On distance matrices of wheel graphs with odd number of vertices

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(In memory of Miroslav Fiedler)

June 8, 2020

Abstract

Let W_n denote the wheel graph having n-vertices. If i and j are any two vertices of W_n, define

$$d_{ij} := \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 2 & \text{else.} \end{cases}$$

Let D be the $n \times n$ matrix with $(i,j)^{\text{th}}$ entry equal to d_{ij}. The matrix D is called the distance matrix of W_n. Suppose $n \geq 5$ is an odd integer. In this paper, we deduce a formula to compute the Moore-Penrose inverse of D. More precisely, we obtain an $n \times n$ matrix $\widetilde{L}$ and a rank one matrix ww' such that

$$D^{\dagger} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww'.$$

Here, $\widetilde{L}$ is positive semidefinite, $\text{rank}(\widetilde{L}) = n - 2$ and all row sums are equal to zero.

Keywords. Wheel graphs, circulant matrices, Laplacian matrices, distance matrices, Moore-Penrose inverse.

AMS CLASSIFICATION. 05C50

1 Introduction

Let G be a connected graph with vertex set $V := \{1, \ldots, n\}$. Since G is connected, any two vertices i and j in V are now connected by a path in G. Let the minimum length of all such paths be denoted by d_{ij}. The distance matrix of G is then the $n \times n$ symmetric matrix with $(i,j)^{\text{th}}$ off-diagonal entry equal to d_{ij} and all diagonal entries equal to zero. Distance matrices of connected graphs have several interesting properties and have applications in various fields like data communication, chemistry and biology. Distance matrices have a wide literature. For a comprehensive introduction, we refer to the survey article [1] and the monograph [2] and [3]. There are several interesting problems on distance matrices. One of them is the following: If G is a connected graph and D is the distance matrix of G, deduce a formula to compute the determinant and the inverse of D. This problem originates from a well-known result of Graham and Lovász [4]. To introduce this result, we need to recall the notion of the Laplacian matrix of G. Define $S := \text{diag}(s_1, \ldots, s_n)$, where s_i is the degree of the vertex i in G. Suppose A is the adjacency matrix of G. Then the matrix $M := S - A$ is called the Laplacian matrix of G with the following properties:

(M1) M is positive semidefinite.

(M2) All row sums of M are zero.

(M3) $\text{rank}(M) = n - 1$.

Suppose E is the distance matrix of a tree with n-vertices. According to Graham and Lovász [4],

$$E^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau', \tag{1}$$

where L is the Laplacian matrix of the tree and $\tau = (2 - \delta_1, \ldots, 2 - \delta_n)'$ with δ_i equal to the degree of the vertex i. The remarkable feature of this formula is that the inverse can be expressed just by using the adjacency matrix and the vertex degrees of the tree. A question that arises now naturally is how to generalize formula (1) to connected graphs other than trees. In the case of trees, there is an elegant identity that connects the Laplacian with the distance matrix. If $E = [e_{ij}]$ and $L^\dagger = [\beta_{ij}]$, then

$$e_{ij} = \beta_{ii} + \beta_{jj} - 2\beta_{ij}, \tag{2}$$

where $L^\dagger$ is the Moore-Penrose inverse of the Laplacian L. All the known proofs for (1) rely on the relation (2) either directly or indirectly and the properties (M1), (M2) and (M3) of the Laplacian. If the connected graph is not a tree, then the identity (2) does not hold and hence in general it is very difficult to get an elegant formula similar to (1). However, for some special cases like weighted trees, complete graphs, complete bipartite graphs and wheel graphs with even number of vertices, there are formula in the spirit of (1): see [5, 6, 7].

Let W_n be the wheel graph having n-vertices. In this paper, we assume n is an odd integer. Suppose D is the distance matrix of W_n. Define a vector $d := (0, 1, -1, 1, -1, \ldots, -1)' \in \mathbb{R}^n$. Then $Dd = \mathbf{0}$. So, $\det(D) = 0$. We now deduce a formula to compute the Moore-Penrose inverse of D which is similar to (1). Precisely, we obtain a matrix $\widetilde{L}$ and a rank one matrix ww' such that

$$D^\dagger = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww',$$

where $\widetilde{L}$ is positive semidefinite, $\text{rank}(\widetilde{L}) = n - 2$ and all row sums are equal to zero. We also show that if $\widetilde{L}^\dagger = [\theta_{ij}]$, then

$$d_{ij} = \theta_{ii} + \theta_{jj} - 2\theta_{ij}.$$

2 Notation and conventions

- The notation n will *always* denote an odd positive integer which is at least 5 and W_n will stand for the wheel graph with n number of vertices. The center of W_n

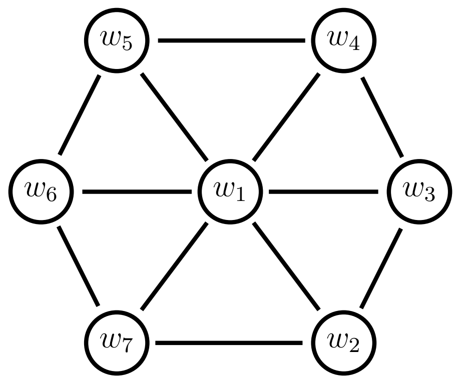

Figure 1: Wheel graph W_7

is labelled w_1. All vertices other than w_1 lie in a cycle of length $n-1$. We label these vertices by $w_2, w_3, \ldots, w_{n-1}$ such that (w_i, w_{i+1}) is an edge. For example, see Figure 1. Since any other labelling of W_n leads to a distance matrix which is permutation similar to D, without loss of generality, we fix this labelling.

- All vectors are assumed to be column vectors unless stated otherwise. The identity matrix of order n is denoted by I. If $k < n$, we use I_k to denote the identity matrix of order k.

- We denote the vector of all ones in $\mathbb{R}^{n-1}$ by $\mathbf{1}$ and the $(n-1)\times(n-1)$ matrix of all ones by J. If $\nu \neq n-1$, we use the notation $\mathbf{1}_\nu$ to denote the vector of all ones in $\mathbb{R}^\nu$ and J_ν to denote the $\nu \times \nu$ matrix of all ones. As usual, we use 0 to denote the scalar zero. To denote the zero vector (row/column), we use the notation $\mathbf{0}$. A matrix with more than one row/column and having all entries equal to zero is denoted by O. If $(x_1, \ldots, x_k)$ is a row vector, then $\mathrm{Circ}(x_1, \ldots, x_k)$ will be the circulant matrix with first row equal to $(x_1, \ldots, x_k)$.

- We reserve the letter u to denote the row vector $(0, 1, 2, \ldots, 2, 1)$ with $n-1$ components. The distance matrix of W_n now has the form

$$D = \begin{bmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & \widetilde{D} \end{bmatrix}, \tag{3}$$

where $\widetilde{D} = \mathrm{Circ}(u)$. We record the equation

$$\widetilde{D}\mathbf{1} = 2(n-3)\mathbf{1} \tag{4}$$

for later use. An $n \times n$ matrix $A = [a_{ij}]$ is an Euclidean distance matrix if there exist $x^1, x^2, \ldots, x^n \in \mathbb{R}^r$ such that $a_{ij} = (x^i - x^j)'(x^i - x^j)$. By Theorem 12 in [8], it follows that D is an Euclidean distance matrix.

- Let $x = (x_1, \ldots, x_{n-1})$. We say that x follows symmetry with respect to the $(\frac{n+1}{2})^{\text{th}}$ coordinate in its last $n-2$ coordinates if x has the form

$$(x_1, x_2, \ldots, x_{\frac{n-1}{2}}, x_{\frac{n+1}{2}}, x_{\frac{n-1}{2}}, \ldots, x_3, x_2),$$

or equivalently, x satisfies the equations

$$x_i = x_{n+1-i} \text{ for all } i = 2, 3, \ldots, n-1.$$

We define

$$\Delta := \{(x_1, \ldots, x_{n-1}) : x_i = x_{n+1-i} \text{ for all } i = 2, 3, \ldots, n-1\}.$$

- We fix m to denote $\frac{n-1}{2}$. For each $k \in \{1, 2, \ldots, m\}$, define $(c_1^k, \ldots, c_{n-1}^k)$ by

$$c_j^k := \begin{cases} 1 & j = k+1 \text{ or } j = n-k \\ 0 & \text{otherwise.} \end{cases}$$

Let $c^k := (c_1^k, \ldots, c_{n-1}^k)$ and $C_k := \text{Circ}(c^k)$. By an easy verification,

$$C_k \widetilde{D} = \text{Circ}(c^k \widetilde{D}). \tag{5}$$

We shall say that $c^1, \ldots, c^m$ are special vectors for W_n and $C_1, C_2, \ldots, C_m$ are special matrices for W_n. Each C_k is symmetric. For $i \in \{1, 2, \ldots, n-1\}$, define

$$v_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

Let $v := (v_1, v_2, \ldots, v_{n-1})$ and $V := \text{Circ}(v)$. If $k \in \{1, 2, \ldots, m-1\}$, then each column of C_k has exactly two ones and remaining entries equal to zero. Further, $k+1$ is odd if and only if $n-k$ is odd. On the other hand, each column of C_m has exactly one entry equal to one and remaining entries equal to zero. Further, the first column of C_m has one in the even position if and only if m is even. Also, each C_k is a Toeplitz matrix. In view of these observations, we get

$$vC_k = \begin{cases} (-1)^k 2v & \text{if } k = 1, \ldots, m-1 \\ (-1)^m v & \text{if } k = m. \end{cases} \tag{6}$$

3 Special Laplacian for W_n

We now associate a special Laplacian $\widetilde{L}$ to W_n. This definition is motivated from numerical computations.

Definition 1. For each $k \in \{1, 2, \ldots, m\}$, define

$$g(k) := \frac{n + (-1)^{(m-k)}}{2}$$

and

$$\alpha_k := \frac{(-1)^{g(k)}(2m^2 - 6(m-k)^2 + 1)}{6(n-1)}. \tag{7}$$

We say that the $n \times n$ matrix $\widetilde{L}$ defined by

$$\widetilde{L} := \begin{bmatrix} \frac{n-1}{2} & \mathbf{0} \\ \mathbf{0} & O \end{bmatrix} + \frac{n(n-2)}{6(n-1)} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I_{n-1} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & O \end{bmatrix} + \sum_{k=1}^{m} \alpha_k \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & C_k \end{bmatrix}$$

is the special Laplacian of W_n.

In the rest of the paper, we reserve the notation $\alpha_1, \ldots, \alpha_m$ for the numbers obtained by substituting $k = 1, \ldots, m$ respectively in the right hand side of the equation (7).

3.1 Illustration for W_5 and W_7

The interconnection between the special Laplacian and the distance matrix for W_5 and W_7 is given now. Later, in our main result, we generalize the result mentioned here to a general n.

- Consider W_5. The special vectors are now $c^1 = (0, 1, 0, 1)$ and $c^2 = (0, 0, 1, 0)$ and the special matrices are given by $\mathrm{Circ}(c^1)$ and $\mathrm{Circ}(c^2)$. We have $\alpha_1 = \frac{1}{8}$ and $\alpha_2 = -\frac{3}{8}$. The special Laplacian for W_5 can now be written easily using the definition:

$$\widetilde{L} = \frac{1}{8} \begin{bmatrix} 16 & -4 & -4 & -4 & -4 \\ -4 & 5 & 1 & -3 & 1 \\ -4 & 1 & 5 & 1 & -3 \\ -4 & -3 & 1 & 5 & 1 \\ -4 & 1 & -3 & 1 & 5 \end{bmatrix}.$$

The distance matrix D of W_5 is given by

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

By setting $w := \frac{1}{4}(0,1,1,1,1)'$, we note that

$$-\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww' = \frac{1}{4}\begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}. \tag{8}$$

The Moore-Penrose inverse of D and the matrix in the right hand side of (8) are equal. This can be verified directly.

- Consider W_7. There are three special vectors now. These are given by

$$c^1 = (0,1,0,0,0,1), \quad c^2 = (0,0,1,0,1,0), \text{ and } \quad c^3 = (0,0,0,1,0,0).$$

Using the special matrices $C_1 = \text{Circ}(c^1)$, $C_2 = \text{Circ}(c^2)$ and $C_3 = \text{Circ}(c^3)$ and the numbers $\alpha_1 = -\frac{5}{36}$, $\alpha_2 = -\frac{13}{36}$ and $\alpha_3 = \frac{19}{36}$, we compute the special Laplacian for W_7:

$$\widetilde{L} = \frac{1}{36}\begin{bmatrix} 108 & -18 & -18 & -18 & -18 & -18 & -18 \\ -18 & 35 & -5 & -13 & 19 & -13 & -5 \\ -18 & -5 & 35 & -5 & -13 & 19 & -13 \\ -18 & -13 & -5 & 35 & -5 & -13 & 19 \\ -18 & 19 & -13 & -5 & 35 & -5 & -13 \\ -18 & -13 & 19 & -13 & -5 & 35 & -5 \\ -18 & -5 & -13 & 19 & -13 & -5 & 35 \end{bmatrix}.$$

The distance matrix D of W_7 is given by

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

By setting $w := \frac{1}{4}(-2,1,1,1,1,1,1)'$, we note that

$$-\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww' = \frac{1}{18}\begin{bmatrix} -24 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & -8 & 2 & 4 & -4 & 4 & 2 \\ 3 & 2 & -8 & 2 & 4 & -4 & 4 \\ 3 & 4 & 2 & -8 & 2 & 4 & -4 \\ 3 & -4 & 4 & 2 & -8 & 2 & 4 \\ 3 & 4 & -4 & 4 & 2 & -8 & 2 \\ 3 & 2 & 4 & -4 & 4 & 2 & -8 \end{bmatrix}.$$

The matrix on the right hand side of the above equation is the Moore-Penrose inverse of D.

4 Main result

We are now ready to state our main result. The Moore-Penrose inverse of D is given by

$$D^{\dagger} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww', \tag{9}$$

where $w := \frac{1}{4}(5-n, 1, \ldots, 1)'$. Furthermore, $\widetilde{L}$ has the following properties:

(i) $\widetilde{L}$ is positive semidefinite.

(ii) $\widetilde{L}\mathbb{1} = \mathbf{0}$. That is, all row/column sums of $\widetilde{L}$ are zero.

(iii) $\text{rank}(\widetilde{L}) = n-2$.

In view of Section 3.1, the result is true for W_5 and W_7. We now proceed to show that the result holds for any odd integer n. In the rest of the paper, we assume $n \geq 9$.

4.1 Some identities

To prove the main result, we need the following identities.

Lemma 1. *Let n be odd and $m = \frac{n-1}{2}$. Define*

$$g(k) := \frac{n + (-1)^{m-k}}{2} \quad k = 1, 2, \ldots, m.$$

Then the following are true.

(I_1) $\sum_{k=1}^{m}(-1)^{g(k)}(2m^2 - 6(m-k)^2 + 1) = \begin{cases} -3m^2 + 3m & \text{if } m \text{ is even} \\ -m^2 + 3m + 1 & \text{if } m \text{ is odd.} \end{cases}$

(I_2) $\sum_{k=1}^{m}(-1)^{k+g(k)}(2m^2 - 6(m-k)^2 + 1) = -3m^2.$

(I_3) $2\sum_{k=1}^{m}\alpha_k - \alpha_m = \dfrac{6m - 4m^2 + 1}{6(n-1)}.$

(I_4) *If $j, j-1, j-2$ belong to $\{1, 2 \ldots, m\}$, then*

$$2\alpha_{j-1} + \alpha_j + \alpha_{j-2} = (-1)^j \frac{2}{n-1}.$$

(I_5)

$$2\sum_{k=1}^{m}(-1)^k\alpha_k - (-1)^m\alpha_m = \frac{2n - n^2}{6(n-1)}.$$

Proof. We begin with the proof of (I_1).

Case 1. Suppose m is even. Then,

$$g(k) = \begin{cases} \frac{n+1}{2} & k \text{ is even} \\ \frac{n-1}{2} & k \text{ is odd.} \end{cases}$$

Since $m + 1 = \frac{n+1}{2}$ and m is assumed to be even, we have

$$(-1)^{g(k)} = \begin{cases} -1 & k \text{ is even} \\ \ \ 1 & k \text{ is odd.} \end{cases} \tag{10}$$

Therefore,

$$\sum_{k=1}^{m} (-1)^{g(k)} = 0. \tag{11}$$

We now use the formula: If p is even, then,

$$1 - 2 + 3 - 4 + 5 - \cdots - p = -\frac{p}{2}.$$

Applying this to (10), we get

$$\sum_{k=1}^{m} (-1)^{g(k)} k = -\frac{m}{2}. \tag{12}$$

If p is even, then we know that

$$1^2 - 2^2 + 3^2 - \cdots - p^2 = -\frac{p(p+1)}{2}.$$

By this formula, we deduce

$$\sum_{k=1}^{m} (-1)^{g(k)} k^2 = -\frac{m(m+1)}{2}. \tag{13}$$

By (11), (12) and (13),

$$\begin{aligned} \sum_{k=1}^{m} (-1)^{g(k)} (2m^2 - 6(m-k)^2 + 1) &= -6 \sum_{k=1}^{m} (-1)^{g(k)} (m-k)^2 \\ &= 12m \sum_{k=1}^{m} (-1)^{g(k)} k - 6 \sum_{k=1}^{m} (-1)^{g(k)} k^2 \\ &= 12m(\frac{-m}{2}) - 6(\frac{-m(m+1)}{2}) \\ &= 3m - 3m^2. \end{aligned}$$

Case 2: Suppose m is odd. Then,

$$(-1)^{g(k)} = \begin{cases} 1 & k \text{ is odd} \\ -1 & k \text{ is even.} \end{cases} \tag{14}$$

Therefore,

$$\sum_{k=1}^{m} (-1)^{g(k)} = 1. \tag{15}$$

If p is odd, then,

$$1 - 2 + 3 - 4 - \cdots + p = \frac{p+1}{2}.$$

In view of this formula, we have

$$\sum_{k=1}^{m} (-1)^{g(k)} k = \frac{m+1}{2}. \tag{16}$$

If p is odd, then

$$1 - 2^2 + 3^2 - 4^2 + \cdots + p^2 = \frac{p(p+1)}{2}.$$

So,

$$\sum_{k=1}^{m} (-1)^{g(k)} k^2 = \frac{m(m+1)}{2}. \tag{17}$$

By (15), (16), and (17),

$$\begin{aligned} \sum_{k=1}^{m} (-1)^{g(k)} (m-k)^2 &= m^2 + \frac{m(m+1)}{2} - m(m+1) \\ &= \frac{m^2 - m}{2}. \end{aligned} \tag{18}$$

Again by (15), (16) and (17), and by (18), we get

$$\begin{aligned} \sum_{k=1}^{m} (-1)^{g(k)} (2m^2 - 6(m-k)^2 + 1) &= 2m^2 - 6(\frac{m^2 - m}{2}) + 1 \\ &= -m^2 + 3m + 1. \end{aligned}$$

This completes the proof of (I_1).

We now prove (I_2). If m is even, then by (10), $(-1)^{k+g(k)} = -1$ for any k. Similarly, if m is odd, then by (14), $(-1)^{k+g(k)} = -1$ for any k. Thus we have,

$$\begin{aligned}\sum_{k=1}^{m}(-1)^{k+g(k)}(2m^2 - 6(m-k)^2 + 1) &= (-1)\sum_{k=1}^{m}(-4m^2 - 6k^2 + 12mk + 1)\\ &= (4m^2 - 1)\sum_{k=1}^{m} 1 + 6\sum_{k=1}^{m} k^2 - 12m\sum_{k=1}^{m} k\\ &= -3m^2.\end{aligned}$$

The proof of (I_2) is complete.

We now prove (I_3). Define $\delta := 2\sum_{k=1}^{m}\alpha_k - \alpha_m$. Suppose m is even. Then by (I_1),

$$\sum_{k=1}^{m}\alpha_k = \frac{-3m^2 + 3m}{6(n-1)}.$$

By definition,

$$\alpha_m = (-1)^{\frac{n+1}{2}}\frac{2m^2+1}{6(n-1)}.$$

Since $m = \frac{n-1}{2}$ and m is even, $m + 1 = \frac{n+1}{2}$ is odd. Hence,

$$\alpha_m = -\frac{2m^2+1}{6(n-1)}.$$

Now,

$$\begin{aligned}\delta &= \frac{1}{6(n-1)}(2(3m - 3m^2) + 2m^2 + 1)\\ &= \frac{6m - 4m^2 + 1}{6(n-1)}.\end{aligned}$$

If m is odd, then by (I_1)

$$\sum_{k=1}^{m}\alpha_k = \frac{-m^2 + 3m + 1}{6(n-1)}.$$

Also, by definition

$$\alpha_m = \frac{2m^2+1}{6(n-1)}.$$

Now,

$$\begin{aligned}\delta &= \frac{1}{6(n-1)}2(-m^2 + 3m + 1) - (2m^2 + 1)\\ &= \frac{6m - 4m^2 + 1}{6(n-1)}.\end{aligned}$$

The proof of (I_3) is complete.

We now prove (I_4). In view of (7),

$$\alpha_{j-1} = \frac{(-1)^{\frac{n+(-1)^{(m-(j-1))}}{2}}(2m^2-6(m-(j-1))^2+1)}{6(n-1)},$$

$$\alpha_j = \frac{(-1)^{\frac{n+(-1)^{(m-j)}}{2}}(2m^2-6(m-j)^2+1)}{6(n-1)},$$

$$\alpha_{j-2} = \frac{(-1)^{\frac{n+(-1)^{(m-(j-2))}}{2}}(2m^2-6(m-(j-2))^2+1)}{6(n-1)}.$$

We note that

$$\begin{aligned}\alpha_j+\alpha_{j-2} &= \frac{(-1)^{\frac{n+(-1)^{(m-j)}}{2}}(-8m^2-12j^2+24mj+24j-24m-22)}{6(n-1)} \\ &= \frac{(-1)^{j-1}(8m^2+12j^2-24mj-24j+24m+22)}{6(n-1)}.\end{aligned} \tag{19}$$

Also,

$$\begin{aligned}2\alpha_{j-1} &= \frac{(-1)^{\frac{n-(-1)^{(m-j)}}{2}}(-8m^2-12j^2+24j+24mj-24m-10)}{6(n-1)} \\ &= \frac{(-1)^{j}(8m^2+12j^2-24mj-24j+24m+10)}{6(n-1)}.\end{aligned} \tag{20}$$

Adding equations (19) and (20), we get

$$2\alpha_{j-1}+\alpha_j+\alpha_{j-2} = (-1)^j\frac{2}{n-1}.$$

This proves (I_4). The proof of (I_5) is direct from (I_2). This completes the proof. □

5 Computation of $\widetilde{L}D$

To prove the inverse formula, it is useful to compute $\widetilde{L}D$ precisely.

Lemma 2.

$$\widetilde{L}D = \begin{bmatrix} \frac{1-n}{2} & \frac{5-n}{2}\mathbf{1}' \\ \frac{1}{2}\mathbf{1} & M \end{bmatrix},$$

where

$$M := \mathrm{Circ}\left(\frac{n(n-2)}{6(n-1)}u - \frac{1}{2}\mathbf{1}' + \sum_{k=1}^{m}\alpha_k c^k \widetilde{D}\right).$$

Proof. Direct multiplication of $\widetilde{L}$ and D gives

$$\widetilde{L}D = \begin{bmatrix} \frac{1-n}{2} & A \\ & \\ B & M \end{bmatrix},$$

where

$$A = \frac{n-1}{2}\mathbf{1}' - \frac{1}{2}\mathbf{1}'\widetilde{D},$$

$$B = \frac{n(n-2)}{6(n-1)}\mathbf{1} + \sum_{k=1}^{m} \alpha_k C_k \mathbf{1},$$

$$M = \frac{n(n-2)}{6(n-1)}\widetilde{D} - \frac{1}{2}\mathbf{1}\mathbf{1}' + \sum_{k=1}^{m} \alpha_k C_k \widetilde{D}.$$

We now simplify A, B and M. Since $\widetilde{D} = \mathrm{Circ}(u)$, $\mathbf{1}\mathbf{1}' = \mathrm{Circ}(\mathbf{1}')$ and $C_k\widetilde{D} = \mathrm{Circ}(c^k\widetilde{D})$, we get

$$M = \mathrm{Circ}(\frac{n(n-2)}{6(n-1)}u - \frac{1}{2}\mathbf{1}' + \sum_{k=1}^{m} \alpha_k c^k \widetilde{D}).$$

To complete the proof, we need to show that

$$A = \frac{5-n}{2}\mathbf{1}' \text{ and } B = \frac{1}{2}\mathbf{1}.$$

By (4), $\widetilde{D}\mathbf{1} = 2(n-3)\mathbf{1}$. So, $A = \frac{n-1}{2}\mathbf{1}' - (n-3)\mathbf{1}'$. This gives

$$A = \frac{5-n}{2}\mathbf{1}'.$$

To simplify B, we make the following observation first. If $1 \leq k \leq m-1$, then the the first row of C_k has exactly two ones and remaining entries equal to zero. On the other hand, the first row of C_m has exactly one entry equal to one and remaining entries equal to zero. Using this observation together with the fact that C_k is circulant, we now get

$$C_k\mathbf{1} = \begin{cases} 2\mathbf{1} & \text{if } k = 1, \dots, m-1 \\ \mathbf{1} & \text{if } k = m. \end{cases}$$

So,

$$\begin{aligned} B &= \frac{n(n-2)}{6(n-1)}\mathbf{1} + 2\sum_{k=1}^{m-1} \alpha_k \mathbf{1} + \alpha_m \mathbf{1} \\ &= \frac{n(n-2)}{6(n-1)}\mathbf{1} + 2\sum_{k=1}^{m} \alpha_k \mathbf{1} - 2\alpha_m \mathbf{1} + \alpha_m \mathbf{1} \qquad (21) \\ &= \frac{n(n-2)}{6(n-1)}\mathbf{1} + 2\sum_{k=1}^{m} \alpha_k \mathbf{1} - \alpha_m \mathbf{1}. \end{aligned}$$

Let

$$\delta = 2\sum_{k=1}^{m} \alpha_k - \alpha_m.$$

Then by (I_3),

$$\delta = \frac{6m - 4m^2 + 1}{6(n-1)}.$$

Hence (21) reduces to

$$B = \frac{n(n-2) + 6m - 4m^2 + 1}{6(n-1)}\mathbf{1}.$$

Since $m = \frac{n-1}{2}$,

$$n(n-2) - 4m^2 + 6m + 1 = 3n - 3.$$

So,

$$B = \frac{1}{2}\mathbf{1}.$$

The proof is complete now. □

5.1 The vectors $c^k\widetilde{D}$

We now compute the vectors $c^1\widetilde{D}, \ldots, c^m\widetilde{D}$ which appear in the matrix M.

Lemma 3. $c^1\widetilde{D} = (2, 2, 3, \underbrace{4, \ldots, 4}_{n-6}, 3, 2)$ *and* $c^1\widetilde{D} \in \Delta$.

Proof. We first note that

$$c^1 = (0, 1, 0, \ldots, 0, 1).$$

So, $c^1\widetilde{D}$ is the sum of the second row and the last row of $\widetilde{D}$. Let x be the second row and y be the last row of $\widetilde{D}$. Then,

$$x = (1, 0, 1, 2, \ldots, 2) \ \text{ and } \ y = (1, 2, \ldots, 2, 1, 0).$$

Now, $c^1\widetilde{D} = x+y = (2, 2, 3, \underbrace{4, \ldots, 4}_{n-6}, 3, 2)$. To verify $c^1\widetilde{D} \in \Delta$ is direct. This completes the proof. □

Lemma 4.

$$c^m\widetilde{D} = (\underbrace{2, \ldots, 2}_{\frac{n-3}{2}}, 1, 0, 1, 2, \ldots, 2),$$

and $c^m\widetilde{D} \in \Delta$.

Proof. We write c^m:

$$c^m = (\underbrace{0, \ldots, 0}_{\frac{n-1}{2}}, 1, 0, \ldots, 0).$$

Put $j = \frac{n+1}{2}$. Then, $c^m \widetilde{D}$ is the j^{th} row of $\widetilde{D}$. This means that if $(r_1, \ldots, r_n)$ is the $(j+1)^{\text{th}}$ row of D, then $c^m \widetilde{D} = (r_2, \ldots, r_n)$. The vertex w_{j+1} in W_n is adjacent to w_1, w_j and w_{j+2}. Thus,

$$r_\nu = \begin{cases} 0 & \text{if } \nu = j+1 \\ 1 & \text{if } \nu = 1, j, j+2 \\ 2 & \text{otherwise.} \end{cases}$$

If $(\theta_1, \ldots, \theta_{n-1}) = (r_2, \ldots, r_n)$, then the above equation gives

$$\theta_i = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i = j-1, j+1 \\ 2 & \text{otherwise.} \end{cases}$$

As $j = \frac{n+1}{2}$ and $c_m \widetilde{D} = (\theta_1, \ldots, \theta_{n-1})$, we conclude that

$$c^m \widetilde{D} = (\underbrace{2, \ldots, 2}_{\frac{n-3}{2}}, 1, 0, 1, 2, \ldots, 2).$$

Again, $c^m \widetilde{D} \in \Delta$ is direct. This completes the proof. ∎

Lemma 5.

$$c^{m-1} \widetilde{D} = (\underbrace{4, \ldots, 4}_{\frac{n-5}{2}}, 3, 2, 2, 2, 3, \underbrace{4, \ldots, 4}_{\frac{n-7}{2}}),$$

and $c^{m-1} \widetilde{D} \in \Delta$.

Proof. Since,

$$c^{m-1} = (\underbrace{0, \ldots, 0}_{\frac{n-3}{2}}, 1, 0, 1, \underbrace{0, \ldots, 0}_{\frac{n-5}{2}}),$$

$c^{m-1} \widetilde{D}$ is the sum of $(\frac{n-1}{2})^{\text{th}}$ and $(\frac{n+3}{2})^{\text{th}}$ rows of $\widetilde{D}$. Let these two rows be $\theta := (\theta_1, \ldots, \theta_{n-1})$ and $\rho := (\rho_1, \ldots, \rho_{n-1})$ respectively.

Suppose $(s_1, \ldots, s_n)$ is the $(\frac{n+1}{2})^{\text{th}}$ row of D. Then, $(s_2, \ldots, s_n)$ is the $(\frac{n-1}{2})^{\text{th}}$ row of $\widetilde{D}$. The vertex $w_{\frac{n+1}{2}}$ is adjacent to w_1, $w_{\frac{n+3}{2}}$ and $w_{\frac{n-1}{2}}$ in W_n. Thus,

$$s_\nu = \begin{cases} 0 & \text{if } \nu = \frac{n+1}{2} \\ 1 & \text{if } \nu = 1, \frac{n+3}{2}, \frac{n-1}{2} \\ 2 & \text{otherwise.} \end{cases}$$

As $(\theta_1, \ldots, \theta_{n-1}) = (s_2, \ldots, s_n)$,

$$\theta_i = \begin{cases} 0 & \text{if } i = \frac{n-1}{2} \\ 1 & \text{if } i = \frac{n+1}{2}, \frac{n-3}{2} \\ 2 & \text{otherwise.} \end{cases} \tag{22}$$

Suppose $(t_1, \ldots, t_n)$ is the $(\frac{n+5}{2})^{\text{th}}$ row of D. Then, $(t_2, \ldots, t_n)$ is the $(\frac{n+3}{2})^{\text{th}}$ row of $\widetilde{D}$. The vertex $w_{\frac{n+5}{2}}$ is adjacent to w_1, $w_{\frac{n+3}{2}}$ and $w_{\frac{n+7}{2}}$. Thus,

$$t_\nu = \begin{cases} 0 & \text{if } \nu = \frac{n+5}{2} \\ 1 & \text{if } \nu = 1, \frac{n+3}{2}, \frac{n+7}{2} \\ 2 & \text{otherwise.} \end{cases}$$

As $(\rho_1, \ldots, \rho_{n-1}) = (t_2, \ldots, t_n)$,

$$\rho_i = \begin{cases} 0 & \text{if } i = \frac{n+3}{2} \\ 1 & \text{if } i = \frac{n+1}{2}, \frac{n+5}{2} \\ 2 & \text{otherwise.} \end{cases} \tag{23}$$

By (22) and (23),

$$(c^{m-1}\widetilde{D})_i = (\theta + \rho)_i = \begin{cases} 2 & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2} \\ 3 & \text{for } i = \frac{n-3}{2}, \frac{n+5}{2} \\ 4 & \text{otherwise.} \end{cases}$$

We now show that $c^{m-1}\widetilde{D} \in \Delta$. Define

$$\Omega_1 := \{\frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}\},$$

$$\Omega_2 := \{\frac{n-3}{2}, \frac{n-5}{2}\},$$

$$\Omega_3 := \{2, \ldots, n-1\} \setminus (\Omega_1 \cup \Omega_2).$$

It is easy to see that, for each $j = 1, 2, 3$,

$$\nu \in \Omega_j \iff n + 1 - \nu \in \Omega_j,$$

and hence $c^{m-1}\widetilde{D} \in \Delta$. The proof is complete. □

Lemma 6. *Let $1 < k < m-1$. Define $q^k := c^k\widetilde{D}$. If $q^k = (q_1^k, \ldots, q_{n-1}^k)$, then*

$$q_j^k = \begin{cases} 2 & \textit{if } j = k+1, n-k \\ 3 & \textit{if } j = k, k+2, n-k-1, n-k+1 \\ 4 & \textit{otherwise.} \end{cases}$$

Furthermore, each $q^k \in \Delta$.

Proof. Let $1 < k < m-1$. Since c^k has 1 in the $(k+1)^{\text{th}}$ and $(n-k)^{\text{th}}$ positions and zeros elsewhere, q^k is the sum of $(k+1)^{\text{th}}$ and $(n-k)^{\text{th}}$ rows of $\widetilde{D}$. Let these rows be $\theta = (\theta_1, \ldots, \theta_{n-1})$ and $\eta = (\eta_1, \ldots, \eta_{n-1})$ respectively. Let $(s_1, \ldots, s_n)$ be the $(k+2)^{\text{th}}$ row of D. The vertex w_{k+2} is adjacent to w_1, w_{k+1} and w_{k+3}.

We now have

$$s_j = \begin{cases} 0 & \text{if } j = k+2 \\ 1 & \text{if } j = 1, k+1, k+3 \\ 2 & \text{otherwise.} \end{cases}$$

As $(\theta_1, \ldots, \theta_{n-1}) = (s_2, \ldots, s_n)$,

$$\theta_j = \begin{cases} 0 & \text{if } j = k+1 \\ 1 & \text{if } j = k, k+2 \\ 2 & \text{otherwise.} \end{cases} \tag{24}$$

Let $(n-k+1)^{\text{th}}$ row of D be $(t_1, \ldots, t_n)$. Then,

$$t_j = \begin{cases} 0 & \text{if } j = n-k+1 \\ 1 & \text{if } j = 1, n-k, n-k+2 \\ 2 & \text{otherwise.} \end{cases}$$

Because $(\eta_1, \ldots, \eta_{n-1}) = (t_2, \ldots, t_n)$,

$$\eta_j = \begin{cases} 0 & \text{if } j = n-k \\ 1 & \text{if } j = n-k-1, n-k+1 \\ 2 & \text{otherwise.} \end{cases} \tag{25}$$

We now compute $\theta + \eta$. Since $1 < k < m-1$, we have $n - 2k > n - 2m + 1$. As $n - 2m + 1 = 2$, $n - 2k > 2$. Thus, $n - k - 1 > k + 1$. Combining this inequality with the fact that $k < m-1$, we have

$$k + 2 \leq n - k - 2.$$

From (24) and (25), we immediately get

$$(\theta + \eta)_j = \begin{cases} 4 & \text{if } j = 1, \ldots, k-1 \\ 3 & \text{if } j = k, k+2 \\ 2 & \text{if } j = k+1. \end{cases} \tag{26}$$

If $k + 2 < j \leq n - k - 2$, then $\theta_j = \eta_j = 2$. So,

$$(\theta + \eta)_j = 4 \quad \text{for all} \quad k + 2 < j \leq n - k - 2. \tag{27}$$

We note that

$$\eta_j = \begin{cases} 1 & \text{if } j = n-k-1, n-k+1 \\ 0 & \text{if } j = n-k. \end{cases} \tag{28}$$

Since $\theta_j = 2$ for all $j > k+2$ and $n-k-1 > k+2$, we have

$$\theta_j = 2 \ \text{ for all } \ j \geq n-k-1. \tag{29}$$

In view of (28) and (29),

$$(\theta+\eta)_j = \begin{cases} 3 & \text{if } j = n-k-1, n-k+1 \\ 2 & \text{if } j = n-k. \end{cases} \tag{30}$$

Finally, from (24) and (25),

$$\theta_j = 2 \ \text{ and } \ \eta_j = 2 \quad \text{for all } j > n-k+1.$$

So,

$$(\theta+\eta)_j = 4 \ \text{ if } n-k+1 < j \leq n-1. \tag{31}$$

By (26), (27), (30) and (31), we get

$$q_j^k = \begin{cases} 2 & \text{if } j = k+1, n-k \\ 3 & \text{if } j = k, k+2, n-k-1, n-k+1 \\ 4 & \text{otherwise.} \end{cases}$$

We now show that $q^k \in \Delta$. For this, we partition the set $\Omega := \{2, \ldots, n-1\}$ into three parts. Define $\Omega_1 := \{k+1, n-k\}$, $\Omega_2 := \{k, k+2, n-k-1, n-k+1\}$ and $\Omega_3 := \Omega \setminus (\Omega_1 \cup \Omega_2)$. Each Ω_i has the property

$$\nu \in \Omega_i \quad \Longleftrightarrow \quad n+1-\nu \in \Omega_i.$$

Therefore, $q^k \in \Delta$. $\square$

5.2 Computation of $\sum_{k=1}^m \alpha_k c^k \widetilde{D}$

To simplify M, we need to compute the linear combination

$$f := \sum_{k=1}^{m} \alpha_k c^k \widetilde{D}.$$

For $1 \leq k \leq m$, define $q^k := c^k \widetilde{D}$. We shall write $q^k := (q_1^k, \ldots, q_{n-1}^k)$ and $f := (f_1, \ldots, f_{n-1})$. Now,

$$\begin{aligned} (f_1, \ldots, f_n) &= \sum_{k=1}^{m} \alpha_k (q_1^k, \ldots, q_{n-1}^k) \\ &= (\sum_{k=1}^{m} \alpha_k q_1^k, \ldots, \sum_{k=1}^{m} \alpha_k q_{n-1}^k). \end{aligned}$$

We now compute f precisely.

Lemma 7.

$$f_1 = \frac{3-n}{n-1}.$$

Proof. By Lemma 3, 4, 5 and 6, we have

$$q_1^k = \begin{cases} 2 & \text{if } k = 1, m \\ 4 & \text{if } k = 2, \ldots, m-1. \end{cases}$$

In view of this,

$$\begin{aligned} f_1 &= \sum_{k=1}^{m} \alpha_k q_1^k \\ &= 2\alpha_1 + 2\alpha_m + 4\sum_{k=2}^{m-1} \alpha_k \\ &= 2\alpha_1 + 2\alpha_m + 4\sum_{k=1}^{m} \alpha_k - 4\alpha_1 - 4\alpha_m \\ &= -2\alpha_1 - 2\alpha_m + 4\sum_{k=1}^{m} \alpha_k. \end{aligned} \tag{32}$$

Let $\delta = 2\sum_{k=1}^{m} \alpha_k - \alpha_m$. By ($I_3$),

$$\delta = \frac{6m - 4m^2 + 1}{6(n-1)}.$$

Therefore,

$$4\sum_{k=1}^{m} \alpha_k - 2\alpha_m = \frac{-8m^2 + 12m + 2}{6(n-1)}. \tag{33}$$

From (7),

$$\alpha_1 = \frac{-4m^2 + 12m - 5}{6(n-1)}. \tag{34}$$

Substituting (33) and (34) in (32) gives

$$f_1 = \frac{2}{n-1} - \frac{2m}{n-1}.$$

Since $m = \frac{n-1}{2}$,

$$f_1 = \frac{3-n}{n-1}.$$

The proof is complete. □

Lemma 8.
$$f_2 = \frac{-n^2+8n-18}{6(n-1)}.$$

Proof. By Lemma 3, 4, 5 and 6,
$$q_2^k = \begin{cases} 2 & \text{if } k = 1, m \\ 3 & \text{if } k = 2 \\ 4 & \text{otherwise.} \end{cases}$$

This gives,
$$\begin{aligned} f_2 &= 2(\alpha_1+\alpha_m) + 3\alpha_2 + 4\sum_{k=3}^{m-1}\alpha_k \\ &= -2\alpha_1 - 2\alpha_m + 4\sum_{k=1}^{m}\alpha_k - \alpha_2 \end{aligned}$$

Put $\delta = 2\sum_{k=1}^{m}\alpha_k - \alpha_m$. By ($\mathrm{I}_3$),
$$\delta = \frac{6m-4m^2+1}{6(n-1)}.$$

Now,
$$f_2 = -2\alpha_1 + 2\delta - \alpha_2.$$

We note that
$$\alpha_1 = \frac{(-4m^2+12m-5)}{6(n-1)} \quad \text{and} \quad \alpha_2 = \frac{4m^2-24m+23}{6(n-1)}.$$

In view of the above equations,
$$\begin{aligned} f_2 &= \frac{-2(-4m^2+12m-5)+2(6m-4m^2+1)-(4m^2-24m+23)}{6(n-1)} \\ &= \frac{-4m^2+12m-11}{6(n-1)}. \end{aligned}$$

As $m = \frac{n-1}{2}$,
$$f_2 = \frac{-n^2+8n-18}{6(n-1)}.$$

□

Lemma 9. *Let $2 < j \leq \frac{n-3}{2}$. Then,*
$$f_j = \begin{cases} \dfrac{-2n^2+10n-18}{6(n-1)} & \textit{if } j \textit{ is even} \\[2ex] \dfrac{-2n^2+10n+6}{6(n-1)} & \textit{if } j \textit{ is odd.} \end{cases}$$

Proof. Let j be such that $2 < j \leq \frac{n-3}{2}$. By Lemma 3,

$$q_j^1 = \begin{cases} 3 & \text{if } j = 3 \\ 4 & \text{otherwise.} \end{cases}$$

By Lemma 4,

$$q_j^m = 2.$$

In view of Lemma 5,

$$q_j^{m-1} = \begin{cases} 3 & \text{if } j = m-1 \\ 4 & \text{otherwise.} \end{cases}$$

From Lemma 6,

$$1 < k < \frac{n-3}{2} \implies q_j^k = \begin{cases} 2 & \text{if } j = k+1 \\ 3 & \text{if } j = k, k+2 \\ 4 & \text{otherwise.} \end{cases}$$

Together, all these equations give

$$q_j^k = \begin{cases} 2 & \text{if } j = k+1 \text{ and } 1 < k < \frac{n-3}{2} \\ 2 & \text{if } k = \frac{n-1}{2} \\ 3 & \text{if } j = 3 \text{ and } k = 1 \\ 3 & \text{if } j = k = \frac{n-3}{2} \\ 3 & \text{if } j = k, k+2 \text{ and } 1 < k < \frac{n-3}{2} \\ 4 & \text{otherwise.} \end{cases}$$

Thus,

$$q_j^k = \begin{cases} 2 & \text{if } k = j-1, m \\ 3 & \text{if } k = j, j-2 \\ 4 & \text{otherwise.} \end{cases} \tag{35}$$

We need to compute

$$f_j = \sum_{k=1}^{m} \alpha_k q_j^k$$

for $2 < j \leq \frac{n-3}{2}$. By (I_4),

$$2\alpha_{j-1} + \alpha_j + \alpha_{j-2} = (-1)^j \frac{2}{n-1}.$$

Define $\Omega := \{j-2, j-1, j, m\}$. By (35), we have

$$\begin{aligned} f_j &= 2\alpha_{j-1} + 2\alpha_m + 3\alpha_j + 3\alpha_{j-2} + 4\sum_{k\notin\Omega}\alpha_k \\ &= -(2\alpha_{j-1} + \alpha_j + \alpha_{j-2}) - 2\alpha_m + 4\sum_{k=1}^{m}\alpha_k \\ &= -(-1)^j\frac{2}{n-1} - 2(\alpha_m - 2\sum_{k=1}^{m}\alpha_k) \\ &= -(-1)^j\frac{2}{n-1} + \frac{6m-4m^2+1}{3(n-1)}. \end{aligned} \tag{36}$$

where the last two equations follow from (I_3) and (I_4). Replacing m by $\frac{n-1}{2}$ in (36), we get

$$f_j = \begin{cases} \dfrac{-2n^2+10n-18}{6(n-1)} & \text{if } j \text{ is even} \\ \dfrac{-2n^2+10n+6}{6(n-1)} & \text{if } j \text{ is odd.} \end{cases}$$

The proof is complete. □

Lemma 10.

$$f_m = \begin{cases} \dfrac{-2n^2+10n-18}{6(n-1)} & \textit{if } m \textit{ is even} \\ \dfrac{-2n^2+10n+6}{6(n-1)} & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. In view of Lemma 3, 4, 5 and 6, we have

$$q_m^k = \begin{cases} 1 & \text{if } k = m \\ 2 & \text{if } k = m-1 \\ 3 & \text{if } k = m-2 \\ 4 & \text{otherwise.} \end{cases} \tag{37}$$

We need to compute

$$f_m = \sum_{k=1}^{m}\alpha_k q_m^k.$$

By (37),

$$\begin{aligned} f_m &= 4\sum_{k=1}^{m-3}\alpha_k + 3\alpha_{m-2} + 2\alpha_{m-1} + \alpha_m \\ &= 4\sum_{k=1}^{m}\alpha_k - \alpha_{m-2} - 2\alpha_{m-1} - 3\alpha_m \\ &= 2(2\sum_{k=1}^{m}\alpha_k - \alpha_m) - \alpha_m - \alpha_{m-2} - 2\alpha_{m-1}. \end{aligned}$$

Define

$$\delta := 2(\sum_{k=1}^{m}\alpha_k - \alpha_m) \text{ and } \gamma := \alpha_m + \alpha_{m-2} + 2\alpha_{m-1}.$$

In view of (I_3) and (I_4),

$$f_m = \frac{6m - 4m^2 + 1}{3(n-1)} - (-1)^m \frac{2}{n-1}.$$

Upon substituting $m = \frac{n-1}{2}$,

$$f_m = \begin{cases} \dfrac{-2n^2 + 10n - 18}{6(n-1)} & \text{if } m \text{ is even} \\ \\ \dfrac{-2n^2 + 10n + 6}{6(n-1)} & \text{if } m \text{ is odd.} \end{cases}$$

The proof is complete. □

Lemma 11.

$$f_{\frac{n+1}{2}} = \begin{cases} \dfrac{-2n^2 + 10n + 6}{6(n-1)} & \textit{if } m \textit{ is even} \\ \\ \dfrac{-2n^2 + 10n - 18}{6(n-1)} & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. In view of Lemma 3, 4, 5 and 6, we have

$$q^k_{\frac{n+1}{2}} = \begin{cases} 0 & \text{if } k = m \\ 2 & \text{if } k = m-1 \\ 4 & \text{otherwise.} \end{cases}$$

We now have

$$\begin{aligned} f_{\frac{n+1}{2}} &= \sum_{k=1}^{m} \alpha_k q_{\frac{n+1}{2}}^k \\ &= 4\sum_{k=1}^{m-2} \alpha_k + 2\alpha_{m-1} \\ &= 4\sum_{k=1}^{m} \alpha_k - 2\alpha_{m-1} - 4\alpha_m. \end{aligned} \tag{38}$$

By (7),

$$\alpha_m = (-1)^{m+1}\frac{2m^2+1}{6(n-1)} \quad \text{and} \quad \alpha_{m-1} = (-1)^m \frac{2m^2-5}{6(n-1)},$$

and hence,

$$2\alpha_{m-1} + 4\alpha_m = (-1)^{m-1}\frac{4m^2+14}{6(n-1)}. \tag{39}$$

Suppose m is even. Then by (I_1), (38) and (39),

$$\begin{aligned} f_{\frac{n+1}{2}} &= \frac{4(-3m^2+3m)+4m^2+14}{6(n-1)} \\ &= \frac{-8m^2+12m+14}{6(n-1)}. \end{aligned}$$

Substituting $m = \frac{n-1}{2}$ gives

$$f_{\frac{n+1}{2}} = \frac{-2n^2+10n+6}{6(n-1)}.$$

Suppose m is odd. Then by (I_1), (38) and (39),

$$\begin{aligned} f_{\frac{n+1}{2}} &= \frac{4(-m^2+3m+1)-(4m^2+14)}{6(n-1)} \\ &= \frac{-8m^2+12m-10}{6(n-1)}. \end{aligned}$$

Upon substituting $m = \frac{n-1}{2}$,

$$f_{\frac{n+1}{2}} = \frac{-2n^2+10n-18}{6(n-1)}.$$

This completes the proof. □

By Lemma 9, 10 and 11, we get

$$2 < j \leq \frac{n+1}{2} \implies f_j = \begin{cases} \dfrac{-2n^2+10n-18}{6(n-1)} & \text{if } j \text{ is even} \\ \\ \dfrac{-2n^2+10n+6}{6(n-1)} & \text{if } j \text{ is odd.} \end{cases} \tag{40}$$

To this end, we have computed $f_1, \dots, f_{\frac{n+1}{2}}$. We now deduce f.

Lemma 12. *Define*

$$f_1 := \frac{3-n}{n-1},$$
$$f_2 := \frac{-n^2+8n-18}{6(n-1)},$$
$$\tau := \frac{-2n^2+10n-18}{6(n-1)},$$
$$\omega := \frac{-2n^2+10n+6}{6(n-1)}.$$

Then,

$$f = (f_1, f_2, \omega, \tau, \omega, \tau, \dots, \tau, \omega, f_2).$$

Proof. We begin with the following observation: If $x, y \in \Delta$ and $\beta \in \mathbb{R}$, then $\beta x + y \in \Delta$. We have shown that $c^1\widetilde{D}, c^2\widetilde{D}, \dots, c^m\widetilde{D} \in \Delta$. Thus,

$$f = \sum_{k=1}^{m} \alpha_k c^k \widetilde{D} \in \Delta.$$

So, by Lemma 7, Lemma 8 and equation (40),

$$f = (f_1, f_2, \omega, \tau, \omega, \tau, \dots, \tau, \omega, f_2).$$

The proof is complete. □

5.3 Simplification of M

Using the values of f_1, f_2, ω and τ, we simplify the expression:

$$M = \mathrm{Circ}(\frac{n(n-2)}{6(n-1)}u - \frac{1}{2}\mathbf{1}' + f).$$

Lemma 13.

$$M = \frac{1}{2}J - 2I + \frac{2}{n-1}\mathrm{Circ}(1, -1, 1, -1, \dots, -1).$$

Proof. Define

$$h := \frac{n(n-2)}{6(n-1)}u - \frac{1}{2}\mathbf{1}' + f.$$

Recall that u is given by $(0, 1, 2, \ldots, 2, 1)$. Now,

$$\begin{aligned} h_1 &= -\frac{1}{2} + \frac{3-n}{n-1} \\ &= -\frac{3}{2} + \frac{2}{n-1}. \end{aligned} \tag{41}$$

Suppose $j = 2, n-1$. Since $f_2 = f_{n-1}$ and $u_2 = u_{n-1} = 1$, we get

$$\begin{aligned} h_j &= \frac{n(n-2)}{6(n-1)} - \frac{1}{2} + \frac{-n^2+8n-18}{6(n-1)} \\ &= \frac{n-3}{n-1} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{2}{n-1}. \end{aligned} \tag{42}$$

If $2 < j < n-1$ is odd, then $f_j = \omega$ and $u_j = 2$. So,

$$\begin{aligned} h_j &= \frac{2n(n-2)}{6(n-1)} + \frac{6+10n-2n^2}{6(n-1)} - \frac{1}{2} \\ &= \frac{n+1}{n-1} - \frac{1}{2} \\ &= \frac{1}{2} + \frac{2}{n-1}. \end{aligned} \tag{43}$$

If $2 < j < n-1$ is even, then $f_j = \tau$ and $u_j = 2$. So,

$$\begin{aligned} h_j &= \frac{2n(n-2)}{6(n-1)} - \frac{1}{2} + \frac{-2n^2+10n-18}{6(n-1)} \\ &= \frac{n-3}{n-1} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{2}{n-1}. \end{aligned} \tag{44}$$

In view of (41), (42), (43) and (44),

$$h = \left(-\frac{3}{2} + \frac{2}{n-1}, \frac{1}{2} - \frac{2}{n-1}, \frac{1}{2} + \frac{2}{n-1}, \ldots, \frac{1}{2} - \frac{2}{n-1}\right).$$

Thus, h can be written

$$h = \frac{1}{2}(-3, 1, \ldots, 1) + \frac{2}{n-1}(1, -1, 1, -1, \ldots, -1).$$

It is easy to see that

$$\mathrm{Circ}(\frac{1}{2}(-3,1,\ldots,1)) = \frac{1}{2}J - 2I.$$

Thus,

$$M = \mathrm{Circ}(h) = \frac{1}{2}J - 2I + \frac{2}{n-1}\mathrm{Circ}(1,-1,1,-1,\ldots,-1).$$

□

6 Inverse formula

We now prove our main result.

Theorem 1. *Let W_n be a wheel graph with n vertices, where n is an odd integer. If D is the distance matrix of W_n given by (3), then*

$$D^{\dagger} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww',$$

where $w = \frac{1}{4}(5-n,1,\ldots,1)'$.

Proof. We recall that $v = (1,-1,1,-1,\ldots,-1)$ and $V = \mathrm{Circ}(v)$. By Lemma 2 and Lemma 13, we have

$$\widetilde{L}D = \begin{bmatrix} \frac{1-n}{2} & \frac{5-n}{2}\mathbf{1}' \\ \frac{1}{2}\mathbf{1} & \frac{1}{2}J - 2I + \frac{2}{n-1}V \end{bmatrix}.$$

We write $\widetilde{L}D$ as

$$\begin{bmatrix} \frac{5-n}{2} - 2 & \frac{5-n}{2}\mathbf{1}' \\ \frac{1}{2}\mathbf{1} & \frac{1}{2}\mathbf{1}\mathbf{1}' - 2I \end{bmatrix} + \frac{2}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}.$$

Thus,

$$\widetilde{L}D = \begin{bmatrix} \frac{5-n}{2} & \frac{5-n}{2}\mathbf{1}' \\ \frac{1}{2}\mathbf{1} & \frac{1}{2}\mathbf{1}\mathbf{1}' \end{bmatrix} - 2I + \frac{2}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}.$$

By an easy verification,

$$2w\mathbf{1}_n' = \begin{bmatrix} \frac{5-n}{2} & \frac{5-n}{2}\mathbf{1}' \\ \frac{1}{2}\mathbf{1} & \frac{1}{2}\mathbf{1}\mathbf{1}' \end{bmatrix}.$$

Thus,

$$\widetilde{L}D + 2I = 2w\mathbf{1}_n' + \frac{2}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}. \tag{45}$$

As,

$$D = \begin{bmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & \widetilde{D} \end{bmatrix},$$

where $\widetilde{D} = \mathrm{Circ}(u)$, by (4) we deduce,

$$Dw = \frac{1}{4}(n-1)\mathbf{1}_n. \tag{46}$$

Define

$$K := -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww'.$$

To complete the proof, we show that KD is symmetric, $DKD = D$ and $KDK = K$. We first compute KD. By (45) and (46),

$$\frac{4}{n-1}ww'D = w\mathbf{1}_n',$$

$$-\frac{1}{2}\widetilde{L}D = -w\mathbf{1}_n' - \frac{1}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} + I.$$

Adding the above two equations, we get

$$KD = I - \frac{1}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}. \tag{47}$$

So, KD is symmetric.

Before proceeding further, we note that, since n is odd, $uv' = 0$ and $\mathbf{1}'v' = 0$. Since $\widetilde{D} = \mathrm{Circ}(u)$ and $V = \mathrm{Circ}(v)$, $\widetilde{D}V = \mathrm{Circ}(uV)$. So, $\widetilde{D}V = O$.

By (47), it follows that

$$\begin{aligned} DKD &= D - \frac{1}{n-1}\begin{bmatrix} 0 & \mathbf{1}'V \\ \mathbf{0} & \widetilde{D}V \end{bmatrix} \\ &= D - \frac{1}{n-1}\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \widetilde{D}V \end{bmatrix} \\ &= D. \end{aligned}$$

We now compute KDK. From (6), we recall the following observation for the special matrices $C_1, \dots, C_m$ for W_n:

$$vC_k = \begin{cases} (-1)^k 2v & \text{if } k = 1, \dots, m-1 \\ (-1)^m v & \text{if } k = m. \end{cases} \tag{48}$$

We claim that

$$\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} \widetilde{L} = O. \tag{49}$$

By a direct computation, we have

$$\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} \widetilde{L} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix},$$

where

$$N := \frac{n(n-2)}{6(n-1)} V + \sum_{k=1}^{m} \alpha_k V C_k.$$

Since $V = \mathrm{Circ}(v)$ and $VC_k = \mathrm{Circ}(vC_k)$,

$$N = \mathrm{Circ}(\frac{n(n-2)}{6(n-1)} v + \sum_{k=1}^{m} \alpha_k v C_k).$$

Using (48),

$$\begin{aligned} \sum_{k=1}^{m} \alpha_k v C_k &= 2\sum_{k=1}^{m-1} (-1)^k \alpha_k v + (-1)^m \alpha_m v \\ &= (2\sum_{k=1}^{m} (-1)^k \alpha_k - (-1)^m \alpha_m) v. \end{aligned}$$

In view of (I_5),

$$\sum_{k=1}^{m} \alpha_k v C_k = \frac{2n - n^2}{6(n-1)} v.$$

This implies $N = O$ and the claim is proved.

As $\mathbf{1}'v' = 0$, we have $V\mathbf{1} = \mathbf{0}$. So,

$$\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} w = \mathbf{0}. \tag{50}$$

From equation (47), we obtain

$$KDK = K + \frac{1}{2(n-1)} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} \widetilde{L} - \frac{4}{(n-1)^2} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} ww'.$$

By (49) and (50), we get $KDK = K$. The proof is complete. □

7 Properties of the special Laplacian matrix

In this section, we obtain certain properties of the special Laplacian matrix. In order to do this, we need a preliminary result. Define $p := (p_1, \ldots, p_{n-2})$ and $q := (q_1, \ldots, q_{n-2})$ by

$$p_k := \begin{cases} -1 & \text{if } k = 1 \\ -2 & \text{if } k \text{ is even} \\ 0 & \text{else;} \end{cases}$$

$$q_k := \begin{cases} -1 & \text{if } k = 1 \\ 0 & \text{if } k \text{ is even} \\ -2 & \text{else.} \end{cases}$$

Define an $n \times (n-2)$ matrix by

$$C := \begin{bmatrix} 2I_{n-2} \\ p \\ q \end{bmatrix}.$$

We shall find a matrix X such that $\widetilde{L}DX = C$. Define a vector $y := (y_1, \ldots, y_{n-3})$ by

$$y_k := \begin{cases} -2 & \text{if } k = 1 \\ 0 & \text{if } k \text{ is even} \\ -1 & \text{else;} \end{cases}$$

and let $Y := \text{Circ}(y)$.

Lemma 14. *If*

$$X := \frac{1}{2} \begin{bmatrix} n-7 & (n-5)\mathbf{1}'_{n-3} \\ -\mathbf{1}_{n-3} & 2Y \\ \mathbf{0} & O \end{bmatrix},$$

then $\widetilde{L}DX = C$.

Proof. From (45), we have

$$\widetilde{L}D + 2I = 2w\mathbf{1}'_n + \frac{2}{n-1} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}$$

So,

$$\widetilde{L}DX = 2w\mathbf{1}'_n X - 2X + \frac{2}{n-1} \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix} X. \tag{51}$$

By an easy computation,

$$\mathbf{1}'_{n-3} y = -2 - \frac{n-5}{2} = \frac{1-n}{2},$$

and therefore,

$$\mathbf{1}'_{n-3}Y = \frac{1-n}{2}\mathbf{1}'_{n-3}.$$

Hence

$$\begin{aligned}\mathbf{1}'X &= \frac{1}{2}\left[\begin{array}{cc} n-7+(n-3)(-1) & (n-5)\mathbf{1}'_{n-3}+2\mathbf{1}'_{n-3}Y \end{array}\right] \\ &= \frac{1}{2}\left[\begin{array}{cc} -4 & (n-5)\mathbf{1}'_{n-3}+(1-n)\mathbf{1}'_{n-3} \end{array}\right] \\ &= -2\mathbf{1}'_{n-2}.\end{aligned} \tag{52}$$

Recall that $V = \mathrm{Circ}(v)$, where $v = (1, -1, 1, -1, \dots, -1)$ is a row vector with $n-1$ components. Define a row vector $\widetilde{v}$ with $n-3$ components by

$$\widetilde{v} := (1, -1, 1, -1, \dots, -1).$$

Let

$$R := [\widetilde{v}', -\widetilde{v}'] \text{ and } Q := \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array}\right].$$

Then V can be written

$$V = \left[\begin{array}{cc} \mathrm{Circ}(\widetilde{v}) & R \\ R' & Q \end{array}\right].$$

Therefore

$$\begin{aligned}\left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & V \end{array}\right] X &= \frac{1}{2}\left[\begin{array}{ccc} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{Circ}(\widetilde{v}) & R \\ \mathbf{0} & R' & Q \end{array}\right]\left[\begin{array}{cc} n-7 & (n-5)\mathbf{1}'_{n-3} \\ -\mathbf{1}_{n-3} & 2Y \\ \mathbf{0} & O \end{array}\right], \\ &= \frac{1}{2}\left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & 2\mathrm{Circ}(\widetilde{v}Y) \\ \mathbf{0} & 2R'Y \end{array}\right].\end{aligned} \tag{53}$$

By a direct verification, we see that

$$\widetilde{v}Y = \frac{1-n}{2}\widetilde{v}.$$

This gives

$$R'Y = \frac{1-n}{2}R'$$

From (53), we have

$$\left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & V \end{array}\right] X = \frac{1-n}{2}\left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & \mathrm{Circ}(\widetilde{v}) \\ \mathbf{0} & R' \end{array}\right]. \tag{54}$$

From (51), (52) and (54), we have

$$\widetilde{L}DX = -4w\mathbf{1}'_{n-2} - 2X - \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathrm{Circ}(\widetilde{v}) \\ \mathbf{0} & R' \end{bmatrix}$$
$$= -4w\mathbf{1}'_{n-2} - \begin{bmatrix} n-7 & (n-5)\mathbf{1}'_{n-3} \\ -\mathbf{1}_{n-3} & 2Y \\ \mathbf{0} & O \end{bmatrix} - \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathrm{Circ}(\widetilde{v}) \\ \mathbf{0} & R' \end{bmatrix}$$
$$= -\begin{bmatrix} 5-n & (5-n)\mathbf{1}'_{n-3} \\ \mathbf{1}_{n-3} & J_{n-3} \\ \mathbf{1}_2 & \mathbf{1}_2\mathbf{1}'_{n-3} \end{bmatrix} - \begin{bmatrix} n-7 & (n-5)\mathbf{1}'_{n-3} \\ -\mathbf{1}_{n-3} & 2Y + \mathrm{Circ}(\widetilde{v}) \\ \mathbf{0} & R' \end{bmatrix}.$$

Thus,

$$\widetilde{L}DX = -\begin{bmatrix} -2 & \mathbf{0} \\ \mathbf{0} & J_{n-3} + 2Y + \mathrm{Circ}(\widetilde{v}) \\ \mathbf{1}_2 & \mathbf{1}_2\mathbf{1}'_{n-3} + R' \end{bmatrix}$$
$$= -\begin{bmatrix} -2 & \mathbf{0} \\ \mathbf{0} & \mathrm{Circ}(\mathbf{1}'_{n-3} + 2y' + \widetilde{v}) \\ \mathbf{1}_2 & \mathbf{1}_2\mathbf{1}'_{n-3} + R' \end{bmatrix}. \tag{55}$$

We note that

$$\mathbf{1}'_{n-3} + 2y' + \widetilde{v} = (2,0,2,0,\ldots,2,0) + 2y' = -2(1,0,\ldots,0)' \tag{56}$$

and

$$\mathbf{1}_2\mathbf{1}'_{n-3} + R' = 2\begin{bmatrix} 1 & 0 & 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 1 \end{bmatrix} \tag{57}$$

From (55), (56) and (57), we get

$$\widetilde{L}DX = C.$$

The proof is complete. □

We conclude the paper with the following theorem.

Theorem 2. *The special Laplacian matrix $\widetilde{L}$ has the following properties.*

(i) $\widetilde{L}\mathbf{1}_n = \mathbf{0}$.

(ii) $\mathrm{rank}(\widetilde{L}) = n-2$.

(iii) *If $\widetilde{L}^\dagger = [\theta_{ij}]$, then $d_{ij} = \theta_{ii} + \theta_{jj} - 2\theta_{ij}$.*

(iv) *$\widetilde{L}$ is positive semidefinite.*

Proof. Using Definition 1, we have

$$\begin{aligned}\widetilde{L}\mathbf{1}_n &= \begin{bmatrix} \frac{n-1}{2} \\ \mathbf{0} \end{bmatrix} + \frac{n(n-2)}{6(n-1)} \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} n-1 \\ \mathbf{1} \end{bmatrix} + \sum_{k=1}^{m} \alpha_k \begin{bmatrix} 0 \\ C_k\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{2}\mathbf{1} + B \end{bmatrix}\end{aligned}$$

where $B = \frac{n(n-2)}{6(n-1)}\mathbf{1} + \sum_{k=1}^{m} \alpha_k C_k \mathbf{1}$. From Lemma 2, $B = \frac{1}{2}\mathbf{1}$. Hence $\widetilde{L}\mathbf{1}_n = \mathbf{0}$. The proof of (i) is complete.

We will now prove (ii). Since $\widetilde{L}$ is symmetric and $\widetilde{L}\mathbf{1}_n = \mathbf{0}$, all cofactors of $\widetilde{L}$ are equal. Let the common cofactor of $\widetilde{L}$ be δ. By Theorem 1,

$$D^\dagger = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww'.$$

Using matrix determinant lemma,

$$\begin{aligned}\det(D^\dagger) &= \det(-\frac{1}{2}\widetilde{L}) + \frac{4}{n-1}w'\mathrm{adj}(-\frac{1}{2}\widetilde{L})w \\ &= (-1)^{n-1}\frac{4}{n-1}\frac{1}{2^{n-1}}\delta.\end{aligned}$$

Hence $\delta = 0$. So, $\mathrm{rank}(\widetilde{L}) \leq n-2$. In view of Lemma 14, $\mathrm{rank}(\widetilde{L}) \geq n-2$. Thus, $\mathrm{rank}(\widetilde{L}) = n-2$. This proves (ii).

To prove (iii), we first note that

$$Dw = \frac{n-1}{4}\mathbf{1}_n, \quad D^\dagger\mathbf{1}_n = \frac{4}{n-1}w \ \text{ and } \ \mathbf{1}_n' D^\dagger \mathbf{1}_n = \frac{4}{n-1}. \tag{58}$$

Define

$$P := I - \frac{1}{n}J \ \text{ and } \ G := -\frac{1}{2}PDP.$$

As $\mathbf{1}_n' D^\dagger \mathbf{1}_n > 0$, by Theorem 3.1 in [9], we have

$$D^\dagger = -\frac{1}{2}G^\dagger + \frac{1}{\mathbf{1}_n' D^\dagger \mathbf{1}_n}(D^\dagger\mathbf{1}_n)(\mathbf{1}_n' D^\dagger). \tag{59}$$

From (58) and (59), we get

$$D^\dagger = -\frac{1}{2}G^\dagger + \frac{4}{n-1}ww'. \tag{60}$$

By our inverse formula,

$$D^\dagger = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww' \tag{61}$$

Equations (60) and (61) imply

$$G^\dagger = \widetilde{L}.$$

This gives

$$\widetilde{L}^\dagger = G = -\frac{1}{2}PDP. \tag{62}$$

By putting $\widetilde{L}^\dagger = [\theta_{ij}]$, we see that equation (62) gives

$$d_{ij} = \theta_{ii} + \theta_{jj} - 2\theta_{ij}.$$

This proves (iii).

Since D is a Euclidean distance matrix, by a well-known theorem of Schoenberg, G is a positive semidefinite matrix. Hence, $\widetilde{L}$ is positive semidefinite. This proves (iv). The proof is complete. □

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