codes are over a prime alphabet, this bound holds for all codes over that alphabet. In the more general case where the code is not linear, numerical evidence indicates that it is unlikely that the additive subsystem codes have a different bound. We have shown that a large class of impure codes already satisfy this bound. We conjecture that all subsystem codes satisfy $k+r \leq n-2d+2$. Next, we give an application of this upper bound.

(b) *Can subsystem codes improve upon MDS stabilizer codes?*

In this subsection, we compare stabilizer codes with subsystem codes. We first need to establish the criteria for the comparison, since subsystem codes cannot be universally better than stabilizer codes. For example, it is known that an $[[n, k, r, d]]_q$ subsystem code can be converted to an $[[n, k, d]]_q$ stabilizer code (see Aly *et al.* (2006), lemma 10 for a proof of this claim); this implies that no $[[n, k, r, d]]_q$ subsystem code can beat an optimal $[[n, k, d']]_q$ stabilizer code in terms of minimum distance, as $d' \geq d$. One of the attractive features of subsystem codes is a potential reduction of the number of syndrome measurements, and we use this criterion as the basis for our comparison.

First, we must highlight a subtle point on the required number of syndrome bits for an \mathbb{F}_q -linear $[[n, k, d]]_q$ code. A complete decoder, will require $n - k$ syndrome bits. Complete decoders are also optimal decoders. A bounded distance decoder on the other hand can potentially decode with fewer syndrome bits. Bounded distance decoders typically decode up to $\lfloor (d-1)/2 \rfloor$. However, to the best of our knowledge, except for the lookup table decoding method, all bounded distance decoders also require $n - k$ syndrome bits. As the complexity of decoding using a lookup table increases exponentially in $n - k$ it is highly impractical for long lengths. We therefore assume that for practical purposes, that we need $n - k$ syndrome bits.

Similarly, for an \mathbb{F}_q -linear $[[n, k, r, d]]_q$ subsystem code, a complete decoder will require $n - k - r$ syndrome measurements, as is shown in Appendix A. We are not aware of any quantum code, stabilizer or subsystem, for which there exists a bounded distance decoder that uses less than $n - k - r$ syndrome measurements to perform bounded distance decoding. The work by Poulin (2005) prompts the following question: Given an optimal $[[k + 2d - 2, k, d]]_q$ MDS stabilizer code, is it possible to find an $[[n, k, r, d]]_q$ subsystem code that uses fewer syndrome measurements?

There exist numerous known examples of subsystem codes that improve upon nonoptimal stabilizer codes. The fact that the stabilizer code is assumed to be optimal makes this question interesting. The Singleton bound $k + r \leq n - 2d + 2$ of an \mathbb{F}_q -linear $[[n, k, r, d]]_q$ subsystem code implies that the number $n - k - r$ of syndrome measurements is bounded by $n - k - r \geq 2d - 2$; thus, for fixed minimum distance d , there exists a trade off between the dimension k and the difference $n - r$ between length and number of gauge qudits.

Corollary 3.7. *Under complete decoding an \mathbb{F}_q -linear $[[n, k, r, d \geq 2]]_q$ Clifford subsystem code cannot use fewer syndrome measurements than an \mathbb{F}_q -linear $[[k + 2d - 2, k, d]]_q$ stabilizer code.*

Proof. Seeking a contradiction, we assume that there exists an $[[n, k, r, d]]_q$ subsystem code that requires fewer syndrome measurements than the optimal $[[k + 2d - 2, k, d]]_q$ MDS stabilizer code. In other words, the number of syndrome measurement yield the inequality $k + 2d - 2 - k > n - k - r$, which is equivalent to $k + r > n - 2d + 2$, but this contradicts the Singleton bound. \square

Poulin (2005) showed by exhaustive computer search that there does not exist an $[[5, 1, r > 0, 3]]_2$ subsystem code. The above result confirms his computer search and shows further that not even allowing longer lengths and more gauge qudits can help in reducing the number of syndrome measurements. In fact, we conjecture that corollary 3.7 holds for bounded distance decoders also.

We wish to caution the reader that gains in error recovery cannot be quantified purely by the number of syndrome measurements. In practice, more complex measures such as the simplicity of the decoding algorithm or the resulting threshold in fault-tolerant quantum computing are more relevant. The drawback is that the comparison of large classes of codes becomes unwieldy when such complex criteria are used.

4. Subsystem codes on a lattice

Bacon gave the first family of subsystem codes generalizing the ideas of Shor's $[[9, 1, 3]]_2$ code (Bacon 2006). Recently, he and Casaccino gave another construction which generalizes this further by considering a pair of classical codes (Bacon & Casaccino 2006). We show that this method is a special case of theorem 2.1. Since this construction is not limited to binary codes and our proofs remain essentially the same, we will immediately discuss a generalization to nonbinary alphabets.

Theorem 4.1. *For $i \in \{1, 2\}$, let $C_i \subseteq \mathbb{F}_q^{n_i}$ be \mathbb{F}_q -linear codes with the parameters $[n_i, k_i, d_i]_q$. Then there exists a Clifford subsystem code with the parameters*

$$[[n_1 n_2, k_1 k_2, (n_1 - k_1)(n_2 - k_2), \min\{d_1, d_2\}]_q$$

that is pure to $d_p = \min\{d_1^\perp, d_2^\perp\}$, where d_i^\perp denotes the minimum distance of C_i^\perp .

Proof. Let C be the classical linear code given by $C = (\mathbb{F}_q^{n_1} \otimes C_2^\perp) \times (C_1^\perp \otimes \mathbb{F}_q^{n_2})$. Then $\dim C = n_1(n_2 - k_2) + n_2(n_1 - k_1)$ and $\text{swt}(C \setminus \{0\}) \geq \min\{d_1^\perp, d_2^\perp\}$. The symplectic dual of C is given by

$$\begin{aligned} C^{\perp_s} &= (C_1^\perp \otimes \mathbb{F}_q^{n_2})^\perp \times (\mathbb{F}_q^{n_1} \otimes C_2^\perp)^\perp \\ &= (C_1 \otimes \mathbb{F}_q^{n_2}) \times (\mathbb{F}_q^{n_1} \otimes C_2). \end{aligned}$$

We have $\dim C^{\perp_s} = k_1 n_2 + n_1 k_2$. The code $D = C \cap C^{\perp_s}$ is given by

$$\begin{aligned} D &= ((\mathbb{F}_q^{n_1} \otimes C_2^\perp) \times (C_1^\perp \otimes \mathbb{F}_q^{n_2})) \cap ((C_1 \otimes \mathbb{F}_q^{n_2}) \times (\mathbb{F}_q^{n_1} \otimes C_2)) \\ &= ((\mathbb{F}_q^{n_1} \otimes C_2^\perp) \cap (C_1 \otimes \mathbb{F}_q^{n_2})) \times ((C_1^\perp \otimes \mathbb{F}_q^{n_2}) \cap (\mathbb{F}_q^{n_1} \otimes C_2)) \\ &= (C_1 \otimes C_2^\perp) \times (C_1^\perp \otimes C_2), \end{aligned}$$

and $\dim D = k_1(n_2 - k_2) + k_2(n_1 - k_1)$. It follows that $\dim C - \dim D = 2(n_1 - k_1)(n_2 - k_2)$ and $\dim C^{\perp_s} - \dim D = 2k_1 k_2$. Using corollary 2.2, we can get a subsystem code with the parameters

$$[[n_1 n_2, k_1 k_2, (n_1 - k_1)(n_2 - k_2), d = \text{swt}(D^{\perp_s} \setminus C)]]_q$$

that is pure to $d_p = \min\{d_1^\perp, d_2^\perp\}$. It remains to show that $d = \min\{d_1, d_2\}$.

Since $D = (C_1 \otimes C_2^\perp) \times (C_1^\perp \otimes C_2)$, we have

$$\begin{aligned} D^{\perp_s} &= (C_1^\perp \otimes C_2)^\perp \times (C_1 \otimes C_2^\perp)^\perp \\ &= ((C_1 \otimes \mathbb{F}_q^{n_2}) + (\mathbb{F}_q^{n_1} \otimes C_2^\perp)) \times ((\mathbb{F}_q^{n_1} \otimes C_2) + (C_1^\perp \otimes \mathbb{F}_q^{n_2})). \end{aligned}$$

In the last equality, we used the fact that vectors $u_1 \otimes u_2$ and $v_1 \otimes v_2$ are orthogonal if and only if $u_1 \perp v_1$ or $u_2 \perp v_2$.

For $i \in \{1, 2\}$, let G_i and H_i respectively denote the generator and parity check matrix of the code C_i . Without loss of generality, we may assume that these matrices are in standard form

$$H_i = [I_{n_i - k_i} \quad P_i] \text{ and } G_i = [-P_i^t \quad I_{k_i}],$$

where P_i^t is the transpose of P_i . Let $H_i^c = [0 \quad I_{k_i}]$. Using these notations, the generator matrices of C and D^{\perp_s} can be written as

$$G_C = \begin{bmatrix} I_{n_1} \otimes H_2 & 0 \\ 0 & H_1 \otimes I_{n_2} \end{bmatrix} \text{ and } G_{D^{\perp_s}} = \begin{bmatrix} G_1 \otimes H_2^c & 0 \\ I_{n_1} \otimes H_2 & 0 \\ 0 & H_1^c \otimes G_2 \\ 0 & H_1 \otimes I_{n_2} \end{bmatrix}.$$

It follows that the minimum distance d is given by

$$\text{swt}(D^{\perp_s} \setminus C) = \min \left\{ \text{wt} \left(\left\langle \begin{array}{c} G_1 \otimes H_2^c \\ I_{n_1} \otimes H_2 \end{array} \right\rangle \setminus \left\langle I_{n_1} \otimes H_2 \right\rangle \right), \right. \\ \left. \text{wt} \left(\left\langle \begin{array}{c} H_1^c \otimes G_2 \\ H_1 \otimes I_{n_2} \end{array} \right\rangle \setminus \left\langle H_1 \otimes I_{n_2} \right\rangle \right) \right\}.$$

Let us compute

$$\text{wt} \left(\left\langle \begin{array}{c} H_1^c \otimes G_2 \\ H_1 \otimes I_{n_2} \end{array} \right\rangle \setminus \left\langle H_1 \otimes I_{n_2} \right\rangle \right).$$

If minimum weight codeword is present in $D^{\perp_s} \setminus C$, it must be expressed as linear combination of at least one row from $[H_1^c \otimes G_2]$ otherwise the codeword is entirely in C . Recall that $H_1 = [I_{n_1 - k_1} \quad P_1]$ and $H_1^c = [0 \quad I_{k_1}]$. Letting $P_1 = (p_{ij})$, we can write

$$\begin{bmatrix} H_1^c \otimes G_2 \\ H_1 \otimes I_{n_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & G_2 & 0 & & & \\ 0 & 0 & \dots & 0 & 0 & G_2 & 0 & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 0 & 0 & \dots & \dots & G_2 & \\ \hline I_{n_2} & 0 & \dots & 0 & p_{11}I_{n_2} & \dots & \dots & p_{1k_1}I_{n_2} & \\ 0 & I_{n_2} & \dots & \dots & p_{21}I_{n_2} & \dots & \dots & p_{2k_1}I_{n_2} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & I_{n_2} & p_{(n_1 - k_1)1}I_{n_2} & \dots & \dots & p_{(n_1 - k_1)k_1}I_{n_2} & \end{bmatrix}.$$

Now observe that any row below the line in the above matrix can have a weight of only one in each of the last k_1 blocks of size n_2 . And any linear combination of them involving less than d_2 and at least one generator from the rows above must have a weight $\geq d_2$. If on the other hand there are more than d_2 rows involved, then the first $n_2(n_1 - k_1)$ columns will have a weight $\geq d_2$. Thus in either case the weight of an element that involves a generator from $[H_1^c \otimes G_2]$ must have a weight $\geq d_2$. On the other hand, the minimum weight of the span of $[H_1^c \otimes G_2]$ is $\text{wt}(C_2) = d_2$, from which we can conclude that

$$\text{wt} \left(\left\langle \begin{array}{c} H_1^c \otimes G_2 \\ H_1 \otimes I_{n_2} \end{array} \right\rangle \setminus \left\langle H_1 \otimes I_{n_2} \right\rangle \right) = d_2.$$

Because of the symmetry in the code we can argue that

$$\text{wt} \left(\left\langle \begin{matrix} G_1 \otimes H_2^c \\ I_{n_1} \otimes H_2 \end{matrix} \right\rangle \setminus \left\langle I_{n_1} \otimes H_2 \right\rangle \right) = d_1$$

and consequently $d = \min\{d_1, d_2\}$, which proves the theorem. \square

(a) *Bacon-Shor codes*

Bacon (2006) proposed one of the first families of subsystem codes based on square lattices. A trivial modification using rectangular lattices instead of square ones gives the following codes, see also Bacon & Casaccino (2006). The relevance of these codes will be seen later in §5. Using the same notation as in theorem 4.1, let $G_i = [1, \dots, 1]_{1 \times i}$ and H_i be the matrix defined as

$$H_i = \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & 1 & \\ & & & & & 1 & 1 \end{bmatrix}_{i-1 \times i}$$

and C , the additive code generated by the following matrix.

$$G = \begin{bmatrix} I_{n_1} \otimes H_{n_2} & 0 \\ 0 & H_{n_1} \otimes I_{n_2} \end{bmatrix}.$$

Observe that G_i generates an $[i, 1, i]_q$ code with distance i . By theorem 4.1, G_{n_1} and G_{n_2} will give us the following family of codes

Corollary 4.2. *There exist $[[n_1 n_2, 1, (n_1 - 1)(n_2 - 1), \min\{n_1, n_2\}]_q$ Clifford subsystem codes.*

5. Subsystem codes and packing

We investigate whether subsystem codes lead to better codes because of the decomposition of the code space. Since the early days of quantum codes, it has recognized that the degeneracy of quantum codes could lead to a more efficient quantum code and allow for a much more compact packing of the subspaces in the Hilbert space. But so far it has not been shown for stabilizer codes. We can derive similar bound for subsystem codes. Aly *et al.* (2006) showed the following theorem for pure subsystem codes.

Theorem 5.1. *A pure $((n, K, R, d))_q$ Clifford subsystem code satisfies*

$$\sum_{j=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{j} (q^2 - 1)^j \leq q^n / KR. \quad (5.1)$$

It is natural to ask if impure subsystem codes also satisfy this bound. We show that they do not by giving an explicit counterexample. This counter example

comes from the codes proposed by Bacon (2006). Recall the Bacon-Shor codes are $[[n^2, 1, (n-1)^2, n]]_2$ subsystem codes. The $[[9, 1, 4, 3]]_2$ is an interesting code. We can check that it satisfies the Singleton bound for subsystem codes as

$$k + r = 1 + 4 = n - 2d + 2 = 9 - 6 + 2.$$

So it is an optimal code. More interestingly, substituting the parameters of the $[[9, 1, 4, 3]]_2$ Bacon-Shor code in the above inequality we get

$$\sum_{j=0}^1 \binom{9}{j} 3^j = 28 > 2^{9-5} = 16.$$

Therefore the $[[9, 1, 4, 3]]_2$ Bacon-Shor code beats the quantum Hamming bound for the pure subsystem codes proving the following result.

Theorem 5.2. *There exist impure $((n, K, R, d))_q$ Clifford subsystem codes that do not satisfy*

$$\sum_{j=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{j} (q^2 - 1)^j \leq q^n / KR.$$

An obvious question is why impure codes can potentially pack more efficiently than the pure codes. Let us understand this by looking at the $[[9, 1, 4, 3]]_2$ code a little more closely. This code encodes information into a subspace, Q where $\dim Q = 2^{k+r} = 2^5$. As it is a subsystem code Q can be decomposed as $Q = A \otimes B$, with $\dim A = 2^k = 2$ and $\dim B = 2^r = 2^4$. In a pure single error correcting code all single errors must take the code space into orthogonal subspaces. In an impure code this is not required two or more distinct errors can take the code space to the same orthogonal space. In the Bacon-Shor code a phase flip error on any of the first three qubits will take the code space to same orthogonal subspace and because of this we cannot distinguish between these errors. However, it is not a problem because we can restore the code space with respect to A even though we cannot restore B . Thus instead of requiring 9 orthogonal subspaces as in a pure code, we only require 3 orthogonal subspaces to correct for any single phase flip error. Considering the bit flip errors and the combinations we need only 9 orthogonal subspaces. Thus with the original code space this means we need to pack ten 2^5 -dimensional subspaces in the $2^n = 2^9$ dimensional ambient space, which is achievable as $10 \cdot 2^5 < 2^9$.

More generally, in a sense degeneracy allows distinct errors to share the same orthogonal subspace and thus pack more efficiently. It must be pointed out though that this better packing is attained at the cost of r gauge qudits compared to a stabilizer code.

In fact there exists another code among the Bacon-Shor codes which also beats the Hamming bound for the subsystem codes. This is the $[[16, 1, 9, 4]]_2$ code. The family of codes given in corollary 4.2 provides us with $[[12, 1, 6, 3]]_2$, yet another example of a code that beats the quantum Hamming bound like the $[[9, 1, 4, 3]]_2$ code. We can check that

$$\sum_{j=0}^1 \binom{12}{j} 3^j = 37 > 2^{12-1-6} = 2^5 = 32.$$

But note that unlike $[[9, 1, 4, 3]]_2$ this code does not meet the Singleton bound for pure subsystem codes as $6 + 1 < 12 - 6 + 2$. Naturally we can ask if there is a systematic method to construct codes that beat the quantum Hamming bound. At the moment we do not know. It appears unlikely that there exist long codes that beat the quantum Hamming bound.

6. Conclusion

We have proved that any \mathbb{F}_q -linear $[[n, k, r, d]]_q$ Clifford subsystem code obeys the Singleton bound $k + r \leq n - 2d + 2$. Furthermore, we have shown earlier that pure Clifford subsystem codes satisfy this bound as well. Our results provide much evidence for the conjecture that the Singleton bound holds for arbitrary subsystem codes.

Pure Clifford subsystem codes obey the Hamming (or sphere packing) bound. In this paper, we have shown the amazing fact that there exist impure Clifford subsystem codes beating the Hamming bound. This is the first illustration of a case when impure codes pack more efficiently than their pure counterparts. One example of a code beating the Hamming bound is provided by the $[[9, 1, 4, 3]]_2$ Bacon-Shor code; this remarkable example also illustrates the following noteworthy facts:

- a) The $[[9, 1, 4, 3]]_2$ code requires $9 - 1 - 4 = 4$ syndrome measurements just like the perfect $[[5, 1, 3]]_2$ code.
- b) Since $k + r \leq n - 2d + 2$ for all prime alphabet codes, $[[9, 1, 4, 3]]_2$ code is also an optimal subsystem code. This is interesting because the underlying classical codes are not MDS. In MDS stabilizer codes, the underlying classical codes are required to be MDS codes.
- c) The Bacon-Shor code is also impure. So unlike MDS stabilizer codes which must be pure, MDS subsystem codes can be impure.
- d) The maximal length of a q -ary stabilizer MDS code is $2q^2 - 2$, (Ketkar *et al.* 2006) whereas for subsystem codes it is larger as the $[[9, 1, 4, 3]]_2$ code indicates. The implication of b)–d) is that optimal subsystem codes can be derived from suboptimal classical codes, unlike stabilizer codes.

We conclude with a few open questions that seem worth investigating.

- i) Do arbitrary $[[n, k, r, d]]_q$ subsystem codes also satisfy $k + r \leq n - 2d + 2$?
- ii) Is the Hamming bound for subsystem codes obeyed asymptotically?
- iii) What is the maximal length of MDS subsystem codes?

The second question is motivated by the fact that binary stabilizer codes obey the quantum Hamming bound asymptotically, see Ashikhmin & Litsyn (1999).

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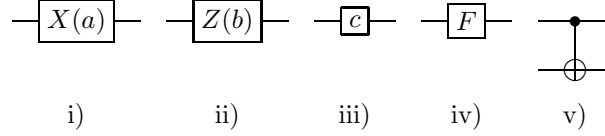
Appendix A. Syndrome measurement for nonbinary \mathbb{F}_q -linear codes

Decoding of nonbinary quantum codes has not been studied as well as binary codes. Encoding of \mathbb{F}_q -linear nonbinary quantum codes was investigated in Grassl *et al.* (2003). The authors suggest that the decoder is simply the encoder running backwards. While that maybe reasonable in quantum communication, it is not preferable in the case of quantum computation.

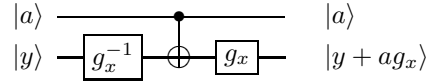
Here we give a method that allows us to measure the syndrome for \mathbb{F}_q -linear nonbinary quantum codes. We also show that an \mathbb{F}_q -linear $[[n, k, r, d]]_q$ code requires $n - k - r$ syndrome measurements. But first we need the definition of the following nonbinary gates, see Grassl *et al.* (2003).

- i) $X(a) |x\rangle = |x + a\rangle$
- ii) $Z(b) |x\rangle = \omega^{\text{tr}_{q/p}(bx)} |x\rangle$, $\omega = e^{j2\pi/p}$
- iii) $M(c) |x\rangle = |cx\rangle$, $c \in \mathbb{F}_q^\times$
- iv) $F |x\rangle = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} \omega^{\text{tr}_{q/p}(xy)} |y\rangle$
- v) $A |x\rangle |y\rangle = |x\rangle |x + y\rangle$

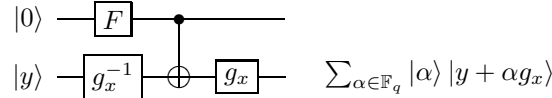
Graphically, these gates are represented below.



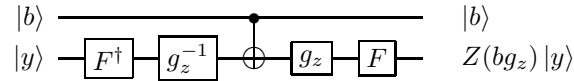
Consider the following circuit.



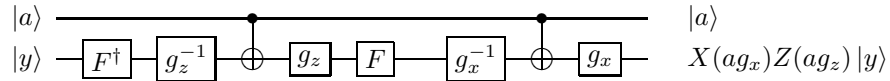
Alternatively, this circuit maps $|a\rangle |x\rangle$ to $|a\rangle X(ag_x) |y\rangle$. Observe that this circuit effectively applies $X(ag_x)$ on the second qudit. Using the linearity, we can analyze the following circuit.



The above circuit maps $|0\rangle |y\rangle$ to $\sum_{\alpha \in \mathbb{F}_q} |\alpha\rangle X(\alpha g_x) |y\rangle$. Using the fact that $FX(b)F^\dagger = Z(b)$, we can show that the following circuit maps $|b\rangle |y\rangle$ to $|b\rangle Z(bg_z) |y\rangle$.



If we wanted to apply a general operator $X(ag_x)Z(ag_z)$ to the second qudit conditioned on the first one, then we can combine the previous circuits as follows.

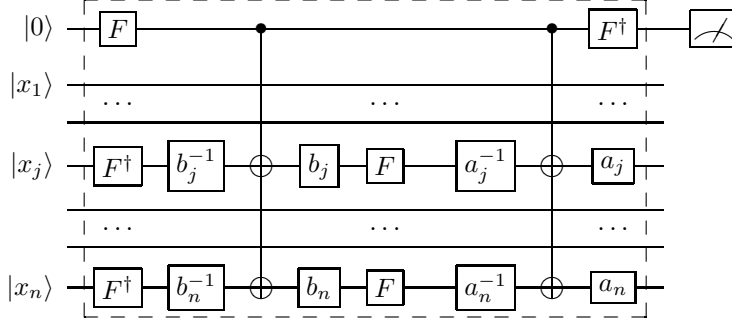


The above implementation is not optimal in terms of gates, but it will suffice for our purposes. Consider an $[[n, k, r, d]]_q$ code. Let E be an error in \mathcal{E} . If E is detectable,

then E does not commute with some element(s) in the stabilizer of the code. Let

$$g = (g_x|g_z) = (0, \dots, 0, a_j, \dots, a_n|0, \dots, 0, b_j, \dots, b_n) \in \mathbb{F}_q^{2n},$$

where $(a_j, b_j) \neq (0, 0)$, be a generator of the stabilizer. Then for all detectable errors that do not commute with a multiple of g , the following circuit gives a nonzero value on measurement.



Note that whenever $(a_i, b_i) = (0, 0)$, then we leave that qudit alone. Similarly if a_i or b_i are zero, then we do not implement the corresponding portion. Let the input to the above circuit be $E|\psi\rangle$, where $|\psi\rangle$ is an encoded state. It can be easily verified that the above circuit maps the state $|0\rangle E|\psi\rangle$ to

$$\sum_{\alpha \in \mathbb{F}_q} F^\dagger |\alpha\rangle X(\alpha g_x) Z(\alpha g_z) E |\psi\rangle.$$

Let $X(g_x)Z(g_z)E = \omega^{\text{tr}_{q/p}(t)} EX(g_x)Z(g_z)$, where $X(g_x)Z(g_z)$ is corresponding matrix representation of g . Then we have $X(\alpha g_x)Z(\alpha g_z)E = \omega^{\text{tr}_{q/p}(\alpha t)} EX(g_x)Z(g_z)$, by lemma 5 in (Ketkar *et al.* 2006). Thus we can write

$$\begin{aligned} \sum_{\alpha \in \mathbb{F}_q} |\alpha\rangle X(\alpha g_x) Z(\alpha g_z) E |\psi\rangle &= \sum_{\alpha \in \mathbb{F}_q} |\alpha\rangle \omega^{\text{tr}_{q/p}(\alpha t)} EX(\alpha g_x) Z(\alpha g_z) |\psi\rangle, \\ &= \left(\sum_{\alpha \in \mathbb{F}_q} |\alpha\rangle \omega^{\text{tr}_{q/p}(\alpha t)} \right) E |\psi\rangle, \end{aligned}$$

where we have made use of the fact that $X(\alpha g_x)Z(\alpha g_z) |\psi\rangle = |\psi\rangle$ as $X(\alpha g_x)Z(\alpha g_z)$ is in the stabilizer. The final state is given by

$$\begin{aligned} \sum_{\alpha \in \mathbb{F}_q} F^\dagger |\alpha\rangle X(\alpha g_x) Z(\alpha g_z) E |\psi\rangle &= \sum_{\alpha \in \mathbb{F}_q} F^\dagger |\alpha\rangle \omega^{\text{tr}_{q/p}(\alpha t)} E |\psi\rangle, \\ &= \sum_{\alpha \in \mathbb{F}_q} \sum_{\beta \in \mathbb{F}_q} \omega^{-\text{tr}_{q/p}(\alpha \beta)} |\beta\rangle \omega^{\text{tr}_{q/p}(\alpha t)} E |\psi\rangle, \\ &= \sum_{\beta \in \mathbb{F}_q} |\beta\rangle \sum_{\alpha \in \mathbb{F}_q} \omega^{\text{tr}_{q/p}(\alpha t - \alpha \beta)} E |\psi\rangle, \\ &= \sum_{\beta \in \mathbb{F}_q} |\beta\rangle \sum_{\alpha \in \mathbb{F}_q} \omega^{\text{tr}_{q/p}(\alpha t - \alpha \beta)} E |\psi\rangle, \\ &= |t\rangle E |\psi\rangle, \end{aligned}$$

where the last equality follows from the property of the characters of \mathbb{F}_q . Next we observe that the error αE , where $\alpha \in \mathbb{F}_q$ gives $|\alpha t\rangle$ on measurement. Strictly speaking we refer to the preimage of $\alpha \overline{E}$ in \mathcal{E} . Hence the syndrome qudit can take q different values. Since every detectable error does not commute with some \mathbb{F}_q -multiple of a stabilizer generator, we have the following lemma on the necessary and sufficient number of syndrome measurements.

Lemma 6.1. *Given an \mathbb{F}_q -linear $[[n, k, r, d]]_q$ Clifford subsystem code, $n - k - r$ syndrome measurements are required for decoding it completely.*

Proof. Let g be a generator of the stabilizer of the subsystem code. By theorem 2.1 and lemma 3.2, for every generator g there exists at least one detectable error that does not commute with g but commutes with all the other generators. This error can be detected only by measuring g . Thus we need to measure all the generators of the stabilizer, equivalently $n - k - r$ syndrome measurements must be performed.

Every correctable error takes the code space into a q^{k+r} -dimensional orthogonal subspace in the q^n -dimensional ambient space, see §2. Each of these errors will give a distinct syndrome. This implies that we can have q^{n-k-r} distinct syndromes. Since each syndrome measurement can have q possible outcomes and there are $n - k - r$ generators, these measurements are sufficient for performing error correction. \square

This parallels the classical case where an $[[n, k, d]]_q$ code requires $n - k$ syndrome bits. A subtle caveat must be issued to the reader. If we choose to perform bounded distance decoding, then it may be possible that the set of correctable errors can be distinguished by a smaller number of syndrome measurements. But even in the case of (classical) bounded distance decoding it is often the case that we need to measure all the syndrome bits.

References

- Aly, S. A., Klappenecker, A. & Sarvepalli, P. K. (2006). Subsystem codes. In *Forty-fourth annual Allerton conference on communication, control, and computing, Illinois, USA*. (eprint:quant-ph/0610153)
- Ashikhmin, A. & Litsyn, S. (1999). Upper bounds on the size of quantum codes. *IEEE Trans. Inform. Theory*, *4*, 1206-1216.
- Bacon, D. (2006). Operator quantum error correcting subsystems for self-correcting quantum memories. *Phys. Rev. A*, **73**(012340).
- Bacon, D. & Casaccino, A. (2006). Quantum error correcting subsystem codes from two classical linear codes. In *Forty-fourth annual Allerton conference on communication, control, and computing, Illinois, USA*.
- Grassl, M. & Beth, T. (1996). Improved decoding of quantum error correcting codes from classical codes. In *Proceedings of the workshop on Physics and Computation, Boston, USA* (pp. 28–31).
- Grassl, M., Rötteler, M. & Beth, T. (2003). Efficient quantum circuits for non-qubit quantum error-correcting codes. *Internat. J. Found. Comput. Sci.*, **14**(5), 757–775.
- Ketkar, A., Klappenecker, A., Kumar, S. & Sarvepalli, P. K. (2006). Nonbinary stabilizer codes over finite fields. *IEEE Trans. Inform. Theory*, **52**(11), 4892–4914. (eprint:quant-ph/0508070)
- Klappenecker, A. & Sarvepalli, P. K. (2006). *Clifford code constructions of operator quantum error-correcting codes*. (eprint:quant-ph/0604161)

- Knill, E. (2006). *On protected realizations of quantum information*. (eprint: quant-ph/0603252)
- Kribs, D. W. (2006). A brief introduction to operator quantum error correction. *Contemporary Mathematics*, **414**, 27–34.
- Kribs, D. W., Laflamme, R. & Poulin, D. (2005). A unified and generalized approach to quantum error correction. *Phys. Rev. Lett.*, **94**(180501).
- Kribs, D. W., Laflamme, R., Poulin, D. & Lesosky, M. (2006). Operator quantum error correction. *Quantum Information & Computation*, **6**, 382–399.
- Poulin, D. (2005). Stabilizer formalism for operator quantum error correction. *Phys. Rev. Lett.*, **95**(230504).