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On a conjecture on the balanced decomposition number

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ABSTRACT

The concept of balanced decomposition number was introduced by Fujita and Nakamigawa in connection with a simultaneous transfer problem. A *balanced colouring* of a graph G is a pair (R, B) of disjoint subsets $R, B \subseteq V(G)$ with $|R| = |B|$. A *balanced decomposition* D of a balanced colouring $C = (R, B)$ of G is a partition of vertices $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ such that $G[V_i]$ is connected and $|V_i \cap R| = |V_i \cap B|$ for $1 \le i \le r$. Let $\mathcal{C}$ be the set of all balanced colourings of G and $\mathcal{D}(C)$ be the set of all balanced decompositions of G for $C \in \mathcal{C}$. Then the *balanced decomposition number* $f(G)$ of G is

$$f(G) = \max_{C\in\mathcal{C}} \min_{D\in\mathcal{D}(C)} \max_{1\le i\le r} |V_i|.$$

Fujita and Nakamigawa conjectured that if G is a 2-vertex connected graph of n vertices, then $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$. In this paper, we confirm this conjecture in the affirmative.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We use $V(G)$ to denote the vertex set of a graph G and $E(G)$, the edge set. The concept of balanced decomposition number was introduced by Fujita and Nakamigawa [9] in connection with a simultaneous transfer problem. A *balanced colouring* C of a graph G is a pair (R, B) of disjoint subsets $R, B \subseteq V(G)$ with $|R| = |B|$. For any vertex subset $S \subseteq V$, the *subgraph induced by* S is the graph $G[S]$ with vertex set S and edge set $E[S] = \{xy \in E : x, y \in S\}$. A *balanced decomposition* D of a balanced colouring $C = (R, B)$ of G is a partition of vertices $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ such that $G[V_i]$ is connected and *balanced*, i.e. $|V_i \cap R| = |V_i \cap B|$, for $1 \le i \le r$. The *size* of the decomposition is $\mathrm{size}(D) = \max_{1\le i\le r} |V_i|$. The *balanced decomposition number* $f(G)$ of a graph G is then defined as

$$f(G) = \max_{C\in\mathcal{C}} \min_{D\in\mathcal{D}(C)} \mathrm{size}(D).$$

In other words, $f(G)$ is the smallest positive integer such that no matter what balanced colouring is given, there is always a balanced decomposition of G into vertex sets of size at most $f(G)$. As a balanced decomposition may not exist for a disconnected graph, we only consider balanced decomposition numbers for connected graphs. For a non-negative integer k, let $f(k, G)$ be defined analogously with the additional restriction that we consider only balanced colourings (R, B) of G where $|R| = k$. It is then the case that $f(G) = \max_k f(k, G)$.

There are a few applications for the balanced decomposition. Theorem 6 in [9] tells us that $f(G) - 1$ is a sharp upper bound for the simultaneous transfer number. The balanced decomposition is also deeply connected to the decomposition of k-linked graphs. For further details and comments, refer to [9].

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Fujita and Nakamigawa [9] proved that $f(G) = 2$ if and only if G is a complete graph of at least two vertices. They also established that $f(T) = n$ for a tree T of n vertices, $f(K_{m,n}) = \lfloor \frac{n-2}{m} \rfloor + 3$ for a complete bipartite graph $K_{m,n}$ with $2 \le m \le n$, and $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$ for an n-cycle C_n. Using this support, they then gave an interesting conjecture which is the main concern of this paper.

Conjecture 1 (*Fujita and Nakamigawa [9]*). *If G is a 2-connected graph of n vertices, then $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$.*

While it is easy to see that $f(1, G) \le \lfloor \frac{n}{2} \rfloor + 1$ for any 2-connected graph of $n \ge 2$ vertices, they in fact proved that $f(2, G) \le \lfloor \frac{n}{2} \rfloor + 1$ for any 2-connected graph of $n \ge 4$ vertices. The conjecture was then confirmed in [8] for generalized θ-graphs, and in [6] for subdivisions of K_4 and serial–parallel graphs. It was also proved in [7] that $f(3, G) \le \lfloor \frac{n}{2} \rfloor + 1$ for any 2-connected graph of $n \ge 6$ vertices.

In the present paper, we prove this conjecture as the following theorem.

Theorem 1. *If G is a 2-connected graph of n vertices, then $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$.*

2. Preliminaries

The *deletion* of a proper subset $S \subset V(G)$ from G is the graph $G - S = G[V(G) \setminus S]$. For a vertex v, we use $G - v$ for $G - \{v\}$. A *cut-vertex* in G is a vertex v such that $G - v$ has more components than G. A graph of at least three vertices is *2-connected* if it is connected and does not contain any cut-vertex. A 2-connected graph is *minimally 2-connected* if $G - e$ is not 2-connected for any edge e. A *block* in G is a maximal connected subgraph without a cut-vertex. An *end-block* is a block containing at most one cut-vertex of G. For a graph G, the *block–cut-vertex structure* is the graph G^* whose vertex set $V(G^*)$ contains all cut-vertices and blocks of G and a cut-vertex v of G is adjacent to a block A of G in the graph G^* whenever $v \in V(A)$. Notice that G^* is always a forest, and in fact a tree if G is connected. A vertex of degree at most 1 in G^* is precisely an end-block in G. We refer the reader to the books [4,12] for more terms relating to graphs.

The following property and its proof technique for 2-connected graphs are useful in our proof of the main theorem.

Lemma 2. *If u and v are two distinct vertices in a 2-connected graph G, then there is an ordering $u = x_1, x_2, \ldots, x_n = v$ of $V(G)$ such that the graphs $G_i = G[x_1, x_2, \ldots, x_i]$ and $G'_i = G - V(G_i)$ are connected for $1 \le i \le n - 1$.*

Proof. We shall construct the ordering by adding the vertices one by one. Initially, choose $x_1 = u$. Then G_1 and G'_1 are connected since G is 2-connected. Assume that $i \ge 2$ and $x_1, x_2, \ldots, x_{i-1}$ are chosen such that G_{i-1} and G'_{i-1} are connected, where v is in G'_{i-1}. By the fact that G is 2-connected, every end-block of G'_{i-1} has a non-cut-vertex adjacent to some vertex in G_{i-1}. Choose such a vertex x_i, which can be assumed to be different from v in the case where G'_{i-1} has at least two vertices. Hence both G_i and G'_i are connected. Continue this process until the ordering is complete. □

The following lemma follows easily from the definition.

Lemma 3. *If H is a connected spanning subgraph of G, then $f(G) \le f(H)$.*

According to Lemma 3, in order to prove an upper bound on $f(G)$ for 2-connected graphs, we only need to consider minimally 2-connected graphs. The properties of minimally 2-connected graphs were studied independently by Dirac [5] and Plummer [11]. A summary of these results appears in [1]. Recent works on acyclic edge colouring [10], strong edge colouring [2] and k-intersection edge colouring [3] make use of these properties and show the potential that this class of graphs has for giving insight into improve colouring bounds of various edge colouring problems.

We now consider some useful properties for minimally 2-connected graphs.

Lemma 4 (*Plummer [11]*). *A 2-connected graph is minimally 2-connected if and only if no cycle in the graph contains a chord.*

The following lemma serves as the induction basis for the proof of the main theorem.

Lemma 5. *If G is a minimally 2-connected graph, then $G - \{u, v\}$ is connected for any edge uv.*

Proof. Suppose to the contrary that $G - \{u, v\}$ is not connected. Since G is 2-connected and $u \ne v$, u and v have neighbours u_i and v_i respectively in every component G_i of $G - \{u, v\}$, for otherwise removal of a single vertex will disconnect G. For component G_i, choose a shortest u_i-v_i path P_i. As there are at least two components G_1 and G_2, we have that u, P_1, v, P_2^{-1}, u is a cycle for which uv is a chord, a contradiction to the assumption that G is minimally 2-connected and Lemma 4. □

3. Proof of the main theorem

We are now ready to prove the main theorem.

By Lemma 3, we may assume that G is minimally 2-connected. Consider any balanced colouring (R, B) of G. We first claim that G has a balanced connected induced subgraph H such that $H' = G - V(H)$ is also connected and balanced. To see this, we consider two cases. For the case where G has some vertex $v \notin R \cup B$, the induced subgraphs $H = G[\{v\}]$ and $H' = G - v$

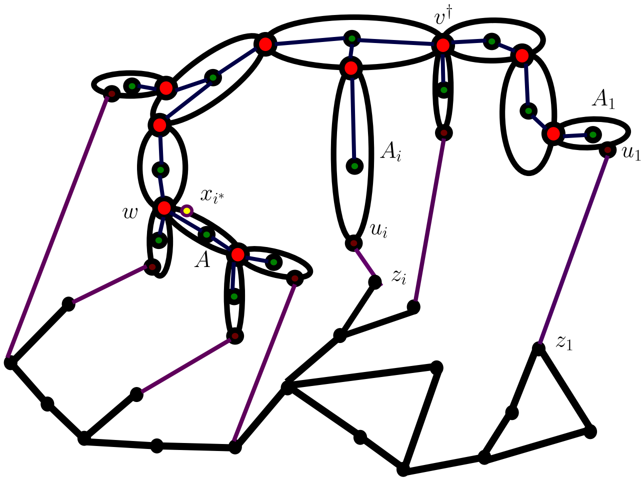

Fig. 1. A representation of H, H' and H'^*.

satisfy the desired properties, since G is 2-connected. For the case where $R \cup B = V(G)$, there is a vertex $u \in R$ adjacent to a vertex $v \in B$. Then $H = G[\{u, v\}]$ and $H' = G - \{u, v\}$ satisfy the desired properties by Lemma 5. We may assume that H is chosen such that $|V(H)| \le \lfloor \frac{n}{2} \rfloor + 1$ and $|V(H)|$ is as large as possible under this condition. It is then the case that $|V(H)| \le \lfloor \frac{n}{2} \rfloor - 1$, for otherwise $\lfloor \frac{n}{2} \rfloor \le |V(H)| \le \lfloor \frac{n}{2} \rfloor + 1$ and so $|V(H')| \le \lfloor \frac{n}{2} \rfloor + 1$ implies the theorem.

If H' is 2-connected, then we may use induction to get a balanced decomposition of H' in which each part has size at most $\lfloor \frac{n'}{2} \rfloor + 1 \le \lfloor \frac{n}{2} \rfloor + 1$. This together with H gives a desired balanced decomposition for G. Hence H' is not 2-connected. Let $\mathfrak{C}$ be the set of all cut-vertices of H' and $\mathfrak{B}$ the set of all blocks of H'. Consider the block–cut-vertex tree H'^* of H' with vertex set $\mathfrak{C} \cup \mathfrak{B}$ and edge set $\{vA : v \in \mathfrak{C}, A \in \mathfrak{B}, v \in A\}$ as a tree rooted at some $v^\dagger \in \mathfrak{C}$. Notice that each vertex $v \in \mathfrak{C}$ has degree at least 2 in H'^*. Further, H'^* has $r \ge 2$ leaves $A_1, A_2, \ldots, A_r$ which are end-blocks of H'. Since G is 2-connected, each A_i contains a non-cut-vertex u_i adjacent to some vertex z_i in H; see Fig. 1.

First, each $u_i \in R \cup B$, for otherwise $H + u_i = G[V(H) \cup \{u_i\}]$ and $(H + u_i)' = H' - u_i$ are balanced connected induced subgraphs of G with $|V(H)| < |V(H + u_i)| \le \lfloor \frac{n}{2} \rfloor$, a contradiction to the choice of H. Secondly, $u_1, u_2, \ldots, u_r$ are all in R or all in B. Otherwise, if some $u_i \in R$ and some $u_j \in B$, then $H + \{u_i, u_j\} = G[V(H) \cup \{u_i, u_j\}]$ and $(H + \{u_i, u_j\})' = H' - \{u_i, u_j\}$ are balanced connected induced subgraphs of G with $|V(H)| < |V(H + \{u_i, u_j\})| \le \lfloor \frac{n}{2} \rfloor + 1$, again a contradiction to the choice of H. We therefore assume that $u_1, u_2, \ldots, u_r \in R$.

For a vertex $v \in \mathfrak{C}$, consider the subtree rooted at v, namely $H_v'^*$ from the tree H'^*. Notice that since H'^* is a rooted tree (rooted at $v^\dagger$), this defines a unique subtree of the block–cut-vertex tree of H'. Let H_v' be the subgraph of H' induced by $\cup\{V(A): A \in \mathfrak{B} \cap V(H_v'^*)\}$. For technical reasons, when $v \in V(H')$ is not a cut-vertex of H', we define H_v' as the graph containing the single vertex v. It is then the case that, for any vertex w in H', the graphs H_w' and $H' - V(H_w')$ are connected induced subgraphs of H'. For any induced subgraph F of H', define $c(F) = |V(F) \cap R| - |V(F) \cap B|$. Then $c(H_{u_i}') = 1$ for $1 \le i \le r$ and $c(H_{v^\dagger}') = 0$.

Since $c(H_{v^\dagger}') = 0$, we may choose a cut-vertex $w \in \mathfrak{C}$ farthest from $v^\dagger$ in the tree H'^* (if it is not unique, pick one arbitrarily) such that $c(H_w') \le 0$. Consider any child A of w in H'^*, which is a block of H'. For the case where A is an end-block A_j, choose $u = u_j$; when A is not an end-block, choose u as some cut-vertex different from w. By Lemma 2, there is an ordering $u = x_1, x_2, \ldots, x_p = w$ of $V(A)$ such that $A_i = A[x_1, x_2, \ldots, x_i]$ and $A_i' = A - V(A_i)$ are connected for $1 \le i \le p-1$. Let H_{A_i}' be the subgraph of H' induced by $\cup_{1 \le j \le i} V(H_{x_j}')$ for $1 \le i \le p-1$. Then H_{A_i}' and $(H_{A_i}')' = H' - V(H_{A_i}')$ are connected with $c(H_{A_i}') = \sum_{1 \le j \le i} c(H_{x_j}')$ for $1 \le i \le p-1$.

If $c(H_{A_i}') \le 0$ for some $2 \le i \le p-1$, then choose a minimum such index i^*. Thus $c(H_{A_{i^*-1}}') > 0 \ge c(H_{A_{i^*}}') = c(H_{A_{i^*-1}}') + c(H_{x_{i^*}}')$ and so $c(H_{x_{i^*}}') < 0$. By the choice of w (being the farthest from $v^\dagger$), the vertex x_{i^*} is not a cut-vertex of H'. Hence $c(H_{x_{i^*}}') = -1$ and so $c(H_{A_{i^*}}') = 0$. This gives two balanced connected induced subgraphs $H_{A_{i^*}}'$ and $(H_{A_{i^*}}')'$ of H'. If both of them are of size at most $\lfloor \frac{n}{2} \rfloor + 1$, then these two subgraphs together with H form a balanced decomposition of G, which implies that $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$. If one of them is of size greater than $\lfloor \frac{n}{2} \rfloor + 1$, then the other one together with H forms a balanced induced subgraph of size at most $\lfloor \frac{n}{2} \rfloor - 1$ but greater than the size of H, a contradiction to the choice of H.

Hence we can assume that $c(H_{A_{p-1}}') > 0$ for any child A of w. Then

$$0 \ge c(H_w') = |R \cap \{w\}| - |B \cap \{w\}| + \sum_{A:A \text{ is a child of } w,\ p=|A|} c(H_{A_{p-1}}').$$

This is possible only when $c(H_w') = 0$, $w \in B$ and w has exactly one child A (since the first difference is at best -1 and each term in the sum is at least $+1$). Thus $w \ne v^\dagger$ and so either we get a contradiction (inductively, from the fact that

$c(H'_{v^\dagger}) = 0$) or H'_w and $H' - H'_w$ are two balanced connected induced subgraphs of H'. The same argument as before leads to $f(G) \le \lfloor \frac{n}{2} \rfloor + 1$. $\square$

4. Conclusion

For positive integers k and n, define $g(k, n) = \max\{f(G): G$ is a k-connected graph of n vertices$\}$. By the fact that $f(T) = n$ for a tree T of n vertices [9], $g(1, n) = n$. By the fact that $f(C_n) = \lfloor \frac{n}{2} \rfloor + 1$ for an n-cycle C_n and the main theorem in this paper, $g(2, n) = \lfloor \frac{n}{2} \rfloor + 1$. It is clear that $g(n-1, n) = 2$. By the following two results, it is the case that $g(k, n) = 3$ for $\lfloor \frac{n}{2} \rfloor \le k \le n-2$.

$f(G) = 2$ if and only if G is a complete graph of at least two vertices [9].

$f(G) = 3$ if and only if $G \ne K_n$ is $\lfloor \frac{n}{2} \rfloor$-connected and has n vertices [8].

It is interesting to note that for a graph on n-vertices with connectivity k, all the above results satisfy $g(k, n) = \lceil \frac{n+k-1}{k} \rceil$. We close the paper by proposing the following conjecture on $g(k, n)$.

Conjecture 2. *If $n \ge k+1$, then $g(k, n) = \lceil \frac{n+k-1}{k} \rceil$. Equivalently, $f(G) \le \lceil \frac{n+k-1}{k} \rceil$ for any k-connected graph on $n \ge k+1$ vertices, and there exists some k-connected graph G on n-vertices such that $f(G) = \lceil \frac{n+k-1}{k} \rceil$.*

Notice that powers of cycles provide extremal graphs in the conjecture above.

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