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On an interconnection between the Lipschitz continuity of the solution map and the positive principal minor property in linear complementarity problems over Euclidean Jordan algebras

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Abstract

In the setting of Euclidean Jordan algebras, we study the Lipschitz continuity of the solution map of linear complementarity problems. We show that if the solution map is Lipschitz continuous and if the linear transformation has the **Q**-property, then the linear transformation has the positive principal minor property.

Keywords: Complementarity; Positive principal minor property; Lipschitzian property; **Q**-property

1. Introduction

A real $n \times n$ matrix M is said to be a **P**-matrix if every principal minor of M is positive. It is well-known that this property can be described in any one of the following ways:

1. For all $q \in R^n$, the linear complementarity problem LCP(M, q) has a unique solution.
2. The map $q \mapsto \text{SOL}(M, q)$ is single valued and Lipschitzian on R^n, where SOL(M, q) denotes the solution set of LCP(M, q).
3. For any $q \in R^n$, SOL(M, q) is nonempty and the set-valued map $q \mapsto \text{SOL}(M, q)$ is Lipschitzian.

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In this article, we study an analog of the above for a linear transformation defined on a Euclidean Jordan algebra, which is a finite dimensional inner product space equipped with a Jordan product, the corrresponding (symmetric) cone of squares. The space S^n of all $n \times n$ real symmetric matrices is an example of a Euclidean Jordan algebra where S^n_+ is the set of all positive semidefinite matrices in S^n is the cone of squares. Given a Jordan frame $\{e_1, \ldots, e_r\}$ in a Euclidean Jordan algebra V and a linear transformation L on V. Gowda et al. [9] introduced the concept of a principal subtransformation for L by restricting L to the eigenspace $V^{(l)} = \{x \in V : x \circ (e_1 + \cdots + e_l) = x\}$. A principal minor of L is then the determinant of this restriction. This is a modified version of the concept of a principal minor in Euclidean Jordan algebra, see [9].

Given a Euclidean Jordan algebra V, a symmetric cone K in V, a linear transformation $L : V \to V$ and $q \in V$, the linear complementarity problem LCP(L, q) is to find a vector $x \in V$ such that

$$x \in K, \quad y := L(x) + q \in K, \text{ and } \langle x, y\rangle = 0.$$

If $K = R^n_+$, the problem is well understood and well-studied, see [2]. A linear complementarity problem is a special case of variational inequality problem which appears in many applications, see [3].

One of the unsolved problems in the linear complementarity theory is the characterization of global uniqueness of solutions: find a necessary and sufficient condition on L so that for all $q \in V$, LCP(L, q) has a unique solution. This is related to the question of global invertibility of the normal map $x \mapsto L(x^+) + x - x^+$ on V. When K is polyhedral, there is a well-known result of Robinson [13] that describes the invertibility of the normal map in terms of the determinants of a certain collection of matrices. In the case when $V = R^n$, says that for a real square matrix M, LCP(M, q) has a unique solution for all $q \in R^n$ iff M is a **P**-matrix. Murthy et al. [12] proved that M is a **P**-matrix iff the LCP(M, q) has a solution for all $q \in V$ and the map $q \mapsto$ SOL(M, q) is Lipschitzian. Gowda and Sznajder [6] further generalized this result to affine variational inequality problems. However, when K is nonpolyhedral, we do not have an analog of this result. In relation to this, Gowda et al. [9], proved the following result: if LCP(L, q) has a unique solution for all $q \in V$ and if $q \mapsto$ SOL(L, q) is Lipschitzian then L has the positive principal minor property. As SOL(L, q) is a single valued map in this case, the normal map is a Lipschitzian homeomorphism and by using techniques from nonsmooth analysis, Gowda et al. [9] proved that the determinant of the linear transformation L and its principal minors are positive. Motivated by the above result, we ask whether this result holds if SOL(L, q) has a solution for all $q \in V$ and if the map $q \mapsto$ SOL(L, q) is Lipschitzian. In this article, we prove this result. In the case of standard LCP, the continuity of the solution map is well-studied, see [2,5]. However when the cone is nonpolyhedral, only few results are known.

In Section 2, we present some background material. In Section 3, we obtain our main result and in the end, we make a note on Lyaounov-like transformations.

2. Preliminaries

2.1. The projection map

Consider a finite dimensional inner product space $(H, \langle\cdot, \cdot\rangle)$ and a closed convex cone K in H. This K induces a (partial) order on H:

$$x \leqslant y \Leftrightarrow y - x \in K.$$

We use the notation $x < y$ when $y - x \in \operatorname{int} K$ (if exists). Corresponding to K, let Π_K denote the metric projection onto K: For an $x \in H$, $x^* = \Pi_K(x)$ iff $x^* \in K$ and $\|x - x^*\| \leqslant \|x - y\|$ for all $y \in K$. Let $K^* := \{x : \langle x, y\rangle \geqslant 0\ \forall x, y \in K\}$ denote the dual cone of K. We then have the Moreau decomposition [9]: Any $x \in H$ can be written as

$$x = \Pi_K(x) - \Pi_{K^*}(-x) \quad \text{with } \langle \Pi_K(x), \Pi_{K^*}(-x)\rangle = 0.$$

Also, $x = x_1 - x_2$ with $x_1 \in K$, $x_2 \in K^*$ and $\langle x_1, x_2\rangle = 0$ if and only if $x_1 = \Pi_K(x)$ and $x_2 = \Pi_{K^*}(-x)$.

Definition 1. Suppose that K is a closed convex cone in H. Assume that K is self-dual, i.e., $K^* = K$. For any $x \in H$, we define the nonnegative part of x and the nonpositive part of x, by

$$x^+ = \Pi_K(x) \quad \text{and} \quad x^- = x^+ - x.$$

The following result is well-known:

Proposition 1. *Let K be a closed convex self-dual cone in H. Then for any element $x \in H$, $x = x^+ - x^-$, $x^+, x^- \geqslant 0$, and $\langle x^+, x^-\rangle = 0$. This decomposition is unique.*

2.2. *Euclidean Jordan algebras*

In this section, we recall some basic concepts of Euclidean Jordan algebras. We refer to Faraut and Korányi [4] for details. A Euclidean Jordan algebra is a triple $(V, \circ, \langle\cdot,\cdot\rangle)$ where $(V, \langle\cdot,\cdot\rangle)$ is a finite dimensional real Hilbert space and $(x, y) \mapsto x \circ y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(a) $x \circ y = y \circ x$ for all $x, y \in V$,
(b) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in V$ where $x^2 := x \circ x$, and
(c) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all $x, y, z \in V$.

In addition, we assume that there is an element $e \in V$ such that $x \circ e = x$ for all $x \in V$. The element e is called the *unit* element in V. Henceforth, we assume that V is a Euclidean Jordan algebra with the unit element and call $x \circ y$ the Jordan product of x and y. In V, the set of squares

$$K = \{x \circ x : x \in V\}$$

is a *symmetric* cone. This means that K is a self-dual closed convex cone and for any two elements $x, y \in \operatorname{int} K$, there exists an invertible linear transformation $\Gamma : V \to V$ such that $\Gamma(K) = K$ and $\Gamma(x) = y$.

For an element $z \in V$, we write

$$z \geqslant 0 \Leftrightarrow z \in K,$$

and $z \leqslant 0$ when $-z \geqslant 0$. We write $z > 0$ if $z \in \operatorname{int} K$.

For $x \in V$, we define

$$m(x) := \min\{k > 0 : \{e, x, x^2, \ldots, x^k\} \text{ is linearly independent}\}$$

and rank of V by $r = \max\{m(x) : x \in V\}$ (by x^2 we mean $x \circ x$).

An element $c \in V$ is an *idempotent* if $c^2 = c$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, \dots, e_m\}$ of primitive idempotents in V is a *Jordan frame* if

$$e_i \circ e_j = 0 \quad \text{if } i \neq j \quad \text{and} \quad e_1 + \cdots + e_m = e.$$

It is easy to note that $\langle e_i, e_j \rangle = \langle e_i \circ e_j, e \rangle = 0$ whenever $i \neq j$.

Theorem 1. *Let V be a Euclidean Jordan algebra with rank r. Then for every $x \in V$, there exists a Jordan frame $\{e_1, \dots, e_r\}$ and real numbers $\lambda_1, \dots, \lambda_r$ such that*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{1}$$

The numbers λ_i are called the eigenvalues of x.

The expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the spectral decomposition of x. Given (1), it can be verified that

$$x^+ = \sum_{i=1}^{r} \lambda_i^+ e_i \quad \text{and} \quad x^- = \sum_{i=1}^{r} \lambda_i^- e_i.$$

It is easy to show that if $x \geqslant 0$, then every eigenvalue of x is nonnegative. We say that an element x is invertible, if every eigenvalue of x is nonzero.

Example 1. Let S^n be the set of all $n \times n$ real symmetric matrices with the inner and Jordan product given by

$$\langle X, Y \rangle := \mathrm{trace}(XY) \quad \text{and} \quad X \circ Y = \frac{1}{2}(XY + YX).$$

In this setting, the cone of squares S^n_+ is the set of all positive semidefinite matrices in S^n. The identity matrix is the unit element. The set $\{E_1, \dots, E_n\}$ is a Jordan frame in S^n where E_i is the diagonal matrix with 1 in the (i, i)-slot and zeros elsewhere. The rank of S^n is n. Given any $X \in S^n$, there exists an orthogonal matrix U with columns $u_1, u_2, \dots, u_n$ and a real diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ such that $X = UDU^{\mathrm{T}}$. Clearly, $X = \sum_{i=1}^{n} \lambda_i u_i u_i^{\mathrm{T}}$.

Example 2. Consider R^n $(n > 1)$, where any element x can be written as

$$x = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$$

with $x_0 \in R$ and $\bar{x} \in R^n - 1$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \bar{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by L^n. In this algebra, the cone of squares, denoted by L^n_+, is called the Lorentz cone (or the second order cone). It is given by

$$L^n_+ = \{x : \|\bar{x}\| \leqslant x_0\}.$$

Definition 2. Let $x \in V$. We define the corresponding *Lyapunov transformation* $L_x : V \to V$ by

$$L_x(z) = x \circ z.$$

Definition 3. We say that x and y operator commute if L_x and L_y commute, i.e.,

$$L_x L_y = L_y L_x.$$

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame [4].

We recall the following from Gowda et al. [9]

Proposition 2. *For $x, y \in V$ the following are equivalent*:

(a) $x \geqslant 0,\ y \geqslant 0$, *and* $\langle x, y\rangle = 0$, *and*
(b) $x \geqslant 0,\ y \geqslant 0$, *and* $x \circ y = 0$.

2.3. Simple Jordan algebras and the structure theorem

A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. The classification theorem [3] says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(a) The algebra S^n of $n \times n$ real symmetric matrices.
(b) The algebra L^n (Example 2).
(c) The algebra H^n of all $n \times n$ complex Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(d) The algebra Q_n of all $n \times n$ quaternion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.
(e) The algebra O_3 of all 3×3 octonion Hermitian matrices with trace inner product and $X \circ Y = \frac{1}{2}(XY + YX)$.

The following result characterizes all Euclidean Jordan algebras.

Theorem 2 (Faraut and Korányi [3]). *Any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Eulcidean Jordan algebras.*

2.4. The Peirce decomposition

Fix a Jordan frame $\{e_1, \ldots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, \ldots, r\}$, define the eigenspaces

$$V_{ii} = \{x \in V : x \circ e_i = x\} = Re_i$$

and when $i \neq j$,

$$V_{ij} := \left\{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\right\}.$$

Theorem 3 (Faraut and Korányi [4]). *The space V is the orthogonal direct sum of spaces V_{ij} $(i \leqslant j)$. Furthermore,*

$$V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj},$$
$$V_{ij} \circ V_{jk} \subset V_{ik} \quad \text{if } i \neq k,$$
$$V_{ij} \circ V_{kl} = \{0\} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset.$$

Thus, given any Jordan frame $\{e_1, \ldots, e_r\}$, we can write any element $x \in V$ as

$$x = \sum_{i=1}^{r} x_i + \sum_{i<j} x_{ij},$$

where $x_i \in Re_i$ and $x_{ij} \in V_{ij}$.

2.5. *Principal subtransformations and principal minors*

Given a Jordan frame $\{e_1, \ldots, e_r\}$ in V, we define

$$V^{(l)} := V(e_1 + \cdots + e_l, 1) = \{x \in V : x \circ (e_1 + \cdots + e_l) = x\}$$

for all $1 \leqslant l \leqslant r$. Corresponding to $V^{(l)}$, we consider the (orthogonal) projection $P^{(l)} : V \to V^{(l)}$. For a given linear transformation $L : V \to V$, the transformation $P^{(l)}L : V^{(l)} \to V^{(l)}$ is a *principal subtransformation* of L corresponding to $\{e_1, \ldots, e_l\}$, and is denoted by $L_{\{e_1,\ldots,e_l\}}$.

We call the determinant of $L_{\{e_1,\ldots,e_l\}}$ a principal minor of L. If all the principal minors of L are positive, then we say that L has *positive principal minor property*, see [9]. This is a modified version of the concept of principal minor of an element in a Euclidean Jordan algebra. Note that for a given Jordan frame $\{e_1, \ldots, e_r\}$ we can permute the objects and select the first l objects (for any $1 \leqslant l \leqslant r$). Thus there are $2^r - 1$ principal subtransformations (minors) corresponding to a Jordan frame. For examples and illustration of this concept, we refer to [9].

2.6. *Automorphisms*

A linear transformation $\Lambda : V \to V$ is said to be an automorphism of V if Λ is invertible and $\Lambda(x) \circ \Lambda(y) = \Lambda(x \circ y)$ for all $x, y \in V$. The set of all automorphisms is denoted by $\mathrm{Aut}(V)$.

A linear transformation $\Gamma : V \to V$ is said to be an automorphism of K if $\Gamma(K) = K$. We denote the set of all automorphisms of K by Aut(K).

2.7. *Definition*

Let $L : V \to V$ be a linear transformation and $q \in V$. The linear complementarity problem $\mathrm{LCP}(L, q)$ is: Find $x \in V$ such that

$$x \in K, \quad y := L(x) + q \in K \quad \text{and} \quad \langle x, y\rangle = 0.$$

In view of Proposition 2, we can replace $\langle x, y\rangle = 0$ by $x \circ y = 0$. This condition is called *complementarity condition*. We denote the set of all solutions of $\mathrm{LCP}(L, q)$ by $\mathrm{SOL}(L, q)$. We say that L has

1. The $\mathbf{R}_0$-property if $\mathrm{SOL}(L, 0) = \{0\}$.
2. The $\mathbf{Q}$-property if $\mathrm{SOL}(L, q) \neq \emptyset$ for all $q \in V$.

3. The Lipschitzian property if the map $q \mapsto \mathrm{SOL}(L, q)$ satisfies the following condition: There exists a constant $C > 0$ such that

$$\mathrm{SOL}(L, q) \subseteq \mathrm{SOL}(L, q') + C\|q - q'\|B \tag{2}$$

for all $q, q' \in V$ satisfying $\mathrm{SOL}(L, q) \neq \emptyset$ and $\mathrm{SOL}(L, q') \neq \emptyset$. Here B is the closed unit ball in V.
4. The positive principal minor property if every principal minor of L is positive.
5. The strong monotonicity property if $\langle v, L(v)\rangle > 0$ for all nonzero $v \in V$.
6. The Lipschitzian $\mathbf{Q}$-property if L has the Lipschitzian and $\mathbf{Q}$-property.

For $G \subseteq V$, we denote by $\mathrm{int}\, G$, the topological interior of the set G in V. The boundary of G is denoted by ∂G.

The following result is well-known, see [9].

Proposition 3. *Let $L : V \to V$ be linear. Then*

1. *L has the $\mathbf{R}_0$-property if and only if $\mathrm{SOL}(L, q)$ is a compact set for all $q \in V$.*
2. *If $\mathrm{SOL}(L, e) = \{0\}$ and if L has the $\mathbf{R}_0$-property, then L has the $\mathbf{Q}$-property.*
3. *If L has the strong monotonicity property, then L has the Lipschitzian property.*

3. Main results

We now prove our main result.

Theorem 4. *If L has the Lipschitzian $\mathbf{Q}$-property, then L has the positive principal minor property.*

The proof of the above theorem is based on the following lemmas. The following lemma is proved for the Euclidean Jordan algebra S^n (Example 5), see Lemma 2.4 [1]. We now generalize this result.

Lemma 1. *If L has the Lipschitzian $\mathbf{Q}$-property, then L is invertible.*

Proof. Suppose $L^{\mathrm{T}}(x) = 0$ for some nonzero $x \in V$. Let $q := -L(e)$ and q_m be a sequence such that $q_m \to q$ and for each m, $\langle q_m, x\rangle \neq 0$. Since L has the Lipschitzian $\mathbf{Q}$-property, there exists $x_m \in \mathrm{SOL}(L, q_m)$ such that

$$e \in x_m + C\|q_m - q\|B$$

for all m. This means that $x_m > 0$ for large m. Let $y_m := L(x_m) + q_m$. Now from the complementarity condition $x_m \circ y_m = 0$, it follows that, for large m, $y_m = 0$ and hence $\langle y_m, x\rangle = 0$. This implies that $\langle x, q_m\rangle = 0$ which is a contradiction. Therefore, L must be invertible. □

Lemma 2. *Let $\{e_1, \ldots, e_r\}$ be a Jordan frame in V and for $x \in V$, let $x = \sum_{i=1}^{r} x_i + \sum_{i<j} x_{ij}$ be the Peirce decomposition of x. Then we have the following*:

(a) *If $x \geqslant 0$, then $P^{(l)}(x) \in K^{(l)}$.*
(b) *If $x > 0$, then $P^{(l)}(x) \in \mathrm{int}\, K^{(l)}$.*

(c) *If* $x \geqslant 0$ *and* $x_k = 0$ *for some index* k, *then* $\sum_{k<j} x_{kj} + \sum_{i<k} x_{ik} = 0$.
(d) *Let* $\{x_n\}$ *be a convergent sequence in* K. *Suppose* $x_n := \sum_{i=1}^{r} x_i^n + \sum_{i<j} x_{ij}^n$ *be the peirce decomposition of* x_n. *If* $x_k^n \to 0$ *for some index* k, *then* $y^n := \sum_{k<j} x_{kj}^n + \sum_{i<k} x_{ik}^n \to 0$.

Proof. The proof of (a), (b) and (c) are given in [7]. Item (d) easily follows from (c) and the fact that K is a closed set. □

Lemma 3. *If* L *has the Lipschitzian* **Q**-*property*, *then every principal subtransformation of* L *with respect to any Jordan frame has the* **Q**-*property*.

Proof. Fix a Jordan frame $\{e_1, \ldots, e_r\}$ in V. Now consider the principal subtransformation corresponding to $\{e_1, \ldots, e_{r-1}\}$. Let $l = r - 1$ and $T := L_{\{e_1,\ldots,e_l\}}$. Suppose $q \in V^{(l)}$. We claim that $\mathrm{LCP}(T, q)$ has a solution. Let $q_k = q + ke_r$ and $q_k' = ke_r$. Since $q_k' \geqslant 0$, $0 \in \mathrm{SOL}(L, q_k')$. Since L has the Lipschitzian **Q**-property, there exists $x_k \in \mathrm{SOL}(L, q_k)$ such that

$$0 \in x_k + C\|q_k' - q_k\|B.$$

From the above inclusion, $\|x_k\| \leqslant C\|q\|$. This means that the sequence $\{x_k\}$ is bounded. Without any loss of generality, let $x_k \to x^*$.

We now write the Peirce decomposition of x_k with respect to the Jordan frame $\{e_1, \ldots, e_r\}$. Let

$$x_k = \sum_{i=1}^{r} x_i^k + \sum_{i<j} x_{ij}^k,$$

where $x_i^k \in Re_i$ and $x_{ij}^k \in V_{ij}$. Let the Peirce decomposition of $L(x_k)$ be

$$L(x_k) = \sum_{i=1}^{r} y_i^k + \sum_{i<j} y_{ij}^k,$$

where $y_i^k \in Re_i$ and $y_{ij}^k \in V_{ij}$. Since x_k is bounded, x_i^k, x_{ij}^k, y_i^k and y_{ij}^k must be bounded. From the complementarity condition $\langle x_k, L(x_k) + q_k\rangle = 0$ and the orthogonality of V_{ij}, it follows that,

$$kx_r^k + p_k = 0$$

for some bounded sequence $\{p_k\}$ in V. Hence $x_r^k \to 0$. Therefore from item (d) of the previous lemma $\sum_{r<j} x_{rj}^k + \sum_{i<r} x_{ir}^k \to 0$. Thus $x^* \in K^{(l)}$.

Let $z_k := L(x_k) + q_k$. By a direct calculation,

$$z_k \circ (e_1 + \cdots + e_l) = P^{(l)}(z_k) = \sum_{i=1}^{l} y_i^k + \sum_{i<j\leqslant l} y_{ij}^k + q, \tag{3}$$

and therefore $P^{(l)}(z_k) \to T(x^*) + q$. Note that $z_k \geqslant 0$ and hence by the previous lemma $P^{(l)}(z_k) \in K^{(l)}$. This means that $T(x^*) + q \in K^{(l)}$.

Since $x_k \circ z_k = 0$, $\langle x_k, z_k \circ (e_1 + \cdots e_l)\rangle = 0$. Therefore, from Eq. (3), $\langle x_k, P^{(l)}(z_k)\rangle = 0$. Now $x_k \to x^*$ and $P^{(l)}(z_k) \to T(x^*) + q$. This implies that $\langle x^*, T(x^*) + q\rangle = 0$. Thus, $x^* \in \mathrm{SOL}(T, q)$. It is easy to note that if $l < r - 1$, then the same argument can be repeated. This completes the proof. □

Lemma 4. *If* L *has the Lipschitzian* **Q**-*property*, *then every principal subtransformation of* L *with respect to any Jordan frame has the Lipschitzian property*.

Proof. Let $\{e_1, \dots, e_r\}$ be a Jordan frame in V. Suppose $l = r - 1$ and $T = L_{\{e_1,\dots,e_l\}}$. We claim that T has the Lipschitzian property. By the above lemma, T has the **Q**-property. Let $p, q \in V^{(l)}$ and $y \in \mathrm{SOL}(T, p)$. Because L has the Lipschitzian property, there exists $C > 0$ satisfying relation (2). We claim that there exists $x^* \in \mathrm{SOL}(T, q)$ such that $\|y - x^*\| \leqslant C\|p - q\|$.

By Peirce decomposition,

$$L(y) = \sum_{i=1}^{r} z_i + \sum_{i<j} z_{ij} \quad \text{and} \quad p = \sum_{i=1}^{r} p_i + \sum_{i<j} p_{ij}.$$

Now it is easy to verify that $T(y) = \sum_{i=1}^{l} z_i + \sum_{i<j\leqslant l} z_{ij}$. We now set

$$q'_k := p - \sum_{i=1}^{r-1} z_{ir} + ke_r.$$

Now $v_k := L(y) + q'_k = T(y) + p + ke_r + z_r$. As z_r is a scalar multiple of e_r, for all large k, $ke_r + z_r \geqslant 0$. Now $T(y) + p \geqslant 0$ and hence for all large k, $v_k \geqslant 0$.

Since $y \in V^{(l)}$, $y \circ (z_r + ke_r) = 0$. From the complementarity condition we have $y \circ (T(y) + p) = 0$. Therefore, $y \circ v_k = 0$. Thus $y \in \mathrm{SOL}(L, q'_k)$ for all large k.

Let $q^*_k = q - \sum_{i=1}^{r-1} z_{ir} + ke_r$. By the Lipschitzian property of L, there exists $x_k \in \mathrm{SOL}(L, q^*_k)$ such that

$$y \in x_k + C\|q'_k - q^*_k\|B. \tag{4}$$

Let the Peirce decomposition of x_k and $L(x_k)$ be

$$x_k = \sum_{i=1}^{r} x_i^k + \sum_{i<j} x_{ij}^k \quad \text{and} \quad L(x_k) = \sum_{i=1}^{r} y_i^k + \sum_{i<j} y_{ij}^k.$$

We note from (4) that x_k must be bounded as $\|q'_k - q^*_k\| = \|p - q\|$. Without any loss of generality, assume that x_k is convergent. From the complementarity condition $\langle x_k, L(x_k) + q^*_k\rangle = 0$, we see that $x_r^k \to 0$. Since $x_k \geqslant 0$, by Lemma 2, it follows that $\sum_{r<j} x_{rj}^k + \sum_{i<r} x_{ir}^k \to 0$. Let $x_k \to x^*$. By using the same argument as in the above lemma, we conclude that $x^* \in \mathrm{SOL}(T, q)$. Now applying the limits in (4), we see that

$$\|y - x^*\| \leqslant C\|p - q\|.$$

Therefore T has the Lipschitzian property. The same argument can be repeated if $l < r - 1$. This completes the proof. □

The next lemma is known for the standard LCP (see Lemma 3 in [5]). By using the same technique in [5], we derive the next lemma.

Lemma 5. *Suppose that L has the Lipschitzian property and* $\mathrm{SOL}(L, q') = \{0\}$ *for some* $q' > 0$. *Then* $\mathrm{SOL}(L, q) = \{0\}$ *for all* $q \geqslant 0$.

Proof. Let $\Omega := \{q > 0 : \mathrm{SOL}(L, q) = \{0\}\}$ and $p \in \Omega$. We claim that Ω is open in $\operatorname{int} K$. Let $u \in B$. By the Lipschitzian property of L,

$$\mathrm{SOL}(L, np + u) \subseteq \mathrm{SOL}(L, np) + C\|u\|B.$$

Let $x_n \in \mathrm{SOL}(L, np + u)$. As $\mathrm{SOL}(L, np) = \{0\}$, $\{x_n\}$ must be bounded. Hence $y_n := L(x_n) + np + u > 0$ for all large n and therefore by the complementarity condition $x_n \circ y_n = 0$, $x_n = 0$

for all large n. Thus $np + u \in \Omega$ for all large n. This means that p is an interior point of Ω. Since p is arbitrary, Ω is an open set in int K.

We now show that Ω is closed in int K. Let $q^* > 0$ be a limit point of Ω. Now there exists a sequence $q_n \in \Omega$ such that $q_n \to q^*$. Since L has the Lipschitzian property,

$$\mathrm{SOL}(L, q^*) \subseteq \mathrm{SOL}(L, q_n) + C\|q_n - q^*\|B.$$

This means that $\mathrm{SOL}(L, q^*) = \{0\}$ and therefore $q^* \in \Omega$. Thus Ω is closed in int K. Because Ω is closed, open and nonempty in int K and int K is connected, $\Omega = \mathrm{int}\, K$.

Let $q \in \partial K$. Then there exists a sequence $\{q_n\}$ converging to q such that $q_n \in \mathrm{int}\, K$ for all n. By the Lipschitzian property of L,

$$\mathrm{SOL}(L, q) \subseteq \mathrm{SOL}(L, q_n) + C\|q - q_n\|B.$$

Therefore if $x \in \mathrm{SOL}(L, q)$, then $\|x\| \leqslant C\|q - q_n\|$ and hence $x = 0$. This completes the proof. □

Lemma 6. *If L has the Lipschitzian* **Q**-*property*, *then* $\mathrm{SOL}(L, q') = \{0\}$ *for some* $q' > 0$.

Proof. We first assume that V is a Euclidean Jordan algebra of rank 2 and prove the result. Since L has the **Q**-property, there exists $x^* > 0$ such that $L(x^*) > 0$. Let $q := L(x^*)$. Let y be a nonzero solution to $\mathrm{LCP}(L, q)$. Supppose $y > 0$. Then by the complementarity condition, we have $L(y) + q = 0$. By Lemma 1, $y = -x^*$. This is a contradiction. Let $y = \alpha e_1$ be the spectral decomposition of y. Then $z := L(y) + q = \beta e_2$. We now put $l = 1$ and $T := L_{\{e_1\}}$. Observe that $P^{(l)}(z) = 0$. Therefore $T(y) + P^{(l)}(q) = 0$. By Lemma 2, $P^{(l)}(q) > 0$ in $V^{(l)}$. Now T is a linear transformation on a one-dimensional space and hence there exists $k > 0$ such that $T(x) = -kx$ for all $x \in V^{(l)}$. But Lemma 3 implies that T has the **Q**-property which means that $T(x) = cx$ for some $c > 0$. This is a contradiction and hence $y = 0$.

Now suppose V is of rank r. We assume that the result is true for any Euclidean Jordan algebra with rank less than r. Since L has the **Q**-property, there exists $x^* > 0$ such that $L(x^*) > 0$. Let $q := L(x^*)$ and $y \in \mathrm{SOL}(L, q)$. If $y > 0$, then by Lemma 1, $y = -x^*$ which is not possible. Let us assume that y has the spectral decomposition $y = \lambda_1 e_1 + \cdots + \lambda_s e_s$ for some $s < r$. Put $l = s$ and $T = L_{\{e_1,\ldots,e_l\}}$. Now it is easy to see that $y \in \mathrm{SOL}(T, P^{(l)}(q))$. Using Lemma 2, $P^{(l)}(q) > 0$. By Lemmas 3 and 4, T has the Lipschitzian **Q**-property. Therefore by our assumption there exists $q'' \in \mathrm{int}\, K^{(l)}$ such that $\mathrm{SOL}(T, q'') = \{0\}$. In view of Lemma 5, $\mathrm{SOL}(T, q) = \{0\}$ for all $q \in \mathrm{int}\, K^{(l)}$. Therefore $y = 0$. This concludes the proof. □

Lemma 7. *If* $\mathrm{SOL}(L, q) = \{0\}$ *for all* $q \geqslant 0$, *then L satisfies the following condition*:

$$x \geqslant 0, x \text{ and } L(x) \text{ operator commute}, \text{ and } x \circ L(x) \leqslant 0 \Rightarrow x = 0.$$

Proof. Since x and $L(x)$ operator commute, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ such that

$$x = \sum_{i=1}^{r} \alpha_i e_i \quad \text{and} \quad L(x) = \sum_{i=1}^{r} \beta_i e_i.$$

As $x \circ L(x) \leqslant 0$, $\alpha_i\beta_i \leqslant 0$ for all $i \in \{1, \ldots, r\}$. Let $q := L(x)^+ - L(x^+)$. Clearly $x^+ \in \mathrm{SOL}(L, q)$. As $x \geqslant 0$, $L(x^+) = L(x) = L(x)^+ - L(x)^-$ and so $q = L(x)^-$. Hence $q \geqslant 0$ and therefore, $x^+ = 0$ by our assumption. Now $x = x^+$ and thus $x = 0$. □

Lemma 8. *Suppose that L has the Lipschitzian property and the* **Q**-*property. Then there exists $q' < 0$ such that* $\mathrm{LCP}(L, q')$ *has a unique solution $x' > 0$.*

Proof. Since L has the **Q**-property, there exists $x > 0$ such that $L(x) > 0$. Let $\Gamma \in \mathrm{Aut(K)}$. Then $\widehat{L}_\Gamma = \Gamma L \Gamma^{\mathrm{T}}$ has the Lipschitzian **Q**-property iff L has the Lipschitzian **Q**-property, see Theorem 5.1 in [7]. Therefore $\widehat{L}_\Gamma(e) > 0$ for some $\Gamma \in \mathrm{Aut(K)}$. Without any loss of generality, let $L(e) > 0$. Now, put $q = -L(e)$. We claim that $\mathrm{LCP}(L, q)$ has a unique solution.

Let x be a solution to $\mathrm{LCP}(L, q)$. Now $x \in \mathrm{SOL}(L^{-1}, -L^{-1}(q))$ if and only if $y := L^{-1}(x) - L^{-1}(q) \in \mathrm{SOL}(L, q)$. Therefore L has the Lipschitzian **Q**-property if and only if L^{-1} has the Lipschitzian **Q**-property. Put $A := L^{-1}$.

By Proposition 6 in [9], x and $L(x - e)$ operator commute. Let the spectral decomposition of x and $L(x - e)$ be

$$x = \sum_{i=1}^{s} \mu_i e_i \quad \text{and} \quad L(x - e) = \sum_{i=s+1}^{r} \mu_i e_i.$$

Put $T := A_{\{e_{s+1},\ldots,e_r\}}$. By Lemmas 4 and 3, T has the Lipschitzian **Q**-property. Let $y := \sum_{i=s+1}^{r} \mu_i e_i$. Then $A(y) = x - e$ and therefore $T(y) = \sum_{i=s+1}^{r} -e_i$. Observe that by Lemmas 6 and 5, $\mathrm{SOL}(T, q) = \{0\}$ for every $q \geqslant 0$ and hence T satisfies all the conditions of Lemma 7. Now y and $T(y)$ operator commute. Further $y \geqslant 0$ and $y \circ T(y) \leqslant 0$. Therefore $y = 0$ by the previous lemma. This means that $L(x - e) = 0$ and hence by Lemma 1, $x = e$. This completes the proof. □

In the next lemma, we use degree theory. The usage of degree theory in complementarity problems is fairly standard; see, for example [3]. Given a bounded open set Ω in V, a continuous function $f : \mathrm{cl}\,\Omega \to V$ such that $0 \notin f(\partial\Omega)$, we can define the (topological) degree of f with respect to Ω at 0; see [11]. We denote this degree by $\deg(f, \Omega, 0)$. Recall that the *normal map* $\phi : V \to V$ is defined by $\phi(v) := L(v^+) - v^-$, see [13]. It is easy to verify that if $\phi(v) = 0$, then $v^+ \in \mathrm{SOL}(L, q)$.

Lemma 9. *Let L satisfy the following conditions*:

1. *L has the* $\mathbf{R}_0$-*property*;
2. *For some* $q' < 0$, $\mathrm{LCP}(L, q')$ *has unique solution* $x' > 0$; *and*
3. $\mathrm{SOL}(L, e) = \{0\}$.

Then $\det L \geqslant 0$.

Proof. Suppose $\det L < 0$. Let $\phi : V \to V$ be the normal map defined by $\phi(v) := L(v^+) - v^-$. If $x > 0$, then it is easy to see that $\phi(x) = L(x)$. By our assumption, there exists $q' < 0$ such that, $\mathrm{LCP}(L, q')$ has a unique solution $x' > 0$ so that $L(x') + q' = 0$. If $F_1 = \phi + q'$, then $F_1(x') = 0$. We claim that x' is the unique zero of F_1. Suppose $F_1(w) = 0$ so that $w^+ \in \mathrm{SOL}(L, q')$. Then $w^+ = x'$. This gives $L(w^+) - w^- + q' = 0$ and $w^- = L(x') + q' = 0$. Hence $w = x'$. Let G be a bounded open set in $\mathrm{int}\,K$ containing x'. On G, $F_1 = L + q'$ so that $\deg(F_1, G, 0) = \mathrm{sgn}\det L = -1$. By the excision property of the degree, it follows that for any open set Δ_1 containing x', $\deg(F_1, \Delta_1, 0) = -1$.

Let $F_2 = \phi + e$. Since $\mathrm{LCP}(L, e)$ has a unique solution, $-e$ is the unique zero of F_2. We note that ϕ is the identity transformation on $\mathrm{int}(-K)$ and now, as before, for any bounded open set Δ_2 containing $-e$, $\deg(F_2, \Delta_2, 0) = 1$. Let $t \in [0, 1]$. Now $H(t, v) = \phi(v) + tq' + (1-t)e$ is a homotopy between F_1 and F_2. Let $S := \{v \in V : H(t, v) = 0 \text{ for some } t \in [0, 1]\}$. Using the $\mathbf{R}_0$-property of L, it is easy to show that S is bounded. Let Ω be any bounded open set containing S. Clearly $-e \in \Omega$ and $x' \in \Omega$. In view of the homotopy invariance property of the degree, $\deg(F_1, \Omega, 0) = \deg(F_2, \Omega, 0)$. This is a contradiction. This completes the proof. □

We now prove our main result:

Theorem 5. *If L has the Lipschitzian* **Q***-property, then L has the positive principal minor property.*

Proof. From Lemma 6, $\mathrm{SOL}(L, q') = \{0\}$ for some $q' > 0$. By the Lipschitzian property of L,

$$\mathrm{SOL}(L, 0) \subseteq \mathrm{SOL}(L, q') + C\|q'\|B$$

and hence $\mathrm{SOL}(L, 0)$ is compact. This means that L has the $\mathbf{R}_0$-property. Now L satisfies all the conditions of the previous lemma. Therefore $\det L \geqslant 0$. In view of Lemma 1, L is invertible and hence $\det L > 0$.

Now consider any principal subtransformation T with respect to a Jordan frame. From Lemmas 3 and 4, T has the Lipschitzian **Q**-property and thus $\det T > 0$. This completes the proof. □

3.1. A note on Lyapunov-like transformations

For some particular linear transformations, using the extra structure, we can obtain specialized results. We say that $L : V \to V$ has the Z-property if

$$x, y \in K, \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leqslant 0.$$

The recent article [10] contains examples and properties of such transformations.

Example 3. Let $V = S^n$. For $A \in R^{n\times n}$, consider the Lyapunov transformation $L_A : S^n \to S^n$ defined by $L_A(X) := AX + XA^{\mathrm{T}}$. It is easy to verify that L_A has the Z-property. Similarly it can be verified easily that the Stein transformation $S_A(X) := X - AXA^{\mathrm{T}}$ has the Z-property.

We say that a linear transformation L is a Lyapunov-like transformation if L and $-L$ have the Z-property, that is,

$$x, y \in K, \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0. \tag{5}$$

For details on Lyapunov-like transformations we refer to [8]. We now prove the following result:

Theorem 6. *Let $L : V \to V$ be a Lyapunov-like transformation. Then the following are equivalent:*

1. *L has the strong monotonicity property.*
2. *L has the Lipschitzian* **Q***-property.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 2.3.1 in [3]. Suppose L has the Lipschitzian **Q**-property. Let c be a primitive idempotent. We claim that $\langle c, L(c)\rangle > 0$. Consider a Jordan frame containing c. Then the one-dimensional linear transformation $L_{\{c\}}$ has the **Q**-property. This means that $\langle L_{\{c\}}(c), c\rangle > 0$. But $L_{\{c\}}(c)(c) = L(c)$ and hence $\langle L(c), c\rangle > 0$. For any x we have the spectral decomposition $x = \sum_{i=1}^{r} \lambda_i e_i$. Since L is a Lyapunov-like transformation, $\langle L(e_i), e_j\rangle = 0$ if $i \neq j$. Therefore $\langle L(x), x\rangle = \sum_{i=1}^{s} \lambda_i^2 \langle L(e_i), e_i\rangle > 0$ for all nonzero $x \in V$. Thus, L has the strong monotonicity property. $\square$

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