ON HARDER-NARASIMHAN REDUCTIONS FOR HIGGS PRINCIPAL BUNDLES

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Abstract. The existence and uniqueness of H-N reduction for the Higgs principal bundles over nonsingular projective variety is shown in this article. We also extend the notion of H-N reduction for (Γ, G)-bundles and ramified G-bundles over a smooth curve.

1. Introduction

Let X be any smooth projective variety over an algebraic closed field k of characteristic zero and G be any reductive algebraic group over k. The problem of the existence and uniqueness of Harder-Narasimhan reductions (henceforth called briefly as H-N reduction) for principal G-bundles was solved by Atiyah and Bott in [3]. The notion of semistability that was used in this paper was using the adjoint representaion, i.e E is semistable iff $E(\mathfrak{g})$ is semistable as a vector bundle.

In an earlier article by A.Ramanathan ([16]) this question of H-N reduction was posed as a problem intrinsically on E for its applicability to the positive characteristic cases as well. The problem in this setting was solved by Behrend [8] and in positive characteristics Behrend had a conjecture which was verified by Mehta and Subramaniam. Biswas and Holla gave a different approach to the solution following essentially the broad setting in Ramanathan's paper in [7].

The aim of this paper is to generalise the methods of Biswas and Holla to give an unified approach to the case of principal bundles with Higgs structure on smooth projective varieties as well as the case of ramified bundles on smooth curves (as defined in [4])(see §2,§5. for definitions). Recall that for the case of Higgs vector bundles on smooth projective varieties the existence and uniqueness of H-N filtrations is proved in §3 of Simpson [18]. The case of ramified bundles similarly generalise the existence and uniqueness of H-N filtrations for parabolic vector bundles (proved in [14]).

By a *principal object* in this introduction we mean a principal Higgs bundle on smooth projective varieties or a ramified bundle on smooth projective curves.(See §1 for the definition of H-N reduction's)

The main theorem of this paper is:

Theorem 1. *Let E be a principal G object on X. Then there exists a canonical H-N reduction (P, σ) where P is a parabolic subgroup of G and $\sigma : X \longrightarrow E/P$ is a section of the associated fibre bundle E/P over X. The H-N reduction is unique in natural sense.*

The brief layout of this paper is as follows; in §2 we define the notions which are necessary for our problem,in §3 we prove the existence of the H-N reduction for Higgs principal bundles, in §4 we prove that for any finite dimensional representation W of G $E(W)$ is Higgs semistable iff E is Higgs semistable, in §5 we prove the uniqueness of the H-N reduction for Higgs principal bundles, in §6 we prove H-N reduction for ramified G-bundles.

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2. Preliminaries

Throughout this paper, unless otherwise stated, we have the following notations :

(1) We work over an algebraically closed field k of characteristic 0.
(2) G will stand for a connected reductive algebraic group (i.e, for the linear algebraic group G, unipotent radical of G, denoted by $R_u(G)$, is trivial).
(3) X is an irreducible smooth projective variety over k of dimension $d \geq 2$.
(4) Let X be embedded in $\mathbb{P}^N$ for some positive integer N, which is equivalent to fixing a very ample line bundle on X, we call it H which we fix it throughout the paper.

Let E be a torsion-free coherent sheaf of rank r over X. Then the degree of E is defined as $\deg(E) = c_1(E) \cdot H^{d-1}$, where $c_1(E) \in \mathrm{H}^2(X, \mathbb{Z})$ is the first Chern class. Let U be an open subscheme containing all points of codimension one, $(codim(X \backslash U) \geq 2)$ such that $E|_U$ is locally free. Then $\wedge^r(E\mid_U)$ is an invertible sheaf on U, corresponding to a divisor class D on U. Then we have $c_1(E\mid_U) = c_1(\wedge^r E\mid_U) = D$. From the functoriality of Chern classes, $c_1(E) = c_1(E\mid_U) = D$.

Let U be an open subset of X such that $codim(X \backslash U) \geq 2$, then $i_*\mathcal{O}_U \cong \mathcal{O}_X$, further if $\mathcal{F}$ is a locally free sheaf on U then $i_*\mathcal{F}$ is a reflexive sheaf over X.

It follows from the above that if two bundles agree at all points of codimension one, they have the same degree. Also, we can talk about the degree of torsion-free sheaf on U(since $codim(X \backslash U) \geq 2$). Let E be a torsion-free sheaf on U, then we define $\deg(E) = \deg(i_*(E))$, where i is the inclusion $i : U \longrightarrow X$. Further, if $0 \to F \to E \to G \to 0$ is an exact sequence of torsion-free sheaves on X, then $\deg(E) = \deg(F) + \deg(G)$, also $\deg E^* = -\deg(E)$. Again from the functoriality of Chern classes, for exact sequence of locally free sheaves over U we have the same. For if, $0 \to F \to E \to G \to 0$ is an exact sequence of locally free sheaves on U, then we have $\deg(i_*E) = \deg(i_*F) + \deg(i_*G)$.Therefore it follows that, $\deg(E) = \deg(F) + \deg(G)$ on U. Now we can define for a torsion-free sheaf E on X,

$$\mu(E) = \frac{\deg(E)}{rank(E)} \in \mathbb{Q}$$

A torsion-free sheaf E is said to be μ-semistable if for all coherent subsheaves $F \subseteq E$ we have

$$\mu(F) = \frac{\deg(F)}{rank(F)} \leq \frac{\deg(E)}{rank(E)} = \mu(E).$$

Note that it is enough to check only for subsheaves whose quotients are torsion-free. Let E be a torsion-free coherent sheaf on X and $\theta : E \to E \otimes \Omega^1_X$, an $\mathcal{O}_X$–linear homomorphism of sheaf of modules, such that $\theta \wedge \theta = 0$ in $E \otimes \Omega^2_X$, then the pair (E, θ) is called *Higgs sheaf on* X. If E is locally free, then the pair (E, θ) is called *Higgs bundle on* X. Let F be a subsheaf of E such that $\theta |_F : F \to F \otimes \Omega^1_X$, then we say $(F, \theta |_F)$ is a Higgs subsheaf of (E, θ). Further, (E, θ) is called *Higgs semistable* if for every Higgs subsheaf $F \subseteq E$ we have, $\mu(F) \leq \mu(E)$. We also have the notion of Chern classes of the principal G bundle E over X; let $c_i(E) \in H^*(X, \mathbb{Z})$ be the Chern classes of the principal bundle E. We will say that $c_i(E) = 0$ for any i if $c_i(E(W)) = 0$ for every $W \in Rep(G)$, where $Rep(G)$ is the category of finite dimensional representations of G ([5] Rem.2.3,).

The aim of this paper is to find Higgs compatible H-N reduction for Higgs principal G bundle E on X. Now we recall some facts which we need in our work.

(1) Let E be a principal G-bundle on X. Let Y be any quasi-projective variety on which G acts from the left, then we define by $E(Y)$ as the associated fiber bundle with fiber type Y which is the following object: $E(Y) = (E \times Y)/G$ for the twisted action of G on $(E \times Y)$ given by $g.(e, y) = (e.g^{-1}, g.y)$ where g, e, y are in G, E, Y respectively.

(2) Let H be a closed subgroup of G. A principal G bundle E on X is said to have a H structure or equivalently a reduction of structure group to H if there exists an open subset U of X which contains all the points of codimension one $codim(X \backslash U) \geq 2$ and a section $\sigma : U \longrightarrow E(G/H)$. In other words a reduction of structure group to H means giving a principal H bundle E_H over an open set U which satisfies $codim(X \backslash U) \geq 2$ such that $E_H(G) \cong E |_U$. Given a reduction of structure group over an open subset U with $codim(X \backslash U) \geq 2$ there is a unique maximal open set over which the reduction extends.

(3) Let H be a closed subgroup of G and U be an open subset of X. A principal G bundle E on U is said to have a H structure or equivalently a reduction of structure group to H if there exists an open subset U' of U which contains all the points of codimension one $codim(U \backslash U') \geq 2$ and a reduction $\sigma : U' \longrightarrow E(G/H)$. In other words, a reduction of structure group to H means giving a principal H bundle E_H on U' such that $E_H(G) \cong E |_{U'}$.

(4) Let $\pi : E \to X$ be a principal G bundle, then E is called a semistable bundle if for every parabolic subgroup P of G and any reduction of structure group (P, σ) where $\sigma : U \longrightarrow E(G/P)$ is a reduction with $codim(X \backslash U) \geq 2$ and for every dominant character χ of P the associated line bundle $i_* L_\chi$ on X has degree ≤ 0 .

(5) Let E be a principal G bundle on X and $\theta \in \mathrm{H}^0(X, E(\mathfrak{g}) \otimes \Omega^1_X)$,($\mathfrak{g}$ is the Lie algebra of G) with $\theta \wedge \theta = 0$ in $E(\mathfrak{g}) \otimes \Omega^2_X$. Then the pair (E, θ) is called a Higgs principal G bundle on X.

Definition 2. *Let (E, θ) be a Higgs principal G bundle on X. Let (H, σ) be a reduction of E to H over an open subset U of X with $codim(X \backslash U) \geq 2$ where H is a connected closed subgroup of G and $\theta_H \in \mathrm{H}^0(U, E_H(\mathfrak{h}) \otimes \Omega^1_U)$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
\mathcal{O}_U & \xrightarrow{\theta} & E(\mathfrak{g})\otimes\Omega^1_U \\
 & \overset{\theta_H}{\searrow} & \uparrow \\
 & & E_H(\mathfrak{h})\otimes\Omega^1_U \\
 & & \uparrow \\
 & & 0
\end{array}
$$

Then the quadruple (H, σ, θ_H, U) is called Higgs reduction of E to H .

Remark 3. *Let (E, θ) be a Higgs principal G bundle over U, where U is an open subset of X with $codim(X\backslash U) \geq 2$. We can define the Higgs reduction of E to a subgroup H in the same way as above.*

Definition 4. *Let (E, θ) be a Higgs principal G bundle over X, it is said to be Higgs semistable if for any Higgs reduction $(P, \sigma, \theta_P, U_P)$ to a parabolic and for any dominant character χ of P the line bundle L_χ on U_P has non-positive degree which is same as saying the degree of the reflexive sheaf i_*L_χ over X has non positive degree.*

Definition 5. *Let (E, θ) be a Higgs principal G bundle over U, where U is an open subset with $codim(X\backslash U) \geq 2$. Then it is said to be Higgs semistable if for any Higgs reduction $(P, \sigma, \theta_P, U_P)$ to a parabolic, where $codim(U\backslash U_P) \geq 2$ and for any dominant character χ of P the line bundle L_χ on U_P has non-positive degree which is same as saying the degree of the reflexive sheaf i_*L_χ over X has non positive degree.*

Lemma 6. *Let (E, θ) be a Higgs principal G bundle over X. Then (E, θ) is Higgs semistable if and only if for any maximal parabolic subgroup P of G and for any Higgs reduction $(P, \sigma, \theta_P, U_P)$ we have $\deg i_*\sigma^*(T_{G/P}) \geq 0$ where $T_{G/P}$ is the relative tangent bundle for the projection $E\mid_U \to E\mid_U (G/P)$ and i is the inclusion $U \hookrightarrow X$.*

Proof: Let P be a maximal parabolic subgroup of G. Then consider the following exact sequence of vector bundles on U_P; $0 \to E_P(\mathfrak{p}) \to E_P(\mathfrak{g}) \to E_P(\mathfrak{g}/\mathfrak{p}) \to 0$ given by the exact sequence $0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p} \to 0$. Since G is reductive there is a non-degenerate bilinear form on $\mathfrak{g}$ invariant under G; therefore we see that $\deg E_P(\mathfrak{g}) = 0$. Therefore, $\deg E_P(\mathfrak{p}) \leq 0$ iff $\deg E_P(\mathfrak{g}/\mathfrak{p}) \geq 0$. So the 'only if' part is trivial. The other way can be proved following the proof of lemma 2.1. in [15]

Lemma 7. *When G is $GL(n, k)$, the Higgs semistability of Higgs principal G bundle E on X is equivalent to the Higgs semistability of the associated Higgs vector bundle $E(k^n)$ by the standard representation.*

Proof: Let P be a maximal parabolic subgroup of G. Let $(P, \sigma_P, \theta_P, U_P)$ be a Higgs reduction of E to P, where P is parabolic and let E_P be the corresponding P bundle. Let us consider the flag $0 \subset k^r \subset k^n$ corresponding to the parabolic subgroup P. Let $V = E_P(k^n), W = E_P(k^r)$; then W is a Higgs subbundle of V on U_P (with Higgs structure induced by θ_P). Note that $E_P(\mathfrak{g}/\mathfrak{p}) = V/W \otimes W^*$ and Higgs structures on both sides are same. (since they are coming from θ and θ_P). Also $\mu(W) \leq \mu(V)$ is equivalent to $\mu(W) \leq \mu(V/W)$ on U_P.

$$
\begin{aligned}
\deg((V/W)\otimes W^*) &= -\deg W.rank(V/W) + rankW.\deg(V/W) \\
&= (\mu(V/W) - \mu(W)).rank(V/W).rank(W)
\end{aligned}
$$

From the above equation it is easy to see that $\mu(W) \leq \mu(V)$ is equivalent to $\deg((V/W) \otimes W^*) \geq 0$. Since $E_P(\mathfrak{g}/\mathfrak{p}) = (V/W) \otimes W^*$ we conclude that $\deg(E_P(\mathfrak{g}/\mathfrak{p})) \geq 0$ iff $\mu(W) \leq \mu(V)$.

Suppose E is Higgs semistable. then we have, $\deg i_{P*}(E_P(\mathfrak{g}/\mathfrak{p})) \geq 0$ on X for any Higgs reduction of $(P, \sigma_P, \theta_P, U_P)$ to a maximal parabolic subgroup P where i_P is the inclusion of U_P to X. Let W be a Higgs torsion-free sheaf of V on X. We can choose an open subset U with $codim(X \backslash U) \geq 2$ such that W is a Higgs subbundle of V on U. Then $W = E_P(k^r)$ for some Higgs reduction (P, σ_P, θ_P) corresponding to a flag $0 \subset k^r \subset k^n$ on U. Then $\deg i_{P*}(E_P(\mathfrak{g}/\mathfrak{p})) \geq 0$ implies that $\mu(W) \leq \mu(V)$ on U. Thus V is Higgs semistable. Converse is also true.

Given a torsion-free coherent sheaf E over X, there is a unique filtration of subsheaves

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots\cdots\cdots \subset E_{l-1} \subset E_l = E$$

which is characterised by the two conditions that all the sheaves $\frac{E_i}{E_{i-1}}, i \in [1, l]$, are semistable torsion-free sheaves and the $\mu\left(\frac{E_i}{E_{i-1}}\right)$'s are strictly decreasing as i increases. This filtration is known as *Harder-Narasimhan filtration* for E The following is the definition of H-N reduction for principal bundles in the literature(cf. [1]).

Definition 8. *Let E be a principal G bundle on X and (P, σ_P, U_P) be a reduction of structure group of E to a parabolic subgroup P of G, then this reduction is called H-N reduction if the following two conditions hold:*

1. *If L is the Levi factor of P then the principal L bundle $E_P \times_P L$ over U_P is a semistable L bundle.*
2. *For any dominant character χ of P with respect to some Borel subgroup $B \subsetneq P$ of G, the associated line bundle L_χ over U_P has degree > 0.*

For $G = GL(n, k)$ a reduction E_P gives a filtration of the rank n vector bundle associated to the standard representation. It is easy to see that E_P is canonical in the above sense iff the corresponding filtration of the associated vector bundle filtration coincides with its Harder Narasimhan filtration.

Lemma 9. *Let $\rho : G \longrightarrow H$ be the homomorphism of connected reductive algebraic groups and let (E, θ) be the Higgs G bundle on X. Then the associated bundle $E_H := E(H)$ gets a Higgs structure.*

Proof: The representation ρ induces a morphism $\rho' : E(\mathfrak{g}) \longrightarrow E(\mathfrak{h})$ of vector bundles on X. where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively. So we define Higgs structure on $E(H)$ denoted by $\theta_H := (\rho' \otimes id) \circ \theta$ where $\rho' \otimes id : E(\mathfrak{g}) \otimes \Omega^1_X \longrightarrow E(\mathfrak{h}) \otimes \Omega^1_X$ and $\theta : \mathcal{O}_X \longrightarrow E(\mathfrak{g}) \otimes \Omega^1_X$.

Now we give a definition of a Higgs compatible H-N reduction for Higgs principal bundles.

Definition 10. *Let (E, θ) be a Higgs principal G bundle on X. Then a Higgs reduction $(E_P, \sigma_P, \theta_P, U_P)$ (Definition 2) is called a Higgs compatible H-N reduction if the following conditions hold:*

1. *The Higgs bundle E_L is Higgs semistable on U_P, where L is the Levi factor of P.*
2. *For all dominant character χ of P with respect to some Borel subgroup $B \subset P \subset G$ the associated line bundle L_χ has positive degree on U_P.*

From now onwards we call a Higgs compatible H-N reduction as a Higgs H-N reduction. Suppose H_1, H_2 are closed connected subgroups of G. Let σ_1, σ_2 be reductions of structure group of E to H_1 and H_2 respectively. To this data we can associate subgroup schemes $\sigma_i^* E(H_i)$ of $E(G)$ and their Lie algebras $\sigma_i^* E(\mathfrak{h}_i)$, which are sub-bundles of $E(\mathfrak{g})$. For details we refer to [7].

We recall some facts which we use in our proof of Higgs H-N reduction for Higgs principal bundles:

(1) Let (E_1, θ_1) and (E_2, θ_2) be two Higgs semistable vector bundles on smooth projective variety X. Then their tensor product $(E_1 \otimes E_2, \theta_1 \otimes id + \theta_2 \otimes id)$ is also Higgs semistable.(cf.[17])

(2) For two torsion-free torsion-free coherent sheaves E_1 and E_2 over X, the equality $\mu_{min}(E_1 \otimes E_2) = \mu_{min}(E_1) + \mu_{min}(E_2)$ is valid. Similarly, we have $\mu_{max}(E_1 \otimes E_2) = \mu_{max}(E_1) + \mu_{max}(E_2)$.(cf.[2])

(3) Let E_1 and E_2 be two torsion-free coherent sheaves over X.(cf.[2])
   - If $\mu_{min}(E_1) > \mu_{max}(E_2)$, then $H^0(X, Hom_{\mathcal{O}_X}(E_1, E_2)) = 0$.
   - If there is a surjective $\mathcal{O}_X$ linear homomorphism $\phi : E_1 \to E_2$, then $\mu_{min}(E_1) \leq \mu_{min}(E_2)$.

3. Proof of existence of a Higgs H-N reduction

For a Higgs G bundle (E, θ) on X (following [7]) we define an integer d_E as follows:

$$d_E = min\{\deg i_{P*}\sigma^* E(\mathfrak{g}/\mathfrak{p}) : (P, \sigma, \theta_P, U_P) \text{ is a Higgs reduction } \}$$

where, $i_P : U_P \to X$ is the inclusion. Since $i_{P*}\sigma^* E(\mathfrak{g}/\mathfrak{p})$ is a quotient of a fixed bundle $E(\mathfrak{g})$ on X, the integer d_E is well defined. Here we use the fact that the degrees of quotients of a fixed torsion-free sheaf is bounded from below.

Proposition 11. *Let $(P, \sigma, \theta_P, U_P)$ be a Higgs reduction of structure group of E to P such that the following conditions hold:*

(1) $\deg(\sigma^* E(\mathfrak{g}/\mathfrak{p})) = d_E$
(2) *P is a maximal among all parabolic subgroups P' satisfying the condition that there is a Higgs reduction $(P', \sigma', \theta_{P'})$ of E to P' such that degree$(\sigma'^* E(\mathfrak{g}/\mathfrak{p})) = d_E$.*

Then the Higgs reduction $(P, \sigma, \theta_P, U_P)$ is a Higgs H-N reduction.

Proof: Let $(P, \sigma, \theta_P, U_P)$ be a Higgs reduction of E to P satisfying the properties stated in the proposition, $i.e$ we have a Higgs reduction (P, σ, θ_P) of E to P on the open subset U_P of X. Let U be the unipotent part of P. First note that E_L as a L bundle on U_P gets a Higgs structure induced by the surjective map $P \longrightarrow L \longrightarrow 0$. We have, $\rho : E_P(\mathfrak{p}) \longrightarrow E_P(\mathfrak{l})$ induced by the above map, where $\mathfrak{p}$ and $\mathfrak{l}$ are Lie algebras of P and L respectively. So we define $\theta_L := (\rho \otimes id) \circ \theta_P$ where $\rho \otimes id : E_P(\mathfrak{p}) \otimes \Omega^1_{U_P} \longrightarrow E_P(\mathfrak{l}) \otimes \Omega^1_{U_P}$ and $\theta_P : \mathcal{O}_{U_P} \longrightarrow E_P(\mathfrak{p}) \otimes \Omega^1_{U_P}$. We first show that the associated Levi bundle E_L is Higgs semistable on U_P.
Suppose that E_L is not a Higgs semistable L bundle on U_P. Therefore, there is a Higgs

reduction $(Q, \tau_Q, \theta_Q, U')$ of E_L to a parabolic subgroup Q of L such that,

$$\deg(\tau^* T_{E_L/Q}) = \deg \tau^* E_L(\mathfrak{l}/\mathfrak{q}) < 0 \tag{1}$$

where, U' is the open subset of X with $codim(U_P \backslash U') \geq 2$ which implies that $\operatorname{codim}(X \backslash U') \geq 2$. It is easy to see that the inverse image of Q under the projection $P \longrightarrow L \longrightarrow 0$ which will be denoted as P_1, is a parabolic subgroup of G. (Since, $G/P_1 \longrightarrow G/P$ is a fiber bundle with fiber $P/P_1 \cong L/Q$, hence P/P_1 is complete. This implies that G/P_1 is complete.) Since (Q, τ_Q) is a reduction of structure group of E_L to Q given by the section $\tau_Q : U' \longrightarrow E_P(L/Q)$ we have a section $\sigma_1 : U' \longrightarrow E_P(P/P_1)$. So we have (P_1, σ_1) a reduction of structure group of E_P to P_1 on U' open subset of U_P, of X with $\operatorname{codim}(X \backslash U') \geq 2$. From now onwards we fix this open subset U' of X. Note that we have two short exact sequences of P and P_1 modules respectively.

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{l} \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{p}_1 \longrightarrow \mathfrak{q} \longrightarrow 0 \tag{3}$$

where $\mathfrak{u}, \mathfrak{p}, \mathfrak{p}_1, \mathfrak{q}$ are Lie algebras of U,P,P_1 and Q respectively. On U' we have, $0 \longrightarrow E_{P_1}(\mathfrak{p}_1) \longrightarrow E_P(\mathfrak{p})$ and $0 \longrightarrow E_{P_1}(\mathfrak{q}) \longrightarrow E_P(\mathfrak{l})$. But we know that $E_P(\mathfrak{l}) = E_L(\mathfrak{l})$ and $E_P(\mathfrak{u}) = E_{P_1}(\mathfrak{u})$ The above exact sequences therefore gives the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{P_1}(\mathfrak{u}) & \longrightarrow & E_{P_1}(\mathfrak{p}_1) & \longrightarrow & E_{P_1}(\mathfrak{q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_P(\mathfrak{u}) & \longrightarrow & E_P(\mathfrak{p}) & \longrightarrow & E_P(\mathfrak{l}) & \longrightarrow & 0
\end{array}$$

This results in the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^0(U, E_{P_1}(\mathfrak{u}) \otimes \Omega^1_{U'}) & \to & \mathrm{H}^0(U, E_{P_1}(\mathfrak{p}_1) \otimes \Omega^1_{U'}) & \xrightarrow{g} & \mathrm{H}^0(U, E_{P_1}(\mathfrak{q}) \otimes \Omega^1_{U'}) & \xrightarrow{g_1} & \mathrm{H}^1(U, E_{P_1}(\mathfrak{u}) \otimes \Omega^1_{U'}) \\
 & & \| & & \downarrow i & & \downarrow j & & \| \\
0 & \to & \mathrm{H}^0(U, E_P(\mathfrak{u}) \otimes \Omega^1_{U'}) & \to & \mathrm{H}^0(U, E_P(\mathfrak{p}) \otimes \Omega^1_{U'}) & \xrightarrow{f} & \mathrm{H}^0(U, E_P(\mathfrak{l}) \otimes \Omega^1_{U'}) & \xrightarrow{f_1} & \mathrm{H}^1(U, E_P(\mathfrak{u}) \otimes \Omega^1_{U'})
\end{array}$$

From the above diagram it follows that there exists $\theta_{P_1} \in \mathrm{H}^0(U', E_{P_1}(\mathfrak{p}_1) \otimes \Omega^1_{U'})$ such that $i(\theta_{P_1}) = \theta_P$.

Therefore we have got a Higgs structure $(P_1, \sigma_1, \theta_{P_1})$ induced by Higgs structure (Q, τ_Q, θ_Q) on U'.

But by the above diagram it is clear that this structure is compatible with the Higgs structure induced by (P, σ, θ_P) i.e, we have the commutative diagram;

$$\begin{array}{ccc}
\mathcal{O}_{U'} & \xrightarrow{\theta_P} & E_P(\mathfrak{p}) \otimes \Omega^1_{U'} \\
 & \overset{\theta_{P_1}}{\searrow} & \uparrow \\
 & & E_{P_1}(\mathfrak{p}_1) \otimes \Omega^1_{U'} \\
 & & \uparrow \\
 & & 0
\end{array}$$

Let $\mathfrak{p}_1, \mathfrak{p}, \mathfrak{l}$ denote the Lie algebras of P_1, P, L respectively. We have the following exact sequence of P_1 modules

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}_1 \longrightarrow \mathfrak{g}/\mathfrak{p}_1 \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow 0 \tag{4}$$

Since$P/P_1 \cong L/Q$ we get $\mathfrak{p}/\mathfrak{p}_1 \cong \mathfrak{l}/\mathfrak{q}$ as P_1 modules.

Since E_{P_1} is a Higgs reduction of E to P_1, we consider the vector bundles associated to E_{P_1} using the P_1 modules in (4). We have the following exact sequence of vector bundles on U'.

$$0 \longrightarrow \tau^* T_{E_L/Q} \longrightarrow \sigma^* T_{E/P_1} \longrightarrow \sigma^* T_{E/P} \longrightarrow 0 \tag{5}$$

Using (1) we conclude that

$$\deg(\sigma_1^* T_{E/P_1}) = \deg(\sigma^* T_{E/P}) + \deg(\tau^* T_{E_L/Q}) < \deg(\sigma^* T_{E/P}) = d_E \tag{6}$$

This contradicts the assumption on d_E. Therefore, E_L must be Higgs semistable.

Now we need to check the second condition in the definition of the Higgs H-N reduction. Let B be a Borel subgroup of G contained in P and $T \subset B$ be the maximal torus. Let Δ be the system of simple roots. Let I denote the set of simple roots defining the parabolic subgroup P. So I defines the roots of the Levi factor L of P. Take any dominant character χ of P whose restriction to T is expressed as

$$\chi \mid_T = \sum_{\alpha \in \Delta - I} c_\alpha \alpha + \sum_{\beta \in I} c_\beta \beta \tag{7}$$

with $c_\alpha, c_\beta \geq 0$.

Let $Z^0(L) \subset T$ be the connected component of the center of L. The characters $\beta \in I$ of T have the property that $\beta \mid_{Z^0(L)}$ is trivial. Hence we see that if χ is a nontrivial character of P of the above form then $c_\alpha > 0$ for some $\alpha \in \Delta - I$.

Also, some positive multiple of a character of $Z^0(L)$ extends to character of L. Hence there are positive integers n_α for each $\alpha \in \Delta - I$ such that $n_\alpha \alpha \mid_{Z^0(L)}$ extends to a character χ_α on L. Now we see that $\chi_\alpha \mid_T$ can be written as $n_\alpha \alpha + \sum_{\beta \in I} n_{\beta,\alpha} \beta$ for some integers $n_{\beta,\alpha}$.

Let $N = \prod_{\alpha \in \Delta - I} n_\alpha$, then we have the following equality.

$$N\chi \mid_T = \sum_{\alpha \in \Delta - I} N c_\alpha \alpha + \sum_{\beta \in I} N c_\beta \beta = \sum_{\alpha \in \Delta - I} c'_\alpha \chi_\alpha \mid_T + \sum_{\beta \in I} c'_\beta \beta \tag{8}$$

for some integer $c'_\beta, \beta \in I$ and c'_α where $\alpha \in \Delta - I$ such that c'_α is a positive multiple of c_α. Hence $N\chi - \sum_{\alpha \in \Delta - I} c'_\alpha \chi_\alpha$ is a character of L whose restriction to $Z^0(L)$ is trivial. Thus,$N\chi = \sum_{\alpha \in \Delta - I} c'_\alpha \chi_\alpha$. Hence it is enough to prove the second condition for the characters of the form χ_α, with $\alpha \in \Delta - I$.

Fix an element $\alpha \in \Delta - I$. Let $P_2 \supset P$ be the parabolic subgroup of G defined by the subset $I_2 = \{\alpha\} \cup I$ of Δ. Let L_2 be its Levi factor and P' be the maximal parabolic subgroup of L_2 defined by the image of P in L_2. Consider the group of all characters of P' which are trivial on the center of L_2. This group is generated by a single dominant character ω of P' with respect to the root system of L_2 defined by its maximal torus T.

Now χ_α defines a character on P' which is trivial on center of L. Hence the restriction of χ_α to T can be written as $\chi_\alpha \mid_T = m\omega \mid_T$ for some integer m.

This enables us to write

$$m\omega \mid_T = n_\alpha \alpha + \sum_{\beta \in I} n_{\beta,\alpha} \beta.$$

A dominant weight is always a nonnegative rational linear combination of simple roots. The fact that $n_\alpha > 0$ implies that $m > 0$ and $n_{\beta,\alpha} \geq 0$. Hence χ_α is a positive multiple of ω.

To verify the second condition of Higgs H-N reduction, we consider the surjective map $P_2 \longrightarrow L_2 \longrightarrow 0$ and the injective map $P \overset{i}{\hookrightarrow} P_2$. We note that $P_2/P \cong L_2/P'$. Let E_{P_2} be an extension of structure group of E_P to P_2, that is $E_P(P_2) = E_{P_2}$ on U_P. So E_P is a reduction of structure group of E_{P_2} to P on U_P. It is easy to see that $E_P(G) = E_{P_2}(G)$ and $P_2 \hookrightarrow G$ gives rise to a reduction $(P_2, \sigma_2)$of E on U_P. Also we have $E_P(\mathfrak{p}) \overset{\rho'}{\hookrightarrow} E_{P_2}(\mathfrak{p}_2) = (E_P(\mathfrak{p}_2))$ induced by i.

Now we define $\theta_{P_2} \colon = (\rho' \otimes id) \circ \theta_P$

where $(\rho' \otimes id) \colon E_P(\mathfrak{p}) \otimes \Omega^1_{U_P} \hookrightarrow E_P(\mathfrak{p}_2) \otimes \Omega^1_{U_P}$.

Since the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}_U & \xrightarrow{\theta_{P_2}} & E_P(\mathfrak{p}_2) \otimes \Omega^1_{U_P} \\ & \overset{\theta_P}{\searrow} & \uparrow \\ & & E_P(\mathfrak{p}) \otimes \Omega^1_{U_P} \\ & & \uparrow \\ & & 0 \end{array}$$

we have a Higgs structure θ_{P_2} on E_{P_2} and hence Higgs reduction $(E_{P_2}, \sigma_2, \theta_{P_2})$ on U_P. As in the first part of the proposition $E_{L_2} = E_{P_2}(L_2)$ is a Higgs L_2 bundle. Using the following exact sequence of P modules,

(9) $$0 \longrightarrow \mathfrak{p}_2/\mathfrak{p} \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow \mathfrak{g}/\mathfrak{p}_2 \longrightarrow 0$$

we have the following exact sequence of vector bundles on U_P.

(10) $$0 \longrightarrow \sigma'^* T_{E_{L_2}/P'} \longrightarrow \sigma^* T_{E/P} \longrightarrow {\sigma_2}^* T_{E/P_2} \longrightarrow 0$$

where σ' is the Higgs reduction of the structure group of E_{L_2} to P' which is $E_P(P')$ for the obvious projection of P to P'. Here $E_P(P')$ gets Higgs structure in the following way.

Let $f : E_P(\mathfrak{p}) \longrightarrow E_P(\mathfrak{p}')$ be induced by the projection $P \to P'$. Define $\theta_{P'} \colon = (f \otimes id) \circ \theta_P$. The section $\theta_{P'}$ is the required Higgs structure on $E_P(P')$. Also note that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{O}_{U_P} & \xrightarrow{\theta_{L_2}} & E_{L_2}(\mathfrak{l}_2) \otimes \Omega^1_{U_P} \\
 & \overset{\theta'_P}{\searrow} & \uparrow \\
 & & E_P(\mathfrak{p}') \otimes \Omega^1_{U_P} \\
 & & \uparrow \\
 & & 0
\end{array}
$$

From the above exact sequence we have

$$\deg(\sigma'^{*}T_{E_{L_2}/P'}) = \deg(\sigma^{*}T_{E/P}) - \deg({\sigma_2}^{*}T_{E/P_2}). \tag{11}$$

Now the assumption that $\deg(\sigma^* T_{E/P}) = d_E$ gives the inequality $\deg(\sigma'^* T_{E_{L_2}/P'}) < 0$. Note that $det(\sigma'^* T_{E_{L_2}/P'})$ is the line bundle associated to the P'-bundle $E_P(\mathrm{P}')$, for the character of P' which is a negative multiple of ω. Hence it follows from the above observation that some positive powers of $L^*_{\chi_\alpha}$ and $\det(\sigma'^* T_{E_{L_2}/P'})$ coincide, proving that $\deg L_{\chi_\alpha} > 0$. Therefore, the second condition of the definition of the canonical Higgs reduction holds. This completes the proof of the proposition . This proposition establishes the existence of a Higgs canonical reduction.

4. The Higgs structure on the adjoint bundle $E_G(\mathfrak{g})$

Let (E, θ) be a principal Higgs G-bundle on X, let $E_G(\mathfrak{g})$ denote its associated vector bundle for the adjoint representation $Ad : G \longrightarrow Gl(\mathfrak{g})$. That gives rise to a morphism of Lie algebras $ad : \mathfrak{g} \longrightarrow End(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}^*$. So we get a morphism of vector bundles $\phi : E_G(\mathfrak{g}) \longrightarrow E_G(\mathfrak{g}) \otimes E_G(\mathfrak{g})^*$. We define $ad(\theta) = (\phi \otimes id) \circ \theta : \mathcal{O}_X \longrightarrow E_G(\mathfrak{g}) \otimes E_G(\mathfrak{g})^* \otimes \Omega^1_X$ which gives Higgs structure on $E_G(\mathfrak{g})$, note that $ad(\theta)$ induces a homomorphism $\theta' : E_G(\mathfrak{g}) \longrightarrow E_G(\mathfrak{g}) \otimes \Omega^1_X$ of vector bundles on X. Here we prove the following proposition only for semiharmonic Higgs principal G bundles i.e. for Higgs principal bundles whose all chern classes are vanishing. Therefore, the uniqueness of the H-N reduction holds only for semiharmonic Higgs principal bundles which we prove in the following section (c.f.§5)

Proposition 12. *A Higgs principal G bundle (E, θ) on X is Higgs semistable iff the associated Higgs vector bundle $(E_G(\mathfrak{g}), ad(\theta))$ is Higgs semistable.*

Proof: The proof of this proposition goes along the same lines as that of [2]. First assuming that $(E_G(\mathfrak{g}), ad(\theta))$ is a Higgs semistable vector bundle, we need to prove that (E, θ) is a Higgs semistable principal G bundle. Suppose not, then there exists an open subset U, with $codim(X \backslash U) \geq 2$ and a maximal parabolic subgroup P of G and a Higgs reduction (P, σ, θ_P, U) such that $\deg(\sigma^*(T_{E/P})) < 0$ (by Lemma 6), where $T_{E/P}$ is the relative tangent bundle for the projection $E \longrightarrow E/P$. Let $E_0 = E_P(\mathfrak{p}) \subset E_G(\mathfrak{g})$ be the subbundle given by the adjoint bundle of the P bundle E_P given by the reduction σ then $\sigma^* T_{E/P} = E_G(\mathfrak{g})/E_0$. Also $(E_P(\mathfrak{p}), ad(\theta_P))$ is a Higgs subbundle of $(E_G(\mathfrak{g}), ad(\theta))$. Since G is reductive, using a bilinear form which is non degenerate on $\mathfrak{g}$, we have the identification $E_G(\mathfrak{g}) = E_G(\mathfrak{g})^*$ which implies that $\deg(E_G(\mathfrak{g})) = 0$.

From the exact sequence of vector bundles on U

$$0 \longrightarrow E_0 \longrightarrow E_G(\mathfrak{g}) \longrightarrow \sigma^* T_{E/P} \longrightarrow 0 \tag{12}$$

we conclude, $\deg(E_G(\mathfrak{g})) = \deg(E_0) + \deg(\sigma^* T_{E/P})$. This implies that $\deg(E_0) > 0$ (since $\deg(\sigma^* T_{E/P}) < 0$, $\mu(E_0) > \mu(E_G(\mathfrak{g}))$ which contradicts the fact that $(E_G(\mathfrak{g}), ad(\theta))$ is Higgs semistable.

Now we assume that (E, θ) is Higgs semistable then we will prove that $(E_G(\mathfrak{g}), ad(\theta))$ is a Higgs semistable vector bundle.

Suppose that $E_G(\mathfrak{g})$ is not a Higgs semistable vector bundle on X and let

$$0 = E_0 \subset E_1 \subset E_2 \cdots \subset E_{k-1} \subset E_k = E_G(\mathfrak{g}) \tag{13}$$

be the Higgs H-N filtration of the Higgs vector bundle $E_G(\mathfrak{g})$ (lemma 3.1. [18]).

For the sake of notational convenience, we denote $E_G(\mathfrak{g})$ by V and we have the Higgs structure θ' induced by $ad(\theta)$ on V. Note that we can choose an open subset U of X such that it contains all points of codimension one (since $codim(X \backslash U) \geq 2$) and every sheaf E_i is locally free on U. We fix this U. For any $x \in U$ we consider $E_{j,x}^{\perp} = \{v \in V_x | < v, E_{j,x} >= 0\}$, where $<,>$ is the non-degenerate bilinear form invariant under G on $V_x \cong \mathfrak{g}$. Let $E_j^{\perp}$ be the kernel of the surjection $V \longrightarrow V^*$ (defined by the form) followed by canonical map $V^* \longrightarrow {E_j}^*$.

We *claim* that $E_j^{\perp} \cong (V/E_j)^*$ over U. Let $v \in E_{j,x}^{\perp}$, then define $f : V_x \longrightarrow k$ by $f(v') =< v, v' >$. Since $f(w) =< v, w >= 0$ for all $w \in E_{j,x}$. So f induces a map $\bar{f} : V_x/E_{j,x} \longrightarrow k$. Thus $E_{j,x}^{\perp} \hookrightarrow (V_x/E_{j,x})^*$, but $\dim(E_{j,x}^{\perp}) = \dim(V_x/E_{j,x})^*$, therefore $E_j^{\perp} \cong (V/E_j)^*$.

We define $W_j := E_{k-j}^{\perp} = (V/E_{k-j})^*$. Since $\theta' \mid_{E_j}$ maps $E_j \longrightarrow E_j \otimes \Omega_X^1$, we can define $\bar{\theta}'_j : (V/E_j) \longrightarrow (V/E_j) \otimes \Omega_X^1$ induced by $\theta' : V \longrightarrow V \otimes \Omega_X^1$. So $((V/E_j)^*, \bar{\theta}'_j)$ is a *Higgs subbundle* of (V^*, θ^1) where $\theta^1 : E_G(\mathfrak{g})^* \longrightarrow E_G(\mathfrak{g})^* \otimes \Omega_X^1$ induced by $ad(\theta)$(since both θ' and θ^1 induced by $ad(\theta)$ and $(V/E_j)^*$ have Higgs structures induced by θ').
Now we *claim* that W_i/W_{i-1} is a Higgs semistable vector bundle and $\mu(W_i/W_{i-1})$ decreases as i increases. To prove this claim, observe that (E, θ) is Higgs semistable vector bundle if and only if (E^*, θ) is Higgs semistable vector bundle. Now we observe that,

$$W_i/W_{i-1} = \frac{(V/E_{k-i})^*}{(V/E_{k-(i-1)})^*} = (E_{k-(i-1)}/E_{k-i})^* \tag{14}$$

For, we have the diagrams:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & E_{k-(i-1)}/E_{k-i} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & E_{k-i} & \longrightarrow & V & \longrightarrow & V/E_{k-i} & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
0 & \longrightarrow & E_{k-(i-1)} & \longrightarrow & V & \longrightarrow & V/E_{k-(i-1)} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

$$0 \longleftarrow (E_{k-(i-1)}/E_{k-i})^* \longleftarrow (V/E_{k-i})^* \longleftarrow (V/E_{k-(i-1)})^* \longleftarrow 0 \tag{15}$$

and hence, $\frac{(V/E_{k-i})^*}{(V/E_{k-(i-1)})^*} = (E_{k-(i-1)}/E_{k-i})^*$. Thus, $W_i/W_{i-1} = (E_{k-(i-1)}/E_{k-i})^*$.

But we know that $(E_{k-(i-1)}/E_{k-i})$ is Higgs semistable. Therefore W_i/W_{i-1} is Higgs semistable. Note that $\mu(W_i/W_{i-1}) = -\mu(E_{k-(i-1)}/E_{k-i})$, $\mu(W_{i+1}/W_i) = -\mu(E_{k-i}/E_{k-(i+1)})$. Hence, $\mu(W_i/W_{i-1})$ decreases as i increases. This completes the proof of the claim.

Therefore we obtain a Higgs H-N filtration of V^* over U.

$$0 = W_0 \subset W_1 \subset W_2 ...... \subset W_{k-1} \subset W_k = V^* \tag{16}$$

such that each W_i is a Higgs subbundle of V and the quotient is Higgs semistable; further, $\mu(W_i/W_{i-1})$ decreases as i increases.

Now since $V \cong V^*$, we conclude that the filtration of V^* over U by the W_j's coincides with the Higgs H-N filtration of V (by the uniqueness of filtration). In other words, we have $E_j = E_{k-j}^{\perp}$ on U for all $0 \leq j \leq k$. Therefore, the above filtration is of the following form:

$$0 = E_{-l-1} \subset E_{-l} \subset E_{-l+1} \subset .... \subset E_{-1} \subset E_0 \subset E_1 \subset ...E_{l-1} \subset E_l = V \tag{17}$$

where E_j is the orthogonal complement of E_{-j-1} for the G-invariant form ¡,¿.

Let $\phi : E_0 \otimes E_0 \longrightarrow V/E_0$, be the composition of the Lie bracket operation with the natural projection $V \longrightarrow E_0$. By the Proposition 2.9 in [2] we have, $\mu_{min}(E_0 \otimes E_0) = 2\mu_{min}(E_0) = 2\mu(E_0/E_{-1})$ and $\mu_{max}(V/E_0) = \mu(E_1/E_0)$.

From the fact that E_{-1} is the orthogonal part of E_0, the form ¡,¿ induces a non-degenerate quadratic form on E_0/E_{-1}. Consequently, we have $E_0/E_{-1} \cong (E_0/E_{-1})^*$ which implies that $\mu(E_0/E_{-1}) = 0$. We have $\mu_{min}(E_0 \otimes E_0) = 2\mu(E_0/E_{-1}) = 0 > \mu(E_1/E_0) = \mu_{max}(V/E_0)$. Therefore , it follows that $\mathrm{H}^0(X, Hom_{Higgs}(E_0 \otimes E_0, V/E_0)) = 0$; in particular, we have $\phi = 0$. In other words, E_0 is closed under the Lie bracket. Consider

$$\phi_j : E_{-j} \otimes E_{-1} \longrightarrow V/E_{-j-1} \text{ where } j \geq 0$$

defined using the Lie bracket operation and the projection $V \longrightarrow V/E_{-j-1}$.

Repeating the above argument and using the property that $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ we deduce that $\phi_j = 0$. In other words, we have $[E_{-j}, E_{-1}] \subset E_{-j-1}$ for any $j \geq 0$.

Using the above inclusion we conclude that E_{-1} is a nilpotent Lie sub-algebra bundle of E_0, and $E_{-1,x}$ is also an ideal of $E_{0,x}$ for any $x \in X$.

Now by Lemma 2.11 of [2], we see that over the open set U of X, the sub-algebra bundle E_0 is a bundle of parabolic sub-algebras, and it gives a reduction $\sigma : U \longrightarrow E_G/P$ of the structure group of E_G to a parabolic subgroup P of G.

Using the above lemma we have a reduction (P, σ). Let us denote the principal P bundle over U obtained in the above lemma by E_P and $E_0 = E_P(\mathfrak{p})$. Since E_0 is a Higgs subbundle of $E_G(\mathfrak{g})$ we have, $\theta' \mid_{E_0}: E_P(\mathfrak{p}) \longrightarrow E_P(\mathfrak{p}) \otimes \Omega^1_U$ where $\theta' : E(\mathfrak{g}) \to E(\mathfrak{g}) \otimes \Omega^1_X$ induced by $ad(\theta) : \mathcal{O}_X \to E_G(\mathfrak{g}) \otimes E_G(\mathfrak{g})^* \otimes \Omega^1_X$.

The section θ factors through $E_P(\mathfrak{p}) \otimes \Omega^1_U$, call it θ_P. Therefore we conclude that $(E_P, \sigma_P, \theta_P)$ is a *Higgs reduction* of E to P on U. Let χ_0 be the character on P associated to the action of P on its Lie algebra $\mathfrak{p}$. Then it is clear that χ_0 is a dominant character of P with respect to the Borel subgroup B. Hence from the Higgs semistability of E_G we deduce that $\deg(\sigma^* E_G(\chi_0)) = \deg(E_P(\mathfrak{p})) = \deg(E_0) \le 0$ which contradicts the fact that $\mu(E_0) > \mu(E) = 0$. This completes the proof of the proposition.

Now we will prove the lemma that for every finite dimensional linear representation of the group G the associated bundle is also Higgs semistable with some constraints. The proof of the lemma goes along the same lines as that of the Lemma 5 of [1]

Lemma 13. *A principal Higgs G-bundle (E_G, θ) over X is Higgs semistable if and only if for every finite dimensional linear representation $\rho : G \to Aut(V)$, such that $\rho(Z_0(G))$ is contained in the center $Z(Aut(V))$, the associated Higgs vector bundle $E_G(V)$ is Higgs semistable.*

Proof: Take the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Now by assumption the associated adjoint bundle is Higgs semistable, therefore by the previous proposition (E_G, θ) is Higgs semistable.

Conversely, suppose that (E_G, θ) is Higgs semistable. By our earlier discussion $E_G(V)$ gets Higgs section canonically; call it θ_V. Let $V = \bigoplus_{i=1}^n V_i$ be the decomposition of the G module V into irreducible submodules.
Consider the character $\bigwedge^{top} Hom(V_i, V_j)$ of G given by the top exterior power. By assumption $Z_0(G)$ acts trivially on the line $\bigwedge^{top} Hom(V_i, V_j)$. Since G is connected, G is a quotient(semi-direct product) of $Z_0(G) \times [G, G]$, where $[G, G]$ denotes commutator subgroup of G. Any character of $[G, G]$ is trivial. So the action of G on $\bigwedge^{top} Hom(V_i, V_j)$ is trivial.

Let us denote the Higgs vector bundle $E_G(V_i)$ by $(\mathcal{W}_i, \theta_i)$ associated to E_G for the G module V_i. So we have

$$\bigoplus_{i=1}^{n} \mathcal{W}_i = \mathcal{W}, \qquad \bigoplus_{i=1}^{n} \theta_i = \theta_V \tag{18}$$

By the earlier observation, the line bundle $\bigwedge^{top} Hom(\mathcal{W}_i, \mathcal{W}_j)$, which is associated to E_G for the G module $\bigwedge^{top} Hom(V_i, V_j)$ is trivial. Therefore, we have $\mu(\mathcal{W}_i) = \mu(\mathcal{W}_j)$. This means that $\mathcal{W}$ is Higgs semistable iff each $\mathcal{W}_i$ is Higgs semistable.

So, our lemma is reduced to proving this only for irreducible representations $\rho : G \longrightarrow Aut(V)$. Let $\mathcal{W} = E_G(V)$, we know that $End(\mathcal{W}) = \mathcal{W} \otimes \mathcal{W}^*$ and $\mathcal{W} = E_{GL(V)}(V)$, then by Lemma 7, $\mathcal{W}$ is Higgs semistable iff $E_{GL(V)}$ is Higgs semistable. Again by proposition 12 $E_{GL(V)}$ is Higgs semistable iff $E_{GL(V)}(M_n(k)) = E_{GL(V)}(V \otimes V^*) = \mathcal{W} \otimes \mathcal{W}^* = End(\mathcal{W})$ is Higgs semistable. Therefore, it is enough to prove that $End(\mathcal{W})$ is Higgs semistable. Since the G module V is irreducible, from Schur's lemma it follows that the action of the center $Z(G)$ of G on $End(V)$ is trivial. We note that the adjoint representation of G on its Lie algebra $\mathfrak{g}$ gives a faithful representation of $G/Z(G)$. The group $G/Z(G)$ is also reductive. Therefore, the $G/Z(G)$ module $End(V)$ is a submodule of the $G/Z(G)$ module $\bigoplus_{j \in S} \mathfrak{g}^{\otimes j}$, where S is a finite collection of nonnegative integers possibly with repetitions.

Therefore, the Higgs vector bundle $End(\mathcal{W})$ is a Higgs subbundle of $\bigoplus_{j \in S} (E_G(\mathfrak{g}))^{\otimes j}$.
Since $(E_G(\mathfrak{g}))$ is Higgs semistable of degree zero we have $(E_G(\mathfrak{g}))^{\otimes j}$ is Higgs semistable of degree zero for every j. Thus, $\bigoplus_{j \in S} (E_G(\mathfrak{g}))^{\otimes j}$ is Higgs semistable of degree zero. But $End(\mathcal{W})$ being a Higgs subbundle of degree zero, is also Higgs semistable. This completes the proof.

Lemma 14. *Suppose (E,θ) is a Higgs principal G-bundle on X. Let $(E_P,\sigma_P,\theta_P,U_P)$ be a reduction, then E_L is Higgs semistable on U_P, where L is a Levi factor of P, if and only if $i_*E_L(\mathfrak{l})$ is a Higgs semistable torsion free sheaf over X.*

Proof: Note that E_L is Higgs semistable on U_P if and only if $E_L(\mathfrak{l})$ is Higgs semistable on U_P (proposition 12). But the semistability of $i_*E_L(\mathfrak{l})$ on X is equivalent of the semistability of $i_*E_L(\mathfrak{l})|_{U_P} = E_L(\mathfrak{l})$ on U_P.

In conclusion, we have the following theorem.

Theorem 15. *Let E_G be the Higgs semistable G bundle. Let $\rho : G \longrightarrow H$ be a homomorphism of connected reductive groups such that $\rho(Z_0(G))$ is contained in the connected component of the center of H containing the identity element. Let $E_H := E_G(H)$ be the Higgs principal H bundle induced by ρ. Then the H bundle E_H is also Higgs semistable.*

Proof: Let $\phi : H \longrightarrow Aut(V)$ be a representation such that $\phi(Z_0(H)) \subseteq Z(Aut(V))$, where $Z_0(H)$ is the connected component of the center of H containing the identity. By using Lemma 13 it is enough to prove that $E_H(V)$ is Higgs semistable. consider the composition $\phi\circ\rho : G \longrightarrow Aut(V)$. Then $\phi\circ\rho(Z_0(G) \subseteq Z(Aut(V))$. Since E_G is Higgs semistable, we have $E_G(V)$ is Higgs semistable. But $E_G(V) = E_H(V)$. This proves the theorem.

5. Proof of the uniqueness of Higgs H-N reduction

Theorem 16. *Any Higgs G bundle admits a unique H-N Higgs reduction. In other words, if we fix a Borel subgroup B of G, then there is an unique H-N Higgs reduction to a parabolic subgroup P containing B.*

Proof: The existence of H-N Higgs reduction is proved in §2. So, it is enough to prove the uniqueness.

Let $(P_1,\sigma_{P_1},\theta_{P_1},U_{P_1})$ and $(P_2,\sigma_{P_2},\theta_{P_2},U_{P_2})$ (where P_1 and P_2 are two parabolic subgroups containing B) be two H-N Higgs reductions of the Higgs principal G bundle (E,θ). Note that we can choose open subset $U \subseteq X$ with $codim(X\backslash U) \geq 2$ such that, the above two Higgs reductions exist on this U. That is we have two H-N Higgs reductions of E on U. We will prove that the two Higgs subbundles $E_{P_1}(\mathfrak{p}_1)$ and$E_{P_2}(\mathfrak{p}_2)$ of $E_G(\mathfrak{g})$ are the same on U. Equivalently we show $i_*(E_{P_1}(\mathfrak{p}_1)) = i_*(E_{P_2}(\mathfrak{p}_2))$ on X, where i is the inclusion of U in X. And these are reflexive sheaves occurs in the H-N. filtration of $E(\mathfrak{g})$. For proving the above stated result we need the following lemma.

Lemma 17. *Let (E,θ) be a principal G bundle on a curve and $(H_1,\sigma_{H_1}),(H_2,\sigma_{H_2})$ be two reductions of E. Suppose the two subbundles $E_{H_1}(\mathfrak{h}_1)$ and $E_{H_2}(\mathfrak{h}_2)$ of $E(\mathfrak{g})$ coincide. Then there exists an element $g\in G$ such that $H_1 = gH_2g^{-1}$. Moreover, if the normalizer of H_1 in G is H_1 itself then the two reductions $\sigma^*_{H_1}E$ and $\sigma^*_{H_2}E$ are the same.*

The above lemma is easily seen to hold for nonsingular projective varieties. By using this lemma we conclude that $(P_1,\sigma_{P_1}) = (P_2,\sigma_{P_2})$(since B is fixed and normalizer of a parabolic subgroup is itself). Then by the definitions of $\theta_{P_1},\theta_{P_2}$ we arrive at the fact that $(P_1,\sigma_{P_1},\theta_{P_1}) = (P_2,\sigma_{P_2},\theta_{P_2})$. So now our aim is to prove that the two Higgs subbundles $E_{P_1}(\mathfrak{p}_1)$ and $E_{P_2}(\mathfrak{p}_2)$ of $E_G(\mathfrak{g})$ are the same. To prove this fact we need to prove one Lemma.

Lemma 18. *Let (V^1, θ_1) and (V^2, θ_2) be two Higgs torsion-free coherent sheaves on X with Higgs filtrations*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots\cdots\cdots \subset V_{k-1} \subset V_k = V^1 \tag{19}$$

$$0 = V'_0 \subset V'_1 \subset V'_2 \subset \cdots\cdots\cdots \subset V'_{l-1} \subset V_l = V^2 \tag{20}$$

where V_i/V_{i-1} and V'_j/V'_{j-1} are Higgs semistable with $\deg(V_i/V_{i-1}) \geq 0$ and $\deg(V'_j/V'_{j-1}) < 0$ for $0 \leq i \leq k$ and $0 \leq j \leq l$. Then $Hom_{Higgs}(V^1, V^2) = 0$.

Proof: Let $\phi := V^1 \longrightarrow V^2$ be a homomorphism of Higgs torsion-free sheaves on X with the above conditions. We need to prove that $\phi = 0$. We choose an open set U of X with $codim(X \backslash U) \geq 2$ such that all the terms in the above two filtrations are locally free on U. So it is enough to prove that $Hom_{Higgs}(V^1, V^2) = 0$ of Higgs vector bundles on U. To prove this fact we use the induction on the number of terms in the filtrations i.e. on k and l.

Let $k = l = 1$. Then $(V^1, \theta_1), (V^2, \theta_2)$ are Higgs semistable. So we have, $\mu(V^1) \leq \mu(\phi(V^1)) \leq \mu(V^2)$. But $\deg(V^1) \geq 0$ and $\deg(V^2) < 0$ by hypothesis, so, $\mu(V^1) > \mu(V^2)$. This leads to a contradiction.

We will prove the lemma for any k and $l = 1$. Assume the lemma for $k' < k$ and $l = 1$. Note that we have

$$\begin{array}{ccccccc} 0 \longrightarrow V_{k-1} & \xrightarrow{i} & V_k & \xrightarrow{\pi} & V_k/V_{k-1} \longrightarrow 0 \\ \phi_{|V_{k-1}} & \searrow & \downarrow \phi & \swarrow \phi_1 & \\ & & V^2 & & \end{array}$$

So, $\phi \circ i = \phi \mid_{V_{k-1}} = 0$. (by the induction on k) Therefore we have $\phi_1 : V_k/V_{k-1} \longrightarrow V^2$ induced by ϕ such that $\phi = \phi_1 \circ \pi$. Again by induction on k we have $\phi_1 = 0$. This forces ϕ to be zero. So we have the lemma for k and $l = 1$. Now assume the lemma for all k and $l' < l$. Then we consider the following,

$$\begin{array}{ccccccc} 0 \longrightarrow V'_{l-1} & \xrightarrow{i'} & V'_l & \xrightarrow{\pi'} & V'_l/V'_{l-1} \longrightarrow 0 \\ & & \uparrow \phi & \nearrow \pi' \circ \phi & \\ & & V^1 & & \end{array}$$

we get $\pi' \circ \phi = 0$, that is $\mathrm{im}(\phi) \subset V'_{l-1}$.

So, $\phi : V^1 \longrightarrow V'_{l-1}$, this implies that $\phi = 0$. Thus, we have proved the lemma for all k and all l.

Let $(P, \sigma_P, \theta_P, U_P)$ be a H-N Higgs reduction of structure group of E to P, we denote this bundle by E_P. The adjoint action of P on $\mathfrak{p}$ preserves the sub-algebra $\mathfrak{u}$. The Higgs L bundle $E_P(L)$, will be denoted by E_L. By the definition of H-N Higgs reduction, E_L is Higgs semistable.

The Higgs vector bundle associated to E_P for the adjoint representation of P on $\mathfrak{p}$ and $\mathfrak{g}/\mathfrak{p}$ are naturally identified with adjoint bundle of E_P and $\sigma^* T_{E/P}$ respectively.

Consider the filtrations of P modules

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots\cdots\cdots \subset V_{k-1} \subset V_k = \mathfrak{u} \tag{21}$$

$$0 = V'_0 \subset V'_1 \subset V'_2 \subset \cdots\cdots\cdots \subset V'_{m-1} \subset V'_m = \mathfrak{g}/\mathfrak{p} \tag{22}$$

such that the quotients $W_i \colon = V_i/V_{i-1}$ and $W'_j \colon = V'_j/V'_{j-1}$ are all irreducible P modules.

It follows that the action of U on $W_i($ resp$W'_j)$ is trivial. Therefore, the action of P on $W_i($resp$W'_j)$ factors through the quotient L. Let $\mathcal{V}_j, ($resp$\mathcal{V}'_j)$ denote the Higgs vector bundle over X associated to the E_P for the P-module $V_j($resp$V'_j)$. Since W_i and W'_j are all irreducible L-modules and E_L is Higgs semistable, $\mathcal{W}_i \colon = E_L(W_i)$ and $\mathcal{W}'_i \colon = E_L(W'_i)$ are Higgs semistable. This result follows by Lemma 13. Let $B \subset P$ and T a maximal torus in B. Let Δ denote the set of simple roots for G. Let $I \subset \Delta$ denote the set of simple roots defining the parabolic subgroup P. The weights of T on $\mathfrak{g}/\mathfrak{p}$ are of the form

$$\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha \tag{23}$$

with $c_\alpha \leq 0$ and $c_\alpha < 0$ for at least one $\alpha \in \Delta - I$. The weights of T on $\mathfrak{u}$ are of the form $-\gamma$, where γ is a weight on $\mathfrak{g}/\mathfrak{p}$.

From this it follows that the character of P defined by the determinant of the representation of P on W_j (resp, W'_j) is non-trivial and is a non-negative (resp. nonpositive) linear combination of roots in Δ. Now by the second condition in the definition of a Higgs H-N reduction we see that

$$\deg(\mathcal{W}_j) > 0, \qquad \deg(\mathcal{W}'_j) < 0 \tag{24}$$

The adjoint action of P on $\mathfrak{p}/\mathfrak{u}$ factors through L and this is precisely the adjoint representation of L. In other words, the Higgs vector bundle associated to E_P for the P-module $\mathfrak{p}/\mathfrak{u}$ is $E_L(\mathfrak{l})$. The Higgs semistability of E_L implies that $E_L(\mathfrak{l})$ is Higgs semistable. Note that the degree of $E_L(\mathfrak{l})$ is zero.

Consider the exact sequence of Higgs vector bundles,

$$0 \longrightarrow E_P(\mathfrak{p}) \longrightarrow E_G(\mathfrak{g}) \longrightarrow \sigma_P^* T_{E/P} \longrightarrow 0 \tag{25}$$

corresponding to the following exact sequence of P-modules

$$0 \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow 0 \tag{26}$$

From the above observations it follows that the Higgs vector bundle $\sigma_P^* T_{E/P}$ has the Higgs filtration of $\sigma_P^* T_{E/P}$

$$0 = \mathcal{V}'_0 \subset \mathcal{V}'_1 \subset \mathcal{V}'_2 \subset \cdots\cdots\cdots \subset \mathcal{V}'_{m-1} \subset \mathcal{V}'_m = \sigma_P^* T_{E/P} \tag{27}$$

such that each quotient $\mathcal{W}'_i$ is Higgs semistable of negative degree.

In the same way we have

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots\cdots\cdots \subset \mathcal{V}_{k-1} \subset \mathcal{V}_k \subset E_P(\mathfrak{p}) \tag{28}$$

with each quotient W_i is Higgs semistable of positive degree and $E_P(\mathfrak{p})/\mathcal{V}_m$ which is identified with $E_L(\mathfrak{l})$, Higgs semistable of degree zero.

So, we have the Higgs filtration of $E_P(\mathfrak{p})$ such that each quotient is of nonnegative degree.

Now we consider the exact sequences of Higgs vector bundles on X.

$$0 \longrightarrow E_{P_1}(\mathfrak{p}_1) \longrightarrow E_G(\mathfrak{g}) \longrightarrow \sigma_P^* T_{E/P_1} \longrightarrow 0 \tag{29}$$

$$0 \longrightarrow E_{P2}(\mathfrak{p}_2) \longrightarrow E_G(\mathfrak{g}) \longrightarrow \sigma_P^* T_{E/P_2} \longrightarrow 0 \tag{30}$$

From (29) and (30) we have,

$$E_{P_1}(\mathfrak{p}_1) \longrightarrow \sigma^* T_{E/P_2} \tag{31}$$

$$E_{P_2}(\mathfrak{p}_2) \longrightarrow \sigma^* T_{E/P_1} \tag{32}$$

the homomorphisms of Higgs vector bundles.

Now by using Lemma 18 we conclude that the above two homomorphisms are zero. This implies that $E_{P_1}(\mathfrak{p}_1) = E_{P_2}(\mathfrak{p}_2)$. So, finally we have proved the uniqueness of the H-N Higgs reduction.

6. H-N reduction for ramified G bundles over a smooth curve

We assume from here onwards that X is a smooth projective curve of genus g, and $D = \sum_{i=1}^{c} D_i$ is a normal crossing reduced divisor. Unless stated otherwise, all the groups considered will be reductive linear algebraic groups over $\mathbb{C}$ and $N \geq 2$ be a fixed integer.

Definition 19. *A ramified G bundle of type N on X with ramification over D is a smooth variety Q with an action of G and a morphism $\phi : Q \longrightarrow X$ satisfying the following properties:*

1. *The action G on Q is proper.*
2. *The triple (Q, ϕ, X) is a geometric quotient.*
3. *In the complement of the divisor $D \subset X$ the morphism $\phi : Q \longrightarrow X$ is a principal G bundle.*
4. *At the finitely many orbits on which the isotropy is nontrivial, the isotropy is a cyclic subgroup whose order divides N.*

Given a homomorphism $\rho : G \longrightarrow H$ and a ramified G-bundle Q of type N on X, the quotient space $Q \times_G H$ has a natural structure of a ramified H-bundle of type N on X. This construction is called the extension of the structure group of Q to H. Let us denote the quotient space by $Q(H)$(c.f. [6]).

Definition 20. *Let $\phi_1 : Q \longrightarrow X$ and $\phi_2 : Q' \longrightarrow X$ be two ramified G-bundles of type N over X. Let $h : Q \longrightarrow Q'$ be a morphism of varieties. We say h is a morphism of ramified G-bundles if*

1. $\phi_2 \circ h = \phi_1$
2. *The isotropy for the image of the orbits(whose isotropy is nontrivial) is a cyclic subgroup whose order divides N*

We recall some of the definitions stated in [4] and [6].

1. Let H be a closed subgroup of G. A reduction of structure group of a ramified G-bundle Q to H is given by a section $\sigma : X \longrightarrow Q/H$. Let Q_H be the inverse image of the natural projection of Q to Q/H, of the subset of Q/H defined by the image of σ. This Q_H is a ramified H-bundle on X. It is easy to see that $Q_H \times_H G \cong Q$.
2. Let Q_H be a reduction of a ramified G-bundle Q. If W is a finite dimensional H-module, then the associated construction $Q_H \times^H W$, is a parabolic vector bundle over X. This is denoted by $Q(W)_*$.

(3) A ramified G-bundle Q is called semistable if for every reduction of structure group (P, σ) of Q to any parabolic subgroup P of G where $\sigma : X \longrightarrow Q/P$ is a section, and any dominant character χ of P, the parabolic line bundle $Q(\chi)_*$ ($= L_\chi$, as in our first section) has non-positive parabolic degree.
(4) Let Q be a ramified G bundle of type N over X with ramification divisor D. A Kawamata cover of type (N, D) (or simply N) is a Galois cover $p : Y \xrightarrow{\Gamma} X$ where Y is a connected smooth projective variety and Γ is a finite subgroup of $Aut(Y)$. A (Γ, G)-bundle is defined to be a principal G-bundle E together with a lift of the action of Γ on E which commutes with the right action of G on E.
(5) The ramified G-bundle Q over X with ramification divisor D gives rise to a (Γ, G)-bundle E over Y such that for every $y \in (p^* D_i)_{red}$ has a cyclic subgroup $\Gamma_y \subset \Gamma$ of order $k_i N$ as the isotropy group.
(6) Let $[\Gamma, G, N]$ denote the collection of (Γ, G)-bundles on Y satisfying the following two conditions:
   (a) For a general point y of an irreducible component of $(p^* D_i)_{red}$, the action of Γ_y on E_y(fibre over y) is of order N.
   (b) For a general point y of an irreducible component of a ramification divisor for p not contained in $(p^* D)_{red}$, the action of Γ_y on E_y is trivial.
(7) Let $E \in [\Gamma, G, N]$ and $E' = E/\Gamma$. The variety E' is smooth; moreover $\phi : E' \longrightarrow X$ is a ramified G-bundle.
(8) E is a normalisation of the fibre product $Y \times_X E'$. In other words, the G-bundle E on Y can be constructed back from E'.
(9) A ramified G-bundle Q canonically defines a parabolic G-functor $F_Q : Rep(G) \longrightarrow PVect(X, D)$ taking values in a category $PVect(X, D, N)$ for some N. Conversely, given a parabolic G-functor F there exists a ramified G-bundle Q, unique up to isomorphism of ramified G-bundles, such that $F_Q = F$.
(10) A ramified G-bundle Q is semistable if and only if the corresponding functor F_Q is semistable and hence if and only if the induced (Γ, G)-bundle is semistable. A ramified G-bundle Q is stable if and only if the induced (Γ, G)-bundle is stable.

Definition 21. *Let Q be a ramified G-bundle over X. Let (P, σ) be a reduction of structure group of Q to a parabolic subgroup P of G. Then the reduction (P, σ) is called a H-N reduction for the ramified G-bundle Q on X if the following two conditions hold:*

(1) *Let L be the Levi factor of P. Then the ramified L-bundle $Q_P(L)$ is semistable.*
(2) *For any dominant character χ of P with respect to some Borel subgroup $B \subset P$ of G the associated parabolic line bundle $Q_P(\chi)_*$ has parabolic degree > 0.*

6.1. **H-N reduction for $[\Gamma, G, N]$ bundle.** We define the H-N reduction for a $[\Gamma, G, N]$ bundle over any connected smooth projective variety over $\mathbb{C}$(c.f.[4]) and we prove that it exists and is unique.

Definition 22. *A (Γ, G, N) bundle E over Y with a Γ-reduction of structure group (P, σ) over some open subset U with $codim(X \backslash U) \geq 2$ is said to be canonical or* Γ equivariant H-N reduction *if the following two conditions hold:*

(1) *The associated Levi bundle $E_P(L)$, is Γ-semistable.*

(2) *For every dominant character χ of P with respect to some Borel subgroup contained in P, the line bundle $E_P(\chi)$ on U_P has positive degree.*

Proposition 23. *For any (Γ,G)-bundle over Y, Γ-equivariant H-N reduction exists and it is unique with respect to some Borel subgroup B contained in P.*

Proof: We will use the fact that for any G-bundle the canonical reduction (P,σ) exists and the usual semistability is same as Γ-semistability([4], Proposition 4.1). So our main job is to show that the usual H-N reduction (P,σ) is Γ-saturated. Let $\gamma \in \Gamma(\subset Aut(Y))$, we note that γ induces $\tilde{\gamma}$: $E \longrightarrow E$. We denote $\sigma^* E$ by E_P and $\tilde{\gamma}(E_P)$ by $\tilde{E_P^\gamma}$. Clearly $\tilde{E_P^\gamma}$ is a P bundle. [We define P action "$\star$" on $\tilde{E_P^\gamma}$ by $p\star(\gamma(\mathrm{e}))$:=$\gamma(e)\cdot p^{-1} = \gamma(e \cdot p^{-1})$, where $\cdot$ denotes the action of G on E, clearly $\gamma(p \star e) = p \star (\gamma(e))$. This means the free action of "P" on $\tilde{E_P^\gamma}$ commutes with the action of γ. Hence $\tilde{E_P^\gamma}$ is a P-bundle.]

Note that $E_P \cong \tilde{E_P^\gamma}$, by uniqueness of H-N reduction $E_P = \tilde{E_P^\gamma}$. This implies that E_P is Γ saturated (since γ is arbitrary). So E_P is a (Γ, P) reduction. Now we use the fact that that usual semistability of $\tilde{E_P^\gamma}(L)$ is same as Γ semistability of $\tilde{E_P^\gamma}(L)$(c.f. Theorem 4.3, [6]). The second condition for Γ equivariant H-N reduction is satisfied automatically.

So E_P is a Γ *equivariant H-N reduction*, and it is unique.

6.2. **Proof of H-N reduction for ramified G bundle.** We fix a Borel subgroup B. We would like to prove here that for any ramified G-bundle there exists unique H-N reduction. We already know that the existence of the H-N reduction for the (Γ, G, N) bundles. Let $\phi : Q \longrightarrow X$ be a ramified G-bundle on X and let $p : Y \longrightarrow X$ be a Kawamata cover and $E \in [\Gamma, G, N]$ such that $E/\Gamma = Q$. Let (P, σ) be a H-N reduction of E to a parabolic subgroup P of G where $\sigma : Y \longrightarrow E(G/P)$ is the Γ-equivariant section. We will prove that the corresponding reduction $\tau : X \longrightarrow Q/P$ of Q to P (where τ is induced by the Γ-equivariant section σ)is the H-N reduction for the ramified G-bundle Q. Note that the semistablity of $E_P(L)$ is equivalent to the semistablity of the ramified L-bundle $(E_P/\Gamma)(L) = E_P(L)/\Gamma$. Let χ be the dominant character of P with respect to the Borel subgroup $B \subset P$. It is proved in [4] that $\deg(E_P(\chi)) \geq 0$ if and only if $\deg((E/\Gamma)(\chi)_*) \geq 0$. Therefore, by the above discussion it is clear that the reduction $(E_P/\Gamma, \tau)$ is a H-N reduction for the ramified G-bundle $Q = E/\Gamma$.

The uniqueness of the H-N reduction follows from the uniqueness of the H-N reduction of the (Γ, G)-bundle E.

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