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ON SPECTRAL PROPERTIES OF PERTURBED OPERATORS

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ABSTRACT. Farid (1991) has given an estimate for the norm of a perturbation V required to obtain an eigenvector for the perturbed operator $T+V$ within a given ball centered at a given eigenvector of the unperturbed (closed linear) operator T. A similar result is derived from a more general result of the author (1989) which also guarantees that the corresponding eigenvalue is simple and also that the eigenpair is the limit of a sequence obtained in an iterative manner.

1. INTRODUCTION

In a recent paper [3] Farid has considered a method based on contraction mapping theorem instead of the fixed-point theorem approach of Rosenbloom [7] to address the following problem in perturbation theory:

If (λ_0, ϕ_0) is an eigenpair of a densely defined closed linear operator in a Banach space $\mathbf{X}$, and r and ρ are given positive reals, then obtain an estimate for the radius of the disc $\{V \in BL(X): \|V\| \leq \delta\}$ such that the perturbed operator $T+V$ has an eigenpair (λ, ϕ) with

$$\|\phi - \phi_0\| \leq r \quad \text{and} \quad |\lambda - \lambda_0| \leq \rho$$

for every V in $\{V \in BL(X): \|V\| \leq \delta\}$, where $BL(X)$ denotes the space of all bounded linear operators on X.

In this note a result similar to that of Farid [3] is derived from a more general result in Nair [5]. While the results of Farid [3] and Rosenbloom [7] are essentially existential results, ours is an iterative procedure where sequences (λ_k) and (ϕ_k) are obtained in an iterative manner with the property that $\lambda_k \to \lambda$ and $\phi_k \to \phi$ as $k \to \infty$. Moreover, the eigenvalue λ is shown to be a *simple* eigenvalue of $T+V$, and a disc centered at λ_0 is obtained where λ is the only spectral value of $T+V$ lies. The uniqueness of the pair (λ, ϕ) established by Farid [3] is a consequence of the simplicity of λ.

2. THE MAIN RESULT

Let T be a closed linear operator in a Banach space X with a dense domain D. Let λ_0 be an eigenvalue of T with a corresponding eigenvector ϕ_0 with

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$\|\phi_0\| = 1$. The basic assumption in Farid [3] is the following:

(i) λ_0^*, the complex conjugate of λ_0, is an eigenvalue of the adjoint operator T^*, and $\phi_0^* \in X^*$ is a corresponding eigenvector such that $\langle \phi_0, \phi_0^* \rangle = 1$.

(ii) λ_0 does not belong to the spectrum of the operator $\widetilde{T} := T|_Y$, where $Y = \{x \in D : \langle x, \phi_0^* \rangle = 0\}$.

Here and in what follows X^* denotes the adjoint space of X, that is, the space of all conjugate linear functionals on X, and $\langle x, f \rangle$ denotes the complex conjugate of $f(x)$ for $x \in X$ and $x^* \in X^*$. The adjoint operator T^* is defined by $\langle x, T^* f \rangle = \langle Tx, f \rangle$ for all $x \in D$ and $f \in D(T^*) := \{f \in X^* :$ there exists $g \in X^*$ with $\langle Tx, f \rangle = \langle x, g \rangle$ for all $x \in D\}$. First we observe that assumption (i) implies the subspaces

$$X_1 := \{x \in X : \langle x, \phi_0^* \rangle \phi_0 = x\}$$

and

$$X_2 := \{x \in X : \langle x, \phi_0^* \rangle = 0\}$$

are invariant under T, i.e., $Tx \in X_i \cap D$ for every $x \in X_i \cap D$, $i = 1, 2$, with

$$X = X_1 \oplus X_2,$$

and assumption (ii) implies, as a consequence of Theorem 4.2 in Nair [5], that λ is in fact a *simple* eigenvalue of T. Also, we note that the operator $P_0 : X \to X$ defined by

$$P_0 x = \langle x, \phi_0^* \rangle \phi_0, \qquad x \in X,$$

is the projection operator onto X_1 along X_2, and $\|P_0\| = \|\phi_0^*\|$. Let

$$S_0 := (\widetilde{T} - \lambda_0)^{-1} : X_2 \to X_2.$$

With the above notation the main result of Farid [3] is the following.

Theorem (Farid [3, Theorem 2.1]). *For every real number r satisfying*

$$0 < r < \left(\frac{\|S_0(I - P_0)\|}{\|P_0\| \, \|S_0\|} \right)^{1/2}$$

and every bounded linear operator V on X satisfying

$$\|V\| \le \delta(r) := r / (\|P_0\| \, \|S_0\| r^2 + (\|P_0\| \, \|S_0\| + \|S_0(I - P_0)\|) r + \|S_0(I - P_0)\|),$$

the operator $T + V$ has a unique eigenpair (λ, ϕ) such that

$$\langle \phi, \phi_0^* \rangle = 1, \qquad \|\phi - \phi_0\| \le r,$$

and

$$|\lambda - \lambda_0| \le \|V\| (1 + \|\phi - \phi_0\|) \|P_0\|.$$

The main result of this note is the following.

Theorem $*$. *For every real number $r > 0$ and for every bounded linear operator V on X satisfying*

$$\beta_V := \max\{\|P_0 V\|, \|(I - P_0) V\|\} \le \frac{r}{\|S_0\|(1 + r)^2},$$

the operator $T + V$ has a simple eigenvalue λ and a corresponding (unique) *eigenvector* ϕ *such that*

$$\langle \phi, \phi_0^* \rangle = 1, \qquad \|\phi - \phi_0\| \le r,$$
$$|\lambda - \lambda_0| \le \|V\|(\|\phi - \phi_0\| + 1)\|P_0\|,$$

and λ *is the only spectral value of* $T + V$ *lying in the disc*

$$\Delta_0 := \{z : |z - q_0| < \frac{d_0}{2}(1 + \sqrt{1 - 4\mu})\},$$

where

$$d_0 = \frac{(1 - 2\beta_V \|S_0\|)}{\|S_0\|}, \quad q_0 = \lambda_0 + \langle V\phi_0, \phi_0^* \rangle, \quad \mu = \left(\frac{r}{1 + r^2}\right)^2.$$

Moreover,

$$\lambda = \lim_{k \to \infty} \lambda_k, \qquad \phi = \lim_{k \to \infty} \phi_k,$$

where λ_k *and* ϕ_k *are defined iteratively as*

$$\begin{aligned}(\widetilde{T} + \widetilde{V} - \lambda_0 - \langle V\phi_0, \phi_0^* \rangle)\psi_1 &= -(I - P_0)V\phi_0, \\ \phi_1 &= \phi_0 + \psi_1, \\ \lambda_1 &= \lambda_0 + \langle V\psi_1, \phi_0^* \rangle\end{aligned}$$

and, for $k = 1, 2, \ldots,$

$$\begin{aligned}(\widetilde{T} + \widetilde{V} - \lambda_0 - \langle V\phi_0, \phi_0^* \rangle)x_k &= \langle V\psi_k, \phi_0^* \rangle \psi_k, \\ \psi_{k+1} &= \psi_1 + x_k, \\ \phi_{k+1} &= \phi_0 + \psi_{k+1}, \\ \lambda_{k+1} &= \lambda_0 + \langle V\psi_{k+1}, \phi_0^* \rangle.\end{aligned}$$

Here $\widetilde{T} = T|_Y$ *and* $\widetilde{V} = (I - P_0)V_{(I-P_0)X}$.

Remark. We note that

$$\beta_V := \max\{\|P_0 V\|, \|(I - P_0)V\|\} \le c_0 \|V\|,$$

where $c_0 = \max\{\|P_0\|, \|I - P_0\|\}$. Therefore, a sufficient condition for Theorem * to hold is

$$\|V\| \le \omega(r) := \frac{r}{c_0 \|S_0\|(1 + r)^2}.$$

Also,

$$\|P_0\| \, \|S_0\| r^2 + (\|P_0\| \, \|S_0\| + \|S_0(I - P_0\|)r + \|S_0(I - P_0)\| \le c_0 \|S_0\|(1 + r)^2,$$

so that in general,

$$\omega(r) \le \delta(r),$$

and thereby the assumption '$\|V\| \le \delta(r)$' of Farid [3] is weaker than '$\|V\| \le \omega(r)$'. However, if $\|P_0\| = 1 = \|I - P_0\|$, then

$$\beta_V \le \|V\|, \qquad \omega(r) = \delta(r) = \frac{r}{\|S_0\|(1 + r)^2},$$

so in this case the condition in Theorem * is weaker than that of Farid [3], and therefore Theorem * improves the result of Rosenbloom [7] also. Examples with $\beta_V < \|V\|$ can be easily constructed. It is to be noted that if X is a

Hilbert space and T is a normal operator on X, then we have $\phi_0^* = \phi_0$, so that the projection P_0 is orthogonal and therefore $\|P_0\| = 1 = \|I - P_0\|$.

We recall the following from [5] or [4]. If $X = Y_1 \oplus Y_2$ is a decomposition of X into closed subspaces Y_1 and Y_2, B is a bounded linear operator on Y_1, and C is a closed linear operator in Y_2 with domain D_C, then the operator $F: BL(Y_1, Y_2 \cap D_C) \to BL(Y_1, Y_2)$ defined by

$$F(K) = CK - KB, \qquad K \in BL(Y_1, Y_2 \cap D_C)$$

has a bounded inverse on $BL(Y_1, Y_2)$ if and only if $\sigma(B) \cap \sigma(C) = \varnothing$. The *separation* between B and C is defined by

$$sep(B, C) := \begin{cases} 1/\|F^{-1}\| & \text{if } F \text{ has bounded inverse,} \\ 0 & \text{otherwise.} \end{cases}$$

If $E_1 \in BL(Y_1)$ and $E_2 \in BL(Y_2)$, then

$$sep(B + E_1, C + E_2) \geq sep(B, C) - (\|E_1\| + \|E_2\|).$$

Proof (Theorem *). Let (T_{ij}), (V_{ij}), and (A_{ij}), $i, j = 1, 2$, be the 2×2 matrix representations of T, V, and $A = T + V$ respectively with respect to the decomposition $X = X_1 \oplus X_2$ (cf. [8, p. 286]). Then it is seen that

$$\|V_{ij}\| \leq \|P_i V\| \leq \beta_V := \max\{\|P_0 V\|, \|(I - P_0)V\|\}, \qquad i, j = 1, 2,$$

with $P_1 = P_0$ and $P_2 = I - P_0$. Therefore, we have

$$sep(A_{11}, A_{22}) \geq sep(T_{11}, T_{22}) - (\|V_{11}\| + \|V_{22}\|) \geq \frac{(1 - 2\beta_V\|S_0\|)}{\|S_0\|}.$$

Now the condition $\beta_V \leq r/(1+r)^2\|S_0\|$ implies that $2\beta_V\|S_0\| \leq 2r/(1+r)^2 \leq \frac{1}{2}$, so that $sep(A_{11}, A_{22}) > 0$ and consequently the assumption $\sigma(A_{11}) \cap \sigma(A_{22}) = \varnothing$ in Nair [5] is satisfied. Now the quantity ε in [5] is seen to satisfy

$$\begin{aligned} \varepsilon := \frac{\|F^{-1}(A_{12})\|\,\|A_{12}\|}{sep(A_{11}, A_{22})} &\leq \frac{\|A_{12}\|A_{21}\|}{sep(A_{11}, A_{22})^2} \\ &\leq \left(\frac{\beta_V\|S_0\|}{1 - 2\beta_V\|S_0\|}\right)^2 \leq \left(\frac{r}{1 + r^2}\right)^2 \leq \frac{1}{4}. \end{aligned}$$

Writing $\mu = (r/(1 + r^2))^2$ and $g(\mu) = (1 - \sqrt{1 - 4\mu})/2\mu$, it follows from ([5, Theorem 4.3 and relation (4.4)]) that $A := T + V$ has a simple eigenvalue λ and a corresponding eigenvector ϕ such that

$$\begin{aligned} &\langle \phi, \phi_0^* \rangle = 1, \\ &\|\phi - \phi_0\| \leq \alpha g(\mu), \\ &|\lambda - \lambda_0| \leq \frac{\delta_0}{2}(1 - \sqrt{1 - 4\mu}) \end{aligned}$$

and λ is the only spectral value of A lying in the disc

$$\{z : z - \lambda_0| < \frac{\delta_0}{2}(1 + \sqrt{1 - 4\mu})\} \supseteq \Delta_0.$$

Here

$$\delta_0 := sep(A_{11}, A_{22}) \geq \frac{(1 - 2\beta_V\|S_0\|)}{\|S_0\|} = d_0,$$

$$\alpha \leq \frac{\|(I - P_0)V\|}{sep(A_{11}, A_{22})} \leq \frac{\beta_V\|S_0\|}{1 - 2\beta_V\|S_0\|} \leq \frac{r}{1 + r^2} = \sqrt{\mu},$$

and $g(t)$, $0 < t \leq \frac{1}{4}$, satisfies

$$1 \leq g(t) \leq 2,$$
$$g(t_1) \leq g(t_2) \quad \text{for } t_1 \leq t_2,$$
$$\lim_{t\to 0} g(t) = 1, \quad \text{and} \quad \lim_{t\to 1/4} g(t) = 2.$$

It is easily seen that

$$\alpha g(\mu) \leq \sqrt{\mu} g(\mu) \leq r,$$

so that $\|\phi - \phi_0\| \leq r$. Since $\langle \phi, \phi_0^* \rangle = 1$ and $T^*\phi_0^* = \lambda_0^*\phi_0^*$, we have

$$\lambda = \lambda_0 + \langle V(\phi - \phi_0), \phi_0^* \rangle + \langle V\phi_0, \phi_0^* \rangle.$$

Therefore,

$$|\lambda - \lambda_0| \leq \beta_V(\|\phi - \phi_0\| + 1)\|P_0\|.$$

If $\tilde{\phi}$ is another eigenvector of $T + V$ corresponding to the simple eigenvalue λ such that $\langle \tilde{\phi}, \phi_0^* \rangle = 1$, then $\tilde{\phi} = c\phi$ for some constant $c \neq 0$, and therefore $1 = \langle \tilde{\phi}, \phi_0^* \rangle = c\langle \phi, \phi_0^* \rangle = c$. Thus $\tilde{\phi} = \phi$, proving the uniqueness of ϕ.

Lastly, the iterative procedure to obtain (λ_k) and (ϕ_k), and their convergence to λ and ϕ respectively, are the consequences of [5, relations (3.5), (3.6)] and [5, Theorem 4.3], respectively.

Remark. We note that the generalized Rayleigh quotient $q = \langle (T + V)\phi_0, \phi_0^* \rangle$ of $T + V$ at (ϕ_0, ϕ_0^*) satisfies

$$|\lambda - q| \leq \beta_V\|\phi - \phi_0\|.$$

A similar reformulation of the results in Nair [5, 6] and Stewart [9] involving spectral sets and spectral subspaces will show their applicability to more general situations of diagonally dominant infinite matrices than the ones described in [1–3].

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