ODD STATISTICS IN ODD DIMENSIONS
FOR ODD COUPLINGS[*]

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We consider the response of a uniformly accelerated monopole detector that is coupled *non-linearly* to the nth power of a quantum scalar field in $(D + 1)$-dimensional flat spacetime. We show that, when $(D + 1)$ is even, the response of the detector in the Minkowski vacuum is characterized by a Bose-Einstein factor for all n. Whereas, when $(D + 1)$ is odd, we find that a Fermi-Dirac factor appears in the detector response when n is odd, but a Bose-Einstein factor arises when n is even. We emphasize the point that, since, along the accelerated trajectory, the Wightman function and, as a result, the $(2n)$-point function satisfy the Kubo-Martin-Schwinger condition (as required for a scalar field) in *all* dimensions, the appearance of a Fermi-Dirac factor (instead of the expected Bose-Einstein distribution) for odd $(D + 1)$ *and* n reflects a peculiar feature of the detector rather than imply a fundamental change in field theory.

1. Introduction

It has been a quarter of a century now since it was discovered that the response of a uniformly accelerated monopole detector that is coupled to a quantized massless scalar field is characterized by a Planckian distribution when the field is assumed to be in the Minkowski vacuum[2,3]. However, about a decade after the original discovery, it was noticed that this result is true only in even-dimensional flat spacetimes and it was pointed out that a Fermi-Dirac factor (rather than a Bose-Einstein factor) appears in the response of the accelerated detector when the dimensionality of spacetime is odd (see Refs.[4,5,6,7,8,9]; for relatively recent discussions, see Refs.[10,11]). The detector due to Unruh[2] and DeWitt[3] is coupled *linearly* to the quantum scalar field. During the last decade or so, motivated by different reasons, there has been an occasional interest in literature in studying the response of detectors that are coupled *non-linearly* to the quantum field[12,13,14,15,16]. It will be interesting to examine whether the non-linearity of the coupling affects the result in odd-dimensional flat spacetimes that we mentioned above.

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[*]This title is motivated by the title of an unpublished paper by C. R. Stephens[1].
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In this note, we shall consider the response of a uniformly accelerated monopole detector that is coupled to the nth power (with n being a positive integer) of a quantum scalar field in $(D + 1)$-dimensional flat spacetime. As we shall see, the non-linearity of the detector's coupling affects the afore-mentioned result in odd spacetime dimensions in an interesting fashion. We shall show that, when $(D+1)$ is even, a Bose-Einstein factor arises in the response of the detector for all n, whereas, when $(D + 1)$ is odd, a Fermi-Dirac factor appears in the detector response when n is odd, but a Bose-Einstein factor arises when n is even.

We shall adopt units such that $\hbar = c = 1$ and we shall denote the set of $(D + 1)$ coordinates x^μ as $\tilde{x}$.

2. The Non-linearly Coupled Detector

Consider a monopole detector that is moving along a trajectory $\tilde{x}(\tau)$, where τ is the proper time in the frame of the detector. Let the detector interact with a real scalar field Φ through the non-linear interaction Lagrangian[15]

$$\mathcal{L}_{\text{int}} = \bar{c}\, m(\tau)\, \Phi^n\, [\tilde{x}(\tau)]\,, \tag{1}$$

where $\bar{c}$ is a small coupling constant, $m(\tau)$ is the detector's monopole moment and n is a positive integer that denotes the index of non-linearity of the coupling. Let us now assume that the quantum field $\hat{\Phi}$ is initially in the vacuum state $|0\rangle$ and the detector is in its ground state $|E_0\rangle$ corresponding to an energy eigen value E_0. Then, up to the first order in perturbation theory, the amplitude of transition of the non-linearly coupled detector to an excited state $|E\rangle$, corresponding to an energy eigen value $E\ (> E_0)$, is described by the integral (see, for e.g., Ref.[17])

$$\mathcal{A}_n(\mathcal{E}) = \mathcal{M} \int_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\, \langle\Psi|\, \hat{\Phi}^n[\tilde{x}(\tau)]\, |0\rangle\,, \tag{2}$$

where $\mathcal{M} \equiv (i\bar{c}\,\langle E|\,\hat{m}(0)\,|E_0\rangle)$, $\mathcal{E} = (E - E_0) > 0$ and $|\Psi\rangle$ is the state of the quantum scalar field after its interaction with the detector. (Since the quantity $\mathcal{M}$ depends only on the internal structure of the detector and does not depend on its motion, we shall drop this quantity hereafter.)

The transition amplitude $\mathcal{A}_n(\mathcal{E})$ above involves products of the quantum field $\hat{\Phi}$ at the *same* spacetime point and, hence, we will encounter divergences when evaluating this transition amplitude. In order to avoid the divergences, we shall normal order the operators in the matrix element in the transition amplitude $\mathcal{A}_n(\mathcal{E})$ with respect to the Minkowski vacuum[15]. That is, we shall assume that the transition amplitude (2) above is instead given by the expression

$$\bar{\mathcal{A}}_n(\mathcal{E}) = \int_{-\infty}^{\infty} d\tau\, e^{i\mathcal{E}\tau}\, \langle\Psi|\, :\hat{\Phi}^n[\tilde{x}(\tau)]:\, |0\rangle\,, \tag{3}$$

where the colons denote normal ordering with respect to the Minkowski vacuum. Then, the transition probability of the detector to all possible final states $|\Psi\rangle$ of the quantum field is given by

$$\mathcal{P}_n(\mathcal{E}) = \sum_{|\Psi\rangle} |\bar{\mathcal{A}}_n(\mathcal{E})|^2 = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \, e^{-i\mathcal{E}(\tau-\tau')} \, G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right], \qquad (4)$$

where $G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]$ is the $(2n)$-point function defined as

$$G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right] = \langle 0| : \hat{\Phi}^n\left[\tilde{x}(\tau)\right] :: \hat{\Phi}^n\left[\tilde{x}(\tau')\right] : |0\rangle. \qquad (5)$$

In cases wherein the $(2n)$-point function $G^{(n)}\left(\tau, \tau'\right) \left(\equiv G^{(n)}\left[\tilde{x}(\tau), \tilde{x}(\tau')\right]\right)$ is invariant under time translations in frame of the detector, we can define a transition probability rate for the detector as follows:

$$\mathcal{R}_n(\mathcal{E}) = \int_{-\infty}^{\infty} d\bar{\tau} \, e^{-i\mathcal{E}\bar{\tau}} \, G^{(n)}(\bar{\tau}), \qquad (6)$$

where $\bar{\tau} = (\tau - \tau')$.

3. Odd Statistics in Odd dimensions for Odd Couplings

Let us now assume that the quantum scalar field $\hat{\Phi}$ is in the Minkowski vacuum. In such a case, the $(2n)$-point function $G^{(n)}\left(\tilde{x}, \tilde{x}'\right)$ simplifies to

$$G_{\mathrm{M}}^{(n)}\left(\tilde{x}, \tilde{x}'\right) = (n!) \left[G_{\mathrm{M}}^{+}\left(\tilde{x}, \tilde{x}'\right)\right]^n, \qquad (7)$$

where $G_{\mathrm{M}}^{+}\left(\tilde{x}, \tilde{x}'\right)$ denotes the Wightman function in the Minkowski vacuum[a]. In $(D+1)$ spacetime dimensions (and for $(D+1) \geq 3$), the Wightman function for a massless scalar field in the Minkowski vacuum is given by (see, for instance, Refs.[5,8,9])

$$G_{\mathrm{M}}^{+}(\tilde{x}, \tilde{x}') = \mathcal{C}_D \left[(-1)\left((t-t'-i\epsilon)^2 - |\mathbf{x}-\mathbf{x}'|^2\right)\right]^{-(D-1)/2}, \qquad (8)$$

where $\epsilon \to 0^+$, $\left[t, \mathbf{x} \equiv \left(x^1, x^2, \ldots, x^D\right)\right]$ are the Minkowski coordinates and the quantity $\mathcal{C}_D$ is given by

$$\mathcal{C}_D = \left(4\pi^{(D+1)/2}\right)^{-1} \Gamma\left[(D-1)/2\right] \qquad (9)$$

with $\Gamma\left[(D-1)/2\right]$ denoting the Gamma function.

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[a]It ought to be noted here that we would have arrived at the expression (7) for the $(2n)$-point function in the Minkowski vacuum even if we had started with the transition amplitude (2) (instead of the normal ordered amplitude (3)), rewritten the resulting $(2n)$-point function in the transition probability in terms of the two-point functions using Wick's theorem and then replaced the divergent terms that arise (i.e. those two-point functions with coincident points) with the corresponding regularized expressions (for a discussion on this point, also see Ref.[15]).

Now, consider a detector accelerating uniformly along the x^1 direction with a proper acceleration g. The trajectory of such a detector is given by (see, for e.g., Ref.[17])

$$t(\tau) = g^{-1}\sinh(g\tau) \ , \ \ x^1(\tau) = g^{-1}\cosh(g\tau) \ , \ \ x^2 = x^3 = \ldots = x^D = 0, \quad (10)$$

where τ is the proper time in the frame of the detector. On substituting this trajectory in the Minkowski Wightman function (8), we obtain that (see, for instance, Refs.[5,8,9])

$$G_{\mathrm{M}}^{+}(\bar{\tau}) = \left[\mathcal{C}_D \ (g/2i)^{(D-1)}\right] \ \left(\sinh\left[(g\bar{\tau}/2) - i\epsilon\right]\right)^{-(D-1)}. \quad (11)$$

Therefore, along the trajectory of the uniformly accelerated detector, the $(2n)$-point function in the Minkowski vacuum (7) is given by

$$G_{\mathrm{M}}^{(n)}(\bar{\tau}) = (n!) \ [\mathcal{C}_D^n \ (g/2i)^{\alpha}] \ \left(\sinh\left[(g\bar{\tau}/2) - i\epsilon\right]\right)^{-\alpha}, \quad (12)$$

where $\alpha = [(D-1)n]$.

On substituting the $(2n)$-point function (12) in the expression (6) and carrying out the resulting integral, we find that the transition probability rate of the uniformly accelerated, non-linearly coupled detector can be written as (cf. Ref.[18], p. 305, Eq. 3.314; p. 950, Eq. 8.384.1; p. 937, Eqs. 8.331, 8.332.1 and 8.332.2)

$$\mathcal{R}_n(\mathcal{E}) = \mathcal{B}(n,D) \begin{cases} (g^{\alpha}/\mathcal{E}) \ \underbrace{[\exp(2\pi\mathcal{E}/g) - 1]^{-1}}_{\text{Bose-Einstein factor}} \ \prod_{l=0}^{(\alpha-2)/2} \left[l^2 + (\mathcal{E}/g)^2\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } \alpha \text{ is even} \\ g^{(\alpha-1)} \ \underbrace{[\exp(2\pi\mathcal{E}/g) + 1]^{-1}}_{\text{Fermi-Dirac factor}} \ \prod_{l=0}^{(\alpha-3)/2} \left[((2l+1)/2)^2 + (\mathcal{E}/g)^2\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } \alpha \text{ is odd,} \end{cases} \quad (13)$$

where the quantity $\mathcal{B}(n,D)$ is given by

$$\mathcal{B}(n,D) = (2\pi)\,(n!)\ [\mathcal{C}_D^n/\Gamma(\alpha)]. \quad (14)$$

When $(D+1)$ is even, α is even for all n and, hence, a Bose-Einstein factor will always arise in the response of the uniformly accelerated detector in an even-dimensional flat spacetime. Whereas, when $(D+1)$ is odd, evidently, α will be odd or even depending on whether n is odd or even. Therefore, in an odd-dimensional flat spacetime, a Fermi-Dirac factor will arise in the detector response when n is odd (as in the case of the Unruh-DeWitt detector), but a Bose-Einstein factor will appear when n is even!

A few remarks regarding this curious result are in order. Firstly, the temperature associated with the Bose-Einstein and the Fermi-Dirac factors that appear in the response of the non-linearly coupled detector is the standard Unruh temperature,

viz. $(g/2\pi)$. Secondly, the response of the detector is characterized *completely* by either a Bose-Einstein or a Fermi-Dirac distribution *only* in situations wherein $\alpha < 3$. For cases such that $\alpha \geq 3$, the detector response contains, in addition to a Bose-Einstein or a Fermi-Dirac factor, a term which is polynomial in $(\mathcal{E}/g)$. Thirdly, in Figs. 1 and 2, following Unruh[8], we have plotted the transition probability rate of the detector (in fact, the quantity $\bar{\mathcal{R}}_n(\mathcal{E}) = [\mathcal{R}_n(\mathcal{E})/\mathcal{R}_n(0)]$) for a few different values of D for the case $n = 3$ and for a few different values of n for the case wherein $(D + 1) = 3$, respectively. (In plotting these figures, we have set $g = (2\pi)$.) It is
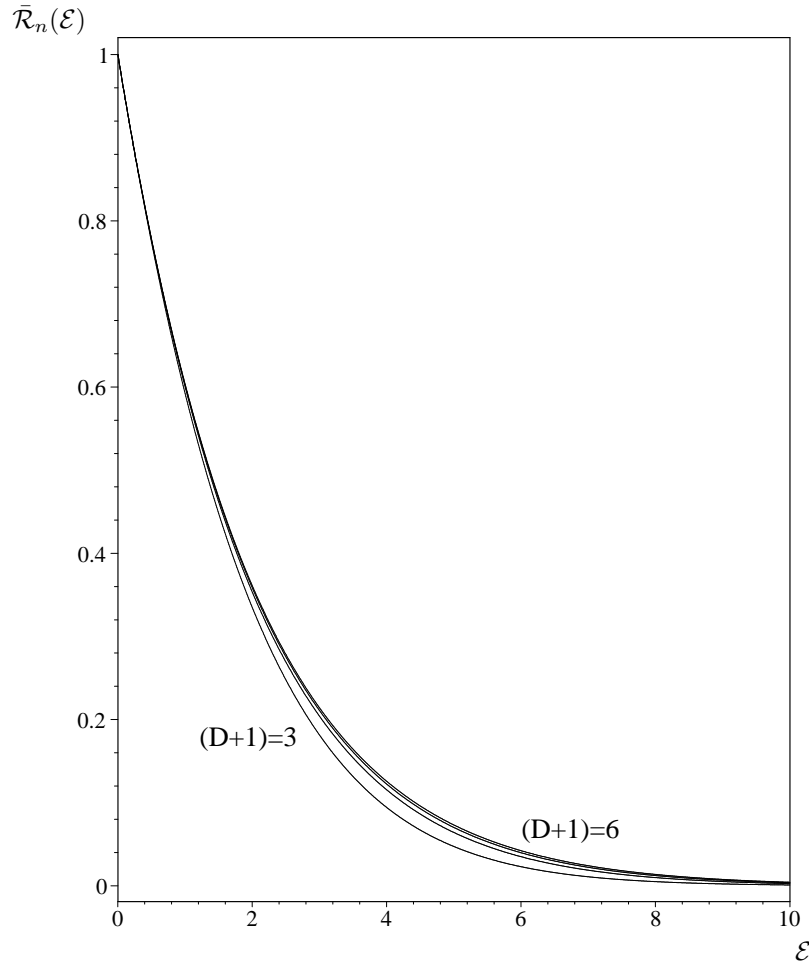

Fig. 1. $\bar{\mathcal{R}}_n(\mathcal{E})$ vs. $\mathcal{E}$ for $n = 3$ and $(D + 1) = 3, 4, 5, 6$.

interesting to note from these two figures that, though the characteristic response of the detector alternates between the Bose-Einstein and the Fermi-Dirac factors as
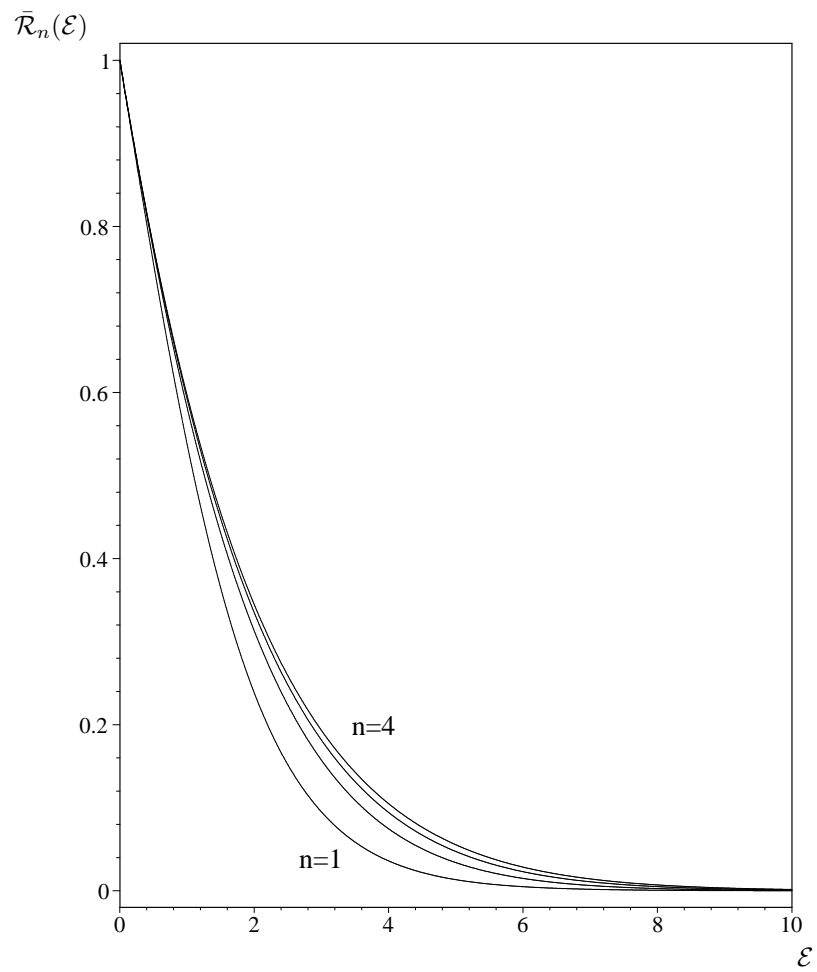
Fig. 2. $\bar{\mathcal{R}}_n(\mathcal{E})$ vs. $\mathcal{E}$ for $(D+1) = 3$ and $n = 1, 2, 3, 4$.

we go from one D to another for odd n (or from one n to another when $(D+1)$ is odd), the complete spectra themselves exhibit a smooth dependence both on the index of non-linearity of the coupling and the dimension of spacetime.

4. *Apparent* nature of the odd statistics

The point that needs to be emphasized regarding the result we have obtained above is the *apparent* nature of the "inversion of statistics" that arises in odd dimensions for odd couplings. It is easy to see that, in the frame of the uniformly accelerated detector, the Wightman function in the Minkowski vacuum (11) is skew-periodic in imaginary proper time with a period corresponding to the inverse of the Unruh temperature, i.e.

$$G_{\mathrm{M}}^{+}(\bar{\tau}) = G_{\mathrm{M}}^{+}\left[-\bar{\tau} + i(2\pi/g)\right]. \tag{15}$$

Also, this property holds for all D. In other words, along the accelerated trajectory, the Wightman function satisfies the Kubo-Martin-Schwinger (KMS) condition as required for a Bosonic field in *all* dimensions (for a discussion on the KMS condition, see, for e.g., Ref.[9], Chaps. 4 and 5). Since the $(2n)$-point function in the Minkowski vacuum is proportional to the nth power of the Wightman function, it is then obvious that, in the frame of the accelerated detector, the $(2n)$-point function will also be skew-periodic in imaginary proper time, thereby satisfying the KMS condition (as required for a Bosonic field) for *all* D and n (cf. Eq. (12)). Had the appearance of the Fermi-Dirac factor for odd $(D+1)$ and n been the manifestation of a truly fundamental change in the field theory, then, in such situations, the $(2n)$-point function along the accelerated trajectory would have been skew *as well as* anti-periodic in imaginary proper time as expected of a Fermionic field (see, for instance, Ref.[7]). The fact that this does not occur then clearly implies that the appearance of the Fermi-Dirac factor (instead of the expected Bose-Einstein factor) for odd $(D+1)$ and n just reflects a peculiar feature of the detector rather than suggest a fundamental shift in the field theory.

5. Outlook

The appearance of a Fermi-Dirac factor (instead of the expected Planckian distribution) is known to occur in the response of a comoving Unruh-DeWitt detector (that is coupled to a massless scalar field) in odd-dimensional de Sitter spacetimes (see, for e.g., Refs.[7,19]; for a recent discussion, see Ref.[20]) and also in the case of a detector stationed at a constant radius in the spacetime of the $(2+1)$ dimensional Banados-Teitelboim-Zanelli (BTZ) black hole[21]. Moreover, it has recently been pointed out that, not only comoving, but even accelerated Unruh-DeWitt detectors in de Sitter spacetime as well as accelerated detectors (with proper acceleration beyond a certain critical value) in anti-de Sitter spacetime exhibit a thermal response[22,23,24,25]. It will be interesting to investigate as to how the non-linearity of the detector's coupling would affect the response of a static detector around the BTZ black hole and also the response of comoving as well as accelerated detectors

in de Sitter and anti-de Sitter spacetimes in different dimensions. Furthermore, it has been shown that a similar "inversion of statistics" occurs in the response of a monopole detector that is coupled linearly to the scalar density of a massless Dirac field in odd-dimensional flat and de Sitter spacetimes[9,11,19,20] and also around the BTZ black hole[26], i.e. the response of the detector exhibits a Bose-Einstein factor instead of the Fermi-Dirac factor expected in such situations. It will be worthwhile to examine as to how detectors coupled non-linearly to the scalar density of spinor fields respond in odd-dimensional spacetimes. We plan to address these issues in some detail in a forthcoming publication[27].

Acknowledgments

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