NON-COMMUTATIVE CARATHÉODORY INTERPOLATION

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ABSTRACT. We prove a Carathéodory-Fejér type interpolation theorem for certain matrix convex sets in $\mathbb{C}^d$ using the Blecher-Ruan-Sinclair characterization of abstract operator algebras. Our results generalize the work of Dmitry S. Kalyuzhnyĭ-Verbovetzkiĭ for the d-dimensional non-commutative polydisc.

1. INTRODUCTION

A classical interpolation problem in function theory is the Carathéodory-Fejér interpolation problem (CFP): Given $n+1$ complex numbers $c_0, c_1, ..., c_n$ does there exist a complex valued analytic function $f(z) = \sum_{j=0}^{\infty} f_j z^j$ defined on the open unit disc $\mathbb{D} \subset \mathbb{C}$ such that $f_j = c_j$ for all $0 \leq j \leq n$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$?

The problem and some of its variants were studied by Carathéodory, Fejér, Schur and Toeplitz in [Sc], [T] and [CF]. A necessary and sufficient condition, commonly referred to as the Schur criterion, for the existence of a solution to the problem is that the matrix

$$\begin{pmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_n & \cdots & c_1 & c_0 \end{pmatrix} \tag{1}$$

is a contraction. A detailed exposition of the CFP and its numerous function-theoretic and engineering applications can be found in the comprehensive book of Foias and Frazho [FF]. The operator theoretic formulation of Sarason [S] has had a major impact on the CFP and the related Pick interpolation problem and the development of opeator theory and the study of non-self adjoint operator algebras generally.

From the operator theory/algebra point of view, the CFP is essentially unchanged if the coefficients $c_0, c_1, ..., c_n$ are taken to be elements of $B(\mathcal{U})$ for some separable Hilbert space $\mathcal{U}$. Indeed, The Schur criterion in this case can be formulated in the following way. Let $p(z) = \sum_{j=0}^{n} c_j z^j : \mathbb{D} \to B(\mathcal{U})$ be

2000 *Mathematics Subject Classification.* 47A57, 47L30 (Primary). 47A13 (Secondary).
Key words and phrases. Interpolation, Carathéodory, Carathéodory-Fejér, Abstract operator algebra, BRS, Matrix convex set, Formal power series.

given. There exists an analytic function $f : \mathbb{D} \to B(\mathcal{U})$ such that $f^{(j)}(0) = c_j$ for $0 \leq j \leq n$ if and only if the norm of $p(T) = \sum_{j=0}^{n} c_j \otimes T^j$ is at most one for every operator T on Hilbert space which is nilpotent of order $n+1$; i.e., $T^{n+1} = 0$.

Several commutative multi-variable generalizations of the CFP have been obtained for different domains for example, the polydisc $\mathbb{D}^d \subset \mathbb{C}^d$, and for different interpolating classes of functions, for example the Schur-Agler class of analytic functions that take contractive operator values on any d-tuple of commuting strict contractions in a manner discussed in [A]. For more details see [EPP], [BLTT]. Some results on the problem for bounded circular domains in $\mathbb{C}^d$ can also be found in [D].

The broad purpose of this article is to extend some of the results in the commutative case to the noncommutative setting of the free algebra on d generators. Some non-commutative generalizations of the CFP have already been studied in [P2], [P3], [Co], [KV], [BGM]. Here we pose the problem for domains that are matrix convex sets (see [EW]) in $\mathbb{C}^d$. The domains considered here include as specific examples, the d-dimensional non-commutative matrix polydisc, which is the domain used in [KV], the d-dimensional non-commutative matrix polyball and the dd'-dimensional non-commutative matrix mixed ball. Using non-commutative matrix (operator) valued analytic functions on matrix-convex sets - formal power series with matrix (operator) coefficients that converge on some non-commutative neighborhood of 0 (see [V1], [V2], [V3], [P1], [P2], [P3], [KVV], [HKMS]) - and the Blecher-Ruan-Sinclair characterization of abstract operator algebras, we prove an interpolation theorem from which a necessary and sufficient condition for the existence of a minimum-norm solution to the CFP follows.

The article is structured as follows: In Section 2, matrix convex sets in $\mathbb{C}^d$ and the interpolating class $\mathcal{A}(\mathcal{K})^\infty$ are introduced and a basic version of the main result is stated. In Section 3, it is established that the interpolating class $\mathcal{A}(\mathcal{K})^\infty$ is an abstract operator algebra. In Section 4, a weak-compactness property of the algebra $\mathcal{A}(\mathcal{K})^\infty$ is proved. Section 5 begins with the definition of the ideal $\mathcal{I}(\mathcal{K}) \subset \mathcal{A}(\mathcal{K})^\infty$ which plays a role analogous to that of the ideal of analytic functions in H^∞ of the unit disc which vanish to order n at 0 in the classical CFP. This section also contains the discussion of why $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ is an abstract operator algebra. It is also shown in this section that the norms of classses in the quotient algebra are attained. In Section 6, completely contractive representations of the algebra $\mathcal{A}(\mathcal{K})^\infty$ are studied and it is also shown that finite-dimensional compressions of operators that give rise to completely contactive representations of $\mathcal{A}(\mathcal{K})^\infty$ lie on the boundary of the underlying matrix convex set. Section 7 contains the matrix and operator versions of the CFP for (certain) matrix-convex sets in $\mathbb{C}^d$ and the main interpolation theorem. The article ends with some examples for which a variant of the main interpolation theorem from Section 7 holds even in the case of infinite initial segments Λ. See Section 8.

2. Matrix Convex Sets in $\mathbb{C}^d$

A basic object of study in this article is a *quantized*, or *non-commutative*, version of a convex set. While the definitions easily extend to convex subsets of arbitrary vector spaces, here the focus is on subsets of $\mathbb{C}^d$. In this section we review the definition of a matrix convex subset of $\mathbb{C}^d$ and introduce our standard assumptions regarding these sets.

2.1. **Non-commutative sets.** Let $M_{m,n} = M_{m,n}(\mathbb{C})$ denote the $m \times n$ matrices over $\mathbb{C}$. In the case that $m = n$, we write M_n instead of $M_{n,n}$. Let $M_n(\mathbb{C}^d)$ denote d-tuples with entries from M_n. Thus, an $X \in M_n(\mathbb{C}^d)$ has the form $X = (X_1, \dots, X_d)$ where each $X_j \in M_n$. A *non-commutative set* $\mathcal{L}$ is a sequence $(\mathcal{L}(n))$ where, for each positive integer n, $\mathcal{L}(n) \subset M_n(\mathbb{C}^d)$ which is *closed with respect to direct sums*; i.e., if $X \in \mathcal{L}(n)$ and $Y \in \mathcal{L}(m)$, then

$$X \oplus Y = (X_1 \oplus Y_1, \dots, X_d \oplus Y_d) \in \mathcal{L}(n+m) \tag{2}$$

where

$$X_j \oplus Y_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}.$$

A non-commutative set $\mathcal{L} = (\mathcal{L}(n))$ is *open* if each $\mathcal{L}(n)$ is open.

2.2. **Convexity.** A *matrix convex set* $\mathcal{K} = (\mathcal{K}(n))$ is a non-commutative set which is *closed with respect to conjugation by an isometry*; i.e., if $\alpha \in M_{m,n}$ and $\alpha^*\alpha = I_n$, and if $X = (X_1, \dots, X_d) \in \mathcal{K}(m)$, then

$$\alpha^* X \alpha = (\alpha^* X_1 \alpha, \dots, \alpha^* X_d \alpha) \in \mathcal{K}(n). \tag{3}$$

Note that, by choosing $n = m$ and α a unitary matrix, the condition (3) implies that each $\mathcal{L}(n)$ is closed with respect to unitary conjugation.

It is a simple matter to combine conditions (2) and (3) to conclude, if $\mathcal{L}$ is matrix convex, then each $\mathcal{L}(n)$ is itself convex.

2.3. **Circled domains.** A subset $\mathcal{U}$ of $M_n(\mathbb{C}^d)$ is *circled* if $e^{i\theta}\mathcal{U} \subseteq \mathcal{U}$ for all $\theta \in \mathbb{R}$. A matrix convex set $\mathcal{K}$ is circled if each $\mathcal{K}(n)$ is circled. As a canonical example of a circled matrix convex set, suppose $\gamma > 0$ and consider the non-commutative γ-neighborhood $N_\gamma = (\mathcal{C}_\gamma(n))$ of $0 \in \mathbb{C}^d$ defined by

$$\mathcal{C}_\gamma(n) = \{X \in M_n(\mathbb{C}^d) : \sum_{j=1}^d X_j X_j^* < \gamma^2\}.$$

For a matrix convex set $\mathcal{K}$, unless otherwise noted, it is assumed there exist $\gamma, \Gamma > 0$ such that

$$N_\gamma \subseteq \mathcal{K} \subseteq N_\Gamma, \tag{4}$$

where the inclusions are interpreted termwise. Equivalently, $\mathcal{K}$ is bounded (contained in some non-commutative neighborhood of 0) and contains a non-commutative neighborhood of 0.

Assumption 1. *In this article, it is typically assumed that $\mathcal{K}$*

(a) *is open;*
(b) *is bounded;*
(c) *is circled;*
(d) *is matrix convex; and*
(e) *contains a non-commutative neighborhood of* 0.

2.4. **Examples of Matrix Convex Sets.** At this point we pause to consider some further examples of open, bounded matrix convex sets which contain a non-commutative neighborhood of 0.

Example 1. *Let* $\mathcal{K}(n) = \{(X_1, X_2, ..., X_d) : X_j \in M_n \ \& \ \|X_j\| < 1\}$ *with* $\gamma < \frac{1}{\sqrt{d}}$ *and* $\Gamma > \sqrt{d}$*.* $\mathcal{K} = (\mathcal{K}(n))$ *is the d-dimensional noncommutative matrix polydisc.*

Example 2. *Let* $\mathcal{K}(n) = \{X = (X_{1,1}, X_{1,2}, ..., X_{d,d'}) : X_{i,j} \in M_n \ \& \ \|X\|_{op} < 1\}$*, where* $\|X\|_{op}$ *is the norm of the operator* $(X_{ij})_{i,j=1}^{d,d'} : (\mathbb{C}^n)^{d'} \to (\mathbb{C}^n)^d$ *with* $\gamma < \frac{1}{\sqrt{dd'}}$ *and* $\Gamma > \sqrt{dd'}$ *is the* $d \times d'$ *non-commutative matrix mixed ball.*

2.5. **The Interpolating class $\mathcal{A}(\mathcal{K})^\infty$.** Let $\mathcal{K}$ denote a matrix convex set satisfying the conditions of Assumption 1. A central object of this article is an algebra of formal power series in non-commuting variables which converge uniformly on the matrix convex set $\mathcal{K}$. These power series are defined in terms of the free semi-group on d letters.

2.5.1. *The Free Semi-group on d Letters.* Let $\mathcal{F}_d$ denote the free semigroup of words generated by d symbols $g_1, ..., g_d$. The product on $\mathcal{F}_d$ is defined by concatenation. Thus, if $w = g_{i_1}...g_{i_m}$ and $w' = g_{j_1}...g_{j_n}$, then the product ww' is given by $g_{i_1}...g_{i_m}g_{j_1}...g_{j_n}$. The empty word $\emptyset$ acts as the identity so that $w\emptyset = w = \emptyset w$. The length of the word $w = g_{i_1}...g_{i_m}$ is m and is denoted $|w|$. The length of $\emptyset$ is zero.

2.5.2. *The Set $\mathcal{A}^\infty$ of Formal Power Series.* A *formal power series* with entries from $\mathbb{C}$ is an expression of the form

$$\sum_{w \in \mathcal{F}_d} f_w w \tag{5}$$

where $f_w \in \mathbb{C}$ (more general coefficients f_w will be considered later). It is convenient to sum f according to its homogeneous of degree j terms; i.e.,

$$f = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} f_j. \tag{6}$$

Give a d-tuple $T = (T_1, \dots, T_d)$ of operators on a common Hilbert space $\mathcal{H}$ and a word $w = g_{i_1}g_{i_2}...g_{i_k} \in \mathcal{F}_d$, $i_1, ..., i_k \in \{1, 2, ..., d\}$, define the evaluation of w at T by

$$T^w = T_{i_1}T_{i_2}...T_{i_k}.$$

Given a formal power series f as above, define

$$f(T) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w T^w \tag{7}$$

provided the sum converges in the operator norm in the indicated order. Convergence in norm is not terribly important here and it is possible to use instead convergence in the strong or weak operator topologies for instance, [BGM].

Fix now a matrix convex set $\mathcal{K} = (\mathcal{K}(n))$ which satisfies the conditions of Assumption 1. We will write $X \in \mathcal{K}$ to denote $X \in \bigcup_{n\in\mathbb{N}} \mathcal{K}(n)$. Let

$$\mathcal{A}(\mathcal{K})^{\infty} = \left\{ f = \sum_{w\in F_d} f_w w : \, f_w \in \mathbb{C}, \text{and for every } X \in \mathcal{K}, \; f(X) \; \text{ converges} \right\}.$$

For $f \in \mathcal{A}(\mathcal{K})^{\infty}$, define

$$\|f\| = \sup\{\|f(X)\| \, : \, X \in \mathcal{K}\}.$$

Of course, as it stands, this supremum can be infinite. Let

$$\mathcal{A}(\mathcal{K})^{\infty} = \left\{ f = \sum_{w\in F_d} f_w w : f \in \mathcal{A}(\mathcal{K})^{\infty}, \|f\| < \infty \right\}.$$

Thus, elements of $\mathcal{A}(\mathcal{K})^{\infty}$ are in some sense analogous to H^{∞} functions on the unit disk $\mathbb{D}$. It is not hard to see that $\|\cdot\|$ is in fact a norm on $\mathcal{A}(\mathcal{K})^{\infty}$ and not just a semi-norm.

It will be shown that $\mathcal{A}(\mathcal{K})^{\infty}$ is an algebra and it will also be necessary - and desirable - to consider formal power series with matrix and operator-valued coefficients. Discsussion of these topics is postponed until after stating a base version of the main result of this paper.

2.6. **The Main Result.** A set $\Lambda \subset \mathcal{F}_d$ is an *initial segment* if its complement is an ideal in the semi-group $\mathcal{F}_d$; i.e., if both $g_j w, w g_j \in \mathcal{F}_d \setminus \Lambda$ $(1 \le j \le d)$, whenever $w \in \mathcal{F}_d \setminus \Lambda$. In the case that $d = 1$ an initial segment is thus a set of the form $\{\emptyset, g_1, g_1^2, \ldots, g_1^m\}$ for some m.

A tuple $X \in \mathcal{K}$ is Λ-*nilpotent* provided $X^v = 0$ whenever $v \in \mathcal{F}_d \setminus \Lambda$. If Λ is a finite initial segment, $X \in \mathcal{K}$ is Λ-nilpotent, and f is as in equation (6), then

$$f(X) = \sum_{w\in\Lambda} f_w X^w.$$

Theorem 1. *Fix a matrix convex set $\mathcal{K}$ satisfying the conditions of Assumption 1. Let Λ, a finite initial segment, and*

$$p = \sum_{w\in\Lambda} p_w w$$

be given. There exists $f \in \mathcal{A}(\mathcal{K})^\infty$ *such that* $f_w = p_w$ *for* $w \in \Lambda$ *and* $\|f\| \leq 1$ *if and only if*

$$\sup\{\|p(X)\| : X \in \mathcal{K}, \quad X \textit{ is } \Lambda - \textit{nilpotent}\} \leq 1.$$

The body of the paper contains a more general version of Theorem 1 allowing for operator-valued coefficients p_w and f_w. A version of the Theorem for the case of infinite initial segments Λ is also presented in Section 8 for some specific noncommutative domains.

3. The operator algebra $\mathcal{A}(\mathcal{K})^\infty$

Broadly speaking, the strategy for proving Theorem 1 is to realize $\mathcal{A}(\mathcal{K})^\infty$ as an operator algebra, note that Λ determines a closed ideal in $\mathcal{A}(\mathcal{K})^\infty$ and then apply the important corollary of the BRS theorem (See [P]) which says that the quotient of an operator algebra by a closed (two-sided) ideal has a completely isometric representation as a subalgebra of the bounded operators on some Hilbert space.

The norm on $\mathcal{A}(\mathcal{K})^\infty$ defined in subsection 2.5.2 naturally generalizes to $m \times n$ matrices with entries from $\mathcal{A}(\mathcal{K})^\infty$ and the resulting sequence of norms makes $\mathcal{A}(\mathcal{K})^\infty$ an abstract operator algebra. This section contains the details of the construction beginning with proving that $\mathcal{A}(\mathcal{K})^\infty$ is an algebra.

3.1. **The Noncommutative Fock Space.** Let $\mathbb{C}\langle g\rangle = \mathbb{C}\langle g_1, \dots, g_d\rangle$ denote the algebra of non-commuting polynomials in the variables $\{g_1, \dots, g_d\}$. Thus elements of $\mathbb{C}\langle g\rangle$ are linear combinations of elements of $\mathcal{F}_d$; i.e., an element of $\mathbb{C}\langle g\rangle$ of degree (at most) k has the form

$$\sum_{j=0}^{k} \sum_{|w|=j} p_w w,$$

where the p_w are complex numbers.

To construct the Fock space, $\mathbb{F}^2$, define an inner product on $\mathbb{C}\langle g\rangle$ by defining

$$\langle w, v\rangle = \begin{cases} 0 & \text{if } w \neq v \\ 1 & \text{if } w = v \end{cases}$$

for $w, v \in \mathcal{F}_d$ and extending by linearity to all of $\mathbb{C}\langle g\rangle$. The completion of $\mathbb{C}\langle g\rangle$ in this inner product is then the Hilbert space $\mathbb{F}^2$.

3.2. **The Creation Operators.** There are natural isometric operators on $\mathbb{F}^2$ called the creation operators which have been studied intensely in part because of their connection to the Cuntz algebra [C]. Given $1 \leq j \leq d$, define $S_j : \mathbb{F}^2 \to \mathbb{F}^2$ by $S_j v = g_j v$ for a word $v \in \mathcal{F}_d$ and extend S_j by linearity to all of $\mathbb{C}\langle g\rangle$. It is readily verified that S_j is an isometric mapping of $\mathbb{C}\langle g\rangle$ into itself and it thus follows that S_j extends to an isometry on all of $\mathbb{F}^2$. In particular $S_j^* S_j = I$, the identity on $\mathbb{F}^2$. Also of note is the identity,

$$\sum_{j=1}^{d} S_j S_j^* = P, \tag{8}$$

where P is the projection onto the orthogonal complement of the one-dimensional subspace of $\mathbb{F}^2$ spanned by $\emptyset$, which follows by observing, for a word $w \in \mathcal{F}_d$ and $1 \leq j \leq d$, that

$$S_j^*(w) = \begin{cases} w' & \text{if } w = g_j w' \\ 0 & \text{otherwise.} \end{cases}$$

Of course, as it stands the tuple $S = (S_1, \ldots, S_d)$ acts on the infinite dimensional Hilbert space $\mathbb{F}^2$. There are however, finite dimensional subspaces which are essentially determined by ideals in $\mathcal{F}_d$ and which are invariant for each S_j^*.

The subset $\Lambda(\ell) = \{w : |w| \leq \ell\}$ of $\mathcal{F}_d$ is a canonical example of a finite initial segment. Moreover, since each S_j^* leaves $\Lambda(\ell)$ invariant, the subspace $\mathbb{F}(\ell)^2$ of $\mathbb{F}^2$ spanned by $\Lambda(\ell)$ is invariant for S^*. Let $V(\ell)$ denote the inclusion of $\mathbb{F}(\ell)^2$ into $\mathbb{F}^2$ and let $S(\ell)$ denote the operator $V(\ell)^* S V(\ell)$. Thus, $S(\ell) = ((S(\ell))_1, \ldots, (S(\ell))_d)$ where $(S(\ell))_j = V(\ell)^* S_j V(\ell)$. Observe, with P denoting both the projection of $\mathbb{F}^2$ and $\mathbb{F}(\ell)^2$ onto the orthogonal complement of the span of $\emptyset$ in $\mathbb{F}^2$ and $\mathbb{F}(\ell)^2$ respectively, equation (8) yields

$$\begin{aligned} P =& V(\ell)^* P V(\ell) \\ =& V(\ell)^* \left(\sum_{j=1}^{d} S_j S_j^* \right) V(\ell) \\ =& \sum_{j=1}^{d} (S(\ell))_j (S(\ell))_j^*. \end{aligned}$$

It follows, for $t < \gamma$, that $tS(\ell) \in \mathcal{C}_\gamma(n)$, where $n = \sum_{j=0}^{\ell} d^j$ is the dimension of $\mathbb{F}(\ell)^2$.

3.3. **The Algebra $\mathcal{A}(\mathcal{K})^\infty$.** In addition to the obvious pointwise addition and multiplication by scalars, there is a natural multiplication on $\mathcal{A}(\mathcal{K})^\infty$ extending multiplication of polynomials which then turns $\mathcal{A}(\mathcal{K})^\infty$ into an algebra over $\mathbb{C}$. Since it will be necessary to consider, in the sequel, matrices with entries from $\mathcal{A}(\mathcal{K})^\infty$ and their products, we define them here.

Let

$$M_{p,q}(\mathcal{A}(\mathcal{K})) = \left\{ f = \sum_{w \in F_d} f_w w : f_w \in M_{p,q}, \text{ and for each } X \in \mathcal{K}, \ f(X) \text{ converges} \right\}.$$

where

$$f(X) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w \otimes X^w.$$

For $f \in M_{p,q}(\mathcal{A}(\mathcal{K}))$, let

$$\|f\| = \|f\|_{p,q} = \sup\{\|f(X)\| : X \in \mathcal{K}\}. \tag{9}$$

Define

$$M_{p,q}(\mathcal{A}(\mathcal{K})^{\infty}) = \left\{ f = \sum_{w \in F_d} f_w w : f \in M_{p,q}(\mathcal{A}(\mathcal{K})), \|f\| < \infty \right\}.$$

The following Lemma plays an important role in the analysis to follow generally, and in proving that $\mathcal{A}(\mathcal{K})^{\infty}$ is an algebra, in particular.

Lemma 1. *Suppose that* $f = \sum_{w \in \mathcal{F}_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^{\infty})$ *and* $X \in \mathcal{K}(n)$. *Let*

$$A_j = \sum_{|w|=j} f_w \otimes X^w.$$

If $0 < r < \sup\{0 < s : sX \in \mathcal{K}(n)\}$, *then*

$$r^j \|A_j\| \leq \|f\|.$$

In particular, there is a $\rho < 1$ *such that* $\|A_j\| \leq \rho^j \|f\|$.

Proof. Because $\mathcal{K}(n)$ is open, convex, and circled, the function $F(z) = f(zX)$ is defined on a neighborhood of the closed unit disk $\{|z| \leq 1\}$. Thus the series,

$$F(z) = \sum_{j=0}^{\infty} A_j z^j$$

has radius of convergence exceeding one. Thus, for each $j \in \mathbb{N}$,

$$A_j = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) e^{-ijt} \, dt.$$

It follows that

$$\|A_j\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|F(e^{it})\| \, dt.$$

Since $\|F(e^{it})\| = \|f(e^{it}X)\|$ and $e^{it}X \in \mathcal{K}(n)$, it follows that $\|F(e^{it})\| \leq \|f\|$ and the lemma follows. □

Given $f = \sum_{w \in \mathcal{F}_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^{\infty})$ and $g = \sum_{w \in \mathcal{F}_d} g_w w \in M_{q,r}(\mathcal{A}(\mathcal{K})^{\infty})$, define the product fg of f and g as the convolution product; i.e.,

$$fg = \sum_{w \in \mathcal{F}_d} \left(\sum_{uv=w} f_u g_v \right) w.$$

This convolution product corresponds to pointwise product, extends the natural product of non-commutative polynomials (formal power series with only finitely many non-zero coefficients), and makes $\mathcal{A}(\mathcal{K})^\infty$ an algebra.

Lemma 2. *If $f \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ and $g \in M_{q,r}(\mathcal{A}(\mathcal{K})^\infty)$ and $X \in \mathcal{K}$, then*
(i) *$fg(X)$ converges;*
(ii) *$fg(X) = f(X)g(X)$;*
(iii) *fg is in $M_{p,r}(\mathcal{A}(\mathcal{K})^\infty)$; and*
(iv) *$\|fg\| \le \|f\|\|g\|$.*

Corollary 1. *$\mathcal{A}(\mathcal{K})^\infty$ is an algebra.*

Proof of Lemma 2. Fix $X \in \mathcal{K}$. As in the proof of Lemma 1, let

$$A_j = \sum_{|w|=j} f_w \otimes X^w,$$
$$B_j = \sum_{|w|=j} g_w \otimes X^w$$
$$C_j = \sum_{|w|=j} (\sum_{uv=w} f_u g_v) \otimes X^w.$$

Observe that $C_j = \sum_{k=0}^{j} A_k B_{j-k}$.

Let $F(z) = f(zX)$ and $G(z) = g(zX)$, both of which are defined in a neighborhood of $\{|z| \le 1\}$. From Lemma 1, there is an $\rho < 1$ such that $\|A_m\| \le \rho^m \|f\|$ and $\|B_k\| \le \rho^k \|g\|$. Hence

$$\|C_j\| \le (j+1)\|f\|\,\|g\|\rho^j.$$

It follows that, for $|z| < \frac{1}{\rho}$, the series

$$\sum_{j=0}^{\infty} (\sum_{k=0}^{j} A_k B_{j-k}) z^j$$

converges absolutely. In particular $fg(X) = \sum_{j=0}^{\infty} C_j$ converges in norm.

For $|z| < \frac{1}{\rho}$, one verifies that

$$\begin{aligned} F(z)G(z) =& \sum_{j=0}^{\infty} \sum_{k=0}^{j} A_k B_{j-k} z^j \\ =& fg(zX). \end{aligned}$$

Choosing $z = 1$ gives $f(X)g(X) = fg(X)$.

Since, for each $X \in \mathcal{K}$, $fg(X) = f(X)g(X)$ it follows that $\|fg(X)\| \le \|f\|\,\|g\|$. Thus $\|fg\| \le \|f\|\,\|g\|$ and $fg \in M_{p,r}(\mathcal{A}(\mathcal{K})^\infty)$. □

3.4. **The Abstract Unital Operator Algebra $\mathcal{A}(\mathcal{K})^\infty$.** In this section, for the convenience of the reader, the definition of an abstract operator algebra is reviewed. Following that, it is shown that $\mathcal{A}(\mathcal{K})^\infty$ with the norms $\|\cdot\|_{p,q}$ on $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ given in equation (9) is an abstract operator algebra.

3.4.1. *Abstract Operator Algebra.* Let V be a complex vector space and $M_{p,q}(V)$ denote the set of all $p \times q$ matrices with entries from V. V is said to be a *matrix normed space* provided that there exist norms $\|\cdot\|_{p,q}$ on $M_{p,q}(V)$ that satisfy

$$\|A \cdot X \cdot B\|_{\ell,r} \leq \|A\|\|X\|_{p,q}\|B\|$$

for all $A \in M_{\ell,p}, X \in M_{p,q}(V), B \in M_{q,r}$.

A matrix normed space V is said to be an *abstract operator space* if

$$\|X \oplus Y\|_{p+\ell,q+r} = \max\{\|X\|_{p,q}, \|Y\|_{\ell,r}\}$$

where $X \in M_{p,q}(V)$ and $Y \in M_{\ell,r}(V)$ and $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

V is an *abstract unital operator algebra* if V is a unital algebra, an abstract operator space and if the product on V is completely contractive i.e. $\|XY\|_p \leq 1$ whenever $\|X\|_p \leq 1$ and $\|Y\|_p \leq 1$ for all $X, Y \in M_p(V)$ and for all p.

3.4.2. *The Abstract Unital Operator Algebra $\mathcal{A}(\mathcal{K})^\infty$.* Consider $\mathcal{A}(\mathcal{K})^\infty$ with $\|\cdot\|_{p,q}$ being the norm on $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ defined in subsection 3.3 (see equation (9)).

Theorem 2. *$\mathcal{A}(\mathcal{K})^\infty$ with the family of norms $\|\cdot\|_{p,q}$, is an abstract unital operator algebra.*

Proof. Let $A \in M_{\ell,p}$, $F \in M_{p,q}(\mathcal{A}^\infty)$, $B \in M_{q,r}$. Interpret A and B as $A\emptyset \in M_{\ell,p}(\mathcal{A}(\mathcal{K})^\infty)$ and $B\emptyset \in M_{q,r}(\mathcal{A}(\mathcal{K})^\infty)$ respectively. For notation ease we will drop the subscripts that go with the norms. It follows from Lemma 2(ii) that for all $X \in \mathcal{K}(n)$,

$$\|AFB(X)\| = \|A(X)F(X)B(X)\| \leq \|A \otimes I_n\| \, \|F(X)\| \, \|B \otimes I_n\| \leq \|A\| \, \|F\| \, \|B\|.$$

Thus,

$$\|AFB\| \leq \|A\| \, \|F\| \, \|B\|. \tag{10}$$

Let $F \in M_{\ell,r}(\mathcal{A}(\mathcal{K})^\infty)$, $G \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$, $X \in \mathcal{K}(n)$. Observe that

$$\|F \oplus G\,(X)\| = \left\| \begin{pmatrix} F(X) & 0 \\ 0 & G(X) \end{pmatrix} \right\| \leq \max\{\|F(X)\|, \|G(X)\|\} \leq \max\{\|F\|, \|G\|\}.$$

Thus,

$$\|F \oplus G\| \leq \max\{\|F\|, \|G\|\} \tag{11}$$

Let $\epsilon > 0$ be given. Without loss of generality assume that $\|F\| \geq \|G\|$. Choose $m \in \mathbb{N}$ and $R \in \mathcal{K}(m)$ such that $\|F(R)\| > \|F\| - \epsilon$. Therefore

$$\|F \oplus G\| \geq \left\| \begin{pmatrix} F(R) & 0 \\ 0 & G(R) \end{pmatrix} \right\| \geq \|F(R)\| > \|F\| - \epsilon. \tag{12}$$

Letting $\epsilon \to 0$ in the inequality (12) and from the inequality (11) it follows that,

$$\|F \oplus G\| = \max\{\|F\|, \|G\|\}. \tag{13}$$

Lastly, complete contractivity of multiplication in $M_p(\mathcal{A}(\mathcal{K})^\infty)$ follows directly from Lemma 2 (iv). Thus $\mathcal{A}(\mathcal{K})^\infty$ is an abstract operator algebra. $\emptyset \in \mathcal{A}(\mathcal{K})^\infty$ is the multiplicative unit. □

4. Weak Compactness and $\mathcal{A}(\mathcal{K})^\infty$

In this section it is shown that every bounded sequence in $\mathcal{A}(\mathcal{K})^\infty$ has a pointwise convergent subsequence. Indeed, $\mathcal{A}(\mathcal{K})^\infty$ has weak compactness properties with respect to bounded pointwise convergence mirroring those for H^∞, the usual space of bounded analytic functions on the unit disk $\mathbb{D}$.

The results easily extend to formal power series with matrix coefficients and it is at this level of generality that they are needed in the sequel.

Proposition 1. *Suppose that $f_m = \sum_{w\in\mathcal{F}_d}(f_m)_w w$ is a $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ sequence. If, for each $X \in \mathcal{K}$ the sequence $f_m(X)$ converges or if for each $w \in \mathcal{F}_d$ the sequence $(f_m)_w$ converges, and if (f_m) is a bounded sequence (so there is a constant c such that $\|f_m\| \le c$ for all m), then there is an $f \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ such that $f_m(X)$ converges to $f(X)$ for each $X \in \mathcal{K}$ and moreover $\|f\| \le c$.*

Proof. If f_m converges pointwise, then, by considering $f_m(S(\ell))$ where $S(\ell)$ is defined in Subsection 3.2, it follows that $(f_m)_w$ converges to some f_w for each word w. Hence, to prove the Proposition it suffices to prove, if $(f_m)_w$ converges to f_w for each w and $\|f_m\| \le c$ for each m, then for each $X \in \mathcal{K}$, the series

$$f(X) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w \otimes X^w$$

converges and $(f_m(X))$ converges to $f(X)$.

For any $X \in \mathcal{K}$,

$$\sum_{|w|=j} (f_m)_w \otimes X^w \to \sum_{|w|=j} f_w \otimes X^w.$$

From Lemma 1, there is a $\rho < 1$ such that for all $j \in \mathbb{N}$,

$$\| \sum_{|w|=j} (f_m)_w \otimes X^w \| \le \rho^j c,$$

an estimate from which the conclusions of the proposition are easily seen to follow. □

Lemma 3. *If $f_m = \sum_{w\in\mathcal{F}_d}(f_m)_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ satisfies $\|f_m\| \le c$ for all $m \in \mathbb{N}$ then,*

(i) $\|(f_m)_w\| \le \frac{c}{\gamma^{|w|}}$ *for all $w \in \mathcal{F}_d$ and for all $m \in \mathbb{N}$;*
(ii) *There exists a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ and $f_w \in M_{p,q}$ such that $(f_{m_k})_w \to f_w$ for all w;*
(iii) *Let $f = \displaystyle\sum_{w\in\mathcal{F}_d} f_w w$. For each $X \in \mathcal{K}$ the sequence $(f_{m_k}(X))$ converges to $f(X)$ and moreover $\|f(X)\| \le c$.*

Proof. To prove item (i), Recall γ from the definition of $\mathcal{K}$. Let $t < \gamma$ and $x \in \mathbb{C}^q$ be a unit vector. For $j = 0, 1, 2, ..., \ell$, the hypothesis $\|f_m\| \leq c$ together with the conclusion of Lemma 1 for $X = tS(\ell)$ imply that

$$\|t^j \sum_{|w|=j} (f_m)_w \otimes S(\ell)^w\| \leq c.$$

Hence

$$\begin{aligned} c^2 &\geq \|t^j \sum_{|w|=j} (f_m)_w x \otimes S(\ell)^w \emptyset\|^2 \\ &= t^{2j} \sum_{|w|=j} \|(f_m)_w x\|^2 \\ &\geq t^{2j} \|(f_m)_w x\|^2. \end{aligned}$$

Since x and ℓ are arbitrary, letting $t \uparrow \gamma$ it follows that $\|(f_m)_w\| \leq \frac{c}{\gamma^{|w|}}$ for all $m \in \mathbb{N}$.

The proof of item (ii) uses a standard diagonal argument. Let $\{w_1, w_2, ...\}$ be an enumeration of words in $\mathcal{F}_d$ which respects length (i.e., if $v \leq w$, the $|v| \leq |w|$). Since $\|(f_m)_{w_1}\| \leq \frac{c}{\gamma^{|w_1|}}$, there exists a subsequence say, $\{f_{1,m}\}$ of $\{f_m\}$ such that $(f_{1,m})_{w_1} \to f_{w_1}$. Since $\|(f_{1,m})_{w_2}\| \leq \frac{c}{\gamma^{|w_2|}}$, there exists a subsequence say, $\{f_{2,m}\}$ of $\{f_{1,m}\}$ and thereby of $\{f_m\}$, such that $(f_{2,m})_{w_2} \to f_{w_2}$. Continue this procedure to obtain a subsequence $\{f_{k,m}\}$ of $\{f_{k-1,m}\}$ and thereby of $\{f_m\}$ such that for all $k \in \mathbb{N}$,

$$(f_{k,m})_{w_k} \to f_{w_k}.$$

Now consider the diagonal sequence $\{f_{m,m}\}$. It follows that $\{f_{m,m}\}$ is a subsequence of $\{f_m\}$ and satisfies $(f_{m,m})_w \to f_w$ for all $w \in \mathcal{F}_d$.

In view of what has already been proved, an application of Proposition 1 proves item (iii). □

5. The operator algebra $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$

In this section we consider the ideal $\mathcal{I}(\mathcal{K})$ of the algebra $\mathcal{A}(\mathcal{K})^\infty$ determined by a finite inital segment Λ. It is shown that the quotient algebra $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ is infact an abstract unital operator algebra. It is also established that norms of classes in the quotient algebra are attained.

5.1. **The Abstract Unital Operator Algebra $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$.** For the initial segment $\Lambda \subset \mathcal{F}_d$, let

$$\mathcal{I}(\mathcal{K}) = \left\{ f = \sum_{w \notin \Lambda} f_w w \quad : \quad \|f\| < \infty \right\} \subset \mathcal{A}(\mathcal{K})^\infty$$

Observe that $\mathcal{I}(\mathcal{K})$ is a closed two-sided ideal in the operator algebra $\mathcal{A}(\mathcal{K})^\infty$. The usual identification of $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}))$ with $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)/M_{p,q}(\mathcal{I}(\mathcal{K}))$

yields the well known fact that the quotient of an abstract operator algebra by a closed two sided ideal is again an abstract operator algebra (see Exercises 13.3 & 16.3 in [P]). We formally record this fact.

Theorem 3. *$\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ is an abstract unital operator algebra.*

5.2. **Attainment of Norms of Classes in $M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K}))$.** Let $p \in M_q(\mathcal{A}(\mathcal{K})^\infty)$. In this section it is shown that there exists $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p+f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\| = \inf\{\|p+g\| : g \in M_q(\mathcal{I}(\mathcal{K}))\}.$$

Let $\{f_m\}$ be a sequence in $M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p + M_q(\mathcal{I}(\mathcal{K}))\| \leq \|p + f_m\| \leq \|p + M_q(\mathcal{I}(\mathcal{K}))\| + \frac{1}{m}$$

It follows that the sequence $\{f_m\}$ is bounded and that $\|p + f_m\| \to \|p + M_q(\mathcal{I}(\mathcal{K}))\|$. An application of Lemma 3 yields a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ and $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$(p + f_{m_k})(X) \to (p+f)(X)$$

for all $X \in \mathcal{K}$.

Proposition 2. *If $p, \{f_{mk}\}, f$ are as above, then $\|p+f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\|$.*

Proof. Let $\epsilon > 0$ be given. Choose $R \in \mathcal{K}$ such that

$$\|p+f\| < \|(p+f)(R)\| + \frac{\epsilon}{4}. \tag{14}$$

Since $\|(p+f_{m_k})(R)\| \to \|(p+f)(R)\|$, there exists $K_1 \in \mathbb{N}$ such that,

$$\|(p+f)(R)\| < \|(p+f_{m_k})(R)\| + \frac{\epsilon}{4} \tag{15}$$

for all $k \geq K_1$. Combining the inequalities from equations 14 and 15, implies that, for all $k \geq K_1$,

$$\|p+f\| < \|p+f_{m_k}\| + \frac{\epsilon}{2}. \tag{16}$$

Since $\|p + f_{m_k}\| \to \|p + M_q(\mathcal{I}(\mathcal{K}))\|$, there exists a Natural number K_2 such that for all $k \geq K_2$,

$$\|p+f_{m_k}\| < \|p + M_q(\mathcal{I}(\mathcal{K}))\| + \frac{\epsilon}{2}. \tag{17}$$

Setting $k = \max\{K_1, K_2\}$ in equations (16) and (17), and letting $\epsilon \to 0$ yields

$$\|p+f\| \leq \|p + M_q(\mathcal{I}(\mathcal{K}))\|.$$

On the other hand, since $f \in M_q(\mathcal{I}(\mathcal{K}))$,

$$\|p+f\| \geq \|p + M_q(\mathcal{I}(\mathcal{K}))\|.$$

□

6. Representations of the operator algebra $\mathcal{A}(\mathcal{K})^\infty$

In this section it is shown that completely contractive representations of the algbera $\mathcal{A}(\mathcal{K})^\infty$, when compressed to finite-dimensional subspaces end up in the boundary of the matrix-convex set $\mathcal{K}$.

6.1. **Completely Contractive/Isometric Representation.** Let $\mathcal{V}$ and $\mathcal{W}$ be abstract operator spaces and $\phi : \mathcal{V} \to \mathcal{W}$ be a linear map. Define $\phi_q : M_q \otimes \mathcal{V} \to M_q \otimes \mathcal{W}$ by $\phi_q = I_q \otimes \phi$, where I_q is the $q \times q$ identity matrix.

The map ϕ is said to be *completely contractive (isometric)* if ϕ_q is a contraction (isometry) for each $q \in \mathbb{N}$.

A *completely contractive (isometric)* representation of an algebra $\mathcal{A}$ is a completely contractive (isometric) algebra homomorphism $\pi : \mathcal{A} \to B(\mathcal{M})$ for some Hilbert space $\mathcal{M}$.

The following Theorem due to Blecher, Ruan and Sinclair guarantees the existence of a completely isometric Hilbert space representation for an abstract unital operator algebra.

Theorem 4 (BRS)**.** *Let $\mathcal{A}$ be an abstract unital operator algebra. There exists a Hilbert space $\mathcal{M}$ and a unital completely isometric algebra homomorphism $\pi : \mathcal{A} \to B(\mathcal{M})$, i.e. a unital operator algebra isomorphism onto $\pi(\mathcal{A})$.*

6.2. **Completely Contractive Representations of $\mathcal{A}(\mathcal{K})^\infty$.** Given a completely contractive unital representation $\pi : \mathcal{A}(\mathcal{K})^\infty \to \mathcal{B}(\mathcal{M})$, let $T_j = \pi(g_j)$ and let $T = (T_1, \dots, T_d)$. For notation convenience, we will write π_T for π. Further, we will also use π_T to denote the map $I_q \otimes \pi : M_q(\mathcal{A}(\mathcal{K})^\infty) \to M_q \otimes B(\mathcal{M})$.

A main result of this section says, for a completely contractive representation π_T of $\mathcal{A}(\mathcal{K})^\infty$, for any $n \in \mathbb{N}$ and finite dimensional subspace $\mathcal{H}$ of $\mathcal{M}$ of dimension n and $0 \le t < 1$ the tuple

$$tZ = tV^*TV = (tV^*T_1V, \dots, tV^*T_dV)$$

is in $\mathcal{K}(n)$. The proof begins with a couple of lemmas. Given $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ and $0 \le r < 1$, let f_r be defined as follows. If

$$f = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} f_j. \tag{18}$$

then

$$f_r = \sum_{j=0}^{\infty} r^j \sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} r^j f_j.$$

Lemma 4. *If π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$ and $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$, then $f_r(T)$ converges in operator norm. Moreover $\pi_T(f_r) = f_r(T)$ and $\|f_r(T)\| \le \|f_r\| \le \|f\|$.*

If in addition π_T is completely isometric, then $\lim_{r \to 1^-} \|f_r(T)\| = \|f\|$.

Proof. Write f as in equation (18). Lemma 1 implies that $\|f_j\| \le \|f\|$. Because π_T is completely contractive $\|f_j(T)\| \le \|f\|$. It follows that $f_r(T)$ converges in norm. Since also the partial sums of f_r converge (to f_r) in the norm of $M_q(\mathcal{A}(\mathcal{K})^\infty)$, it follows that $\pi_T(f_r) = f_r(T)$ and so $\|f_r(T)\| \le \|f_r\|$.

The inequality $\|f_r\| \le \|f\|$ is straightforward because $r\mathcal{K} \subset \mathcal{K}$.

Now suppose that π_T is completely isometric. In this case $\|f_r(T)\| = \|f_r\|$. On the other hand $\lim_{r\to 1^-} \|f_r\| = \|f\|$. ☐

Lemma 5. *Given $A_1, \dots, A_d$ are $k \times k$ matrices. let*

$$L = \sum_{j=1}^{d} A_j g_j.$$

Suppose

$$2 - L(X) - L(X)^* \succ 0$$

for all $X \in \mathcal{K}(\ell)$ and for all $\ell \in \mathbb{N}$. Let Φ_L denote the formal power series,

$$\Phi_L = L(2-L)^{-1} = \sum_{j=0}^{\infty} \frac{L^{j+1}}{2^{j+1}}.$$

(a) If $X \in \mathcal{K}(\ell)$, then $\Phi_L(X)$ converges in norm; i.e., the series

$$\sum_{j=0}^{\infty} \frac{L(X)^{j+1}}{2^{j+1}}$$

converges.

(b) $\|\Phi_L(X)\| < 1$ and hence Φ_L is in $M_k(\mathcal{A}(\mathcal{K})^\infty)$ and has norm at most one.

(c) If π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, then $2 - (L(T) + L(T)^*) \succeq 0$.

Proof. To prove part (a) of the lemma, let $X \in \mathcal{K}(\ell)$ be given. Because $\mathcal{K}(\ell)$ is circled, it follows that $e^{i\theta}X \in \mathcal{K}(\ell)$ for each θ. Hence,

$$2 - e^{i\theta}L(X) - e^{-i\theta}L(X)^* \succ 0 \tag{19}$$

for each θ. For notation ease, let $Y = L(X)$. Thus Y is a $k\ell \times k\ell$ matrix and equation (19) implies that the spectrum of Y lies strictly within the disc; i.e., each eigenvalue of Y has absolute value less than one. Thus,

$$\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{Y}{2}\right)^j = (2-Y)^{-1}$$

converges in norm. It follows that

$$\Phi_L(X) = Y(2-Y)^{-1} = \sum_{j=0}^{\infty} \frac{Y^{j+1}}{2^{j+1}}$$

converges.

To prove (b) observe that $\|Y(2-Y)^{-1}\| < 1$ if and only if

$$(2-Y)^*(2-Y) \succ Y^*Y$$

which is equivalent to $2 - (Y + Y^*) \succ 0$. Thus $\|\Phi_L(X)\| < 1$ which implies that $\Phi_L \in M_k(\mathcal{A}(\mathcal{K})^\infty)$ with $\|\Phi_L\| \le 1$. This completes the proof of (b).

To prove part (c), observe, Since π_T is completely contractive and $\Phi_L \in M_k(\mathcal{A}(\mathcal{K})^\infty)$ with norm at most one, an application of Lemma 4 yields, $\|\Phi_L(rT)\| \le 1$. Arguing as in the proof of part (b), it follows that $2 - (L(rT) + L(rT)^*) \succeq 0$. This inequality holds for all $0 \le r < 1$ and thus the conclusion of part (c) follows. □

Proposition 3. *If $T = (T_1, \dots, T_d)$, and $T_j \in \mathcal{B}(\mathcal{M})$ for some Hilbert space $\mathcal{M}$, and $\pi(g_j) = T_j$ determines a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, then, for each positive integer n and finite dimensional subspace $\mathcal{H}$ of $\mathcal{M}$ of dimension n and each $0 \le t < 1$ the tuple*

$$tZ = tV^*TV = (tV^*T_1V, \dots, tV^*T_dV)$$

is in $\mathcal{K}(n)$, where $V : \mathcal{H} \to \mathcal{M}$ is the inclusion map.

Proof. Let n and $\mathcal{H}$ be given and define Z as in the statement of the proposition. Suppose that L is as in the statement of Lemma 5. From part (c) of the previous lemma, it follows that $2 - (L(T) + L(T)^*) \succeq 0$. Applying $I_k \otimes V^*$ on the left and $I_k \otimes V$ on the right of this inequality gives,

$$2 - (L(Z) + L(Z)^*) = (I_k \otimes V^*)(2 - (L(T) + L(T)^*)(I_k \otimes V) \succeq 0.$$

An application of Theorem 5.4 from [EW] implies that $Z \in \overline{\mathcal{K}(n)}$. Hence $tZ \in \mathcal{K}(n)$ for all $0 \le t < 1$. □

Lemma 6. *Let $\Lambda \subset \mathcal{F}_d$ be a finite initial segment, $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ be as in equation (18) and suppose that π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$ into $B(\mathcal{M})$ and T is $\Lambda-$nilpotent. Then $\|f_r(T)\| \le \sup\{\|f(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}$ for all $0 \le r < 1$. Moreover if $f_w = 0$ for all $w \notin \Lambda$, then $\|f(T)\| \le \sup\{\|f(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}$.*

Proof. Since Λ is finite, $f_r(T) = \sum_{w\in\Lambda} f_w \otimes (rT)^w$. Let $\{e_j\}_{j=1}^q$ be the standard basis of $\mathbb{C}^q$ and $y = \sum_{j=1}^q e_j \otimes h_j \in \mathbb{C}^q \otimes \mathcal{M}$ be a unit vector such that

$$\|f_r(T)\| < \|f_r(T)y\| + \epsilon.$$

Let $\mathcal{H}$ denote the finite-dimensional subspace of $\mathcal{M}$ spanned by the vectors $\{T^w(h_j) : w \in \Lambda, 1 \le j \le q\}$ and $V : \mathcal{H} \to \mathcal{M}$ be the inclusion map. Then $Z = V^*TV$ is Λ-nilpotent and

$$Z^w = \begin{cases} V^*T^wV & \text{if } w \in \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3 implies that $rZ \in \mathcal{K}$. Thus,

$$\begin{aligned}\|f_r(T)\| &< \| \left(\sum_{w\in\Lambda} f_w \otimes r^{|w|}T^w\right) y\| + \epsilon \\ &= \|f_r(Z)y\| + \epsilon \\ &\le \|f_r(Z)\| + \epsilon \\ &\le \sup\{\|f(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\} + \epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ yields the desired inequality. If $f_w = 0$ for all $w \notin \Lambda$, then $f = \sum_{w\in\Lambda} f_w w$ is a non-commutative polynomial in which case we have $\lim_{r\to 1^-} \|f_r(T)\| = \|f(T)\|$ and this completes the proof. □

7. The Caratheodory-Fejer Interpolation Problem (CFP)

The proof of the generalization of Theorem 1 allowing for operator coefficients is proved in this section.

The strategy is to first prove the result for matrix coefficients. This is done in Subsection 7.1 below. Passing from matrix to operator coefficients is then accomplished using well-known facts about the Weak Operator Topology (WOT) and the Strong Operator Topology (SOT) on the space of bounded operators on a separable Hilbert space. The details are in Subsection 7.2.

7.1. **Matrix version of the CFP.** Fix $\Lambda \subset \mathcal{F}_d$, a finite initial segment, and polynomial $p = \sum_{w\in\Lambda} p_w w \in M_q(\mathcal{A}(\mathcal{K})^\infty)$.

Theorem 1 is easily seen to follow from the following proposition.

Proposition 4. *There exists $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ such that $\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\| = \sup\{\|p(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$.*

Proof. From Theorems 3 and 4 it follows that there exists a Hilbert space $\mathcal{M}$ and a completely isometric homomorphism $\pi : \mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}) \to B(\mathcal{M})$. As before, identify $M_q(\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}))$ with $M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K}))$. Let π_q denote the map $I_q \otimes \pi : M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K})) \to M_q \otimes B(\mathcal{M})$. Let R be the d-tuple $(R_1, R_2, ..., R_d)$, where $R_j = \pi(g_j + \mathcal{I}(\mathcal{K})) \in B(\mathcal{M})$, for $1 \le j \le d$. Observe that R is Λ-nilpotent. Let $\eta : \mathcal{A}(\mathcal{K})^\infty \to \mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ be the quotient map. The composition map $\pi \circ \eta : \mathcal{A}(\mathcal{K})^\infty \to B(\mathcal{M})$ is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$. Also since $\pi(\eta(g_j)) = R_j$, consistent with the notation introduced earlier, we will use π_R to denote the map $\pi \circ \eta$.

It follows from Theorem 2 that there exists $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\|. \tag{20}$$

The fact that π is completely isometric implies that

$$\|p + M_q(\mathcal{I}(\mathcal{K}))\| = \|\pi_q(p + M_q(\mathcal{I}(\mathcal{K}))\| = \|p(R)\| \tag{21}$$

Since π_R is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, Lemma 6 implies that

$$\|p(R)\| \leq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda-\text{nilpotent}\}. \tag{22}$$

Combining the equations (20), (21) and (22), it follows that

$$\|p+f\| \leq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda-\text{nilpotent}\}.$$

But the definition of $\|p+f\|$ implies that

$$\|p+f\| \geq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda-\text{nilpotent}\}$$

and this completes the proof. □

7.2. **Operator Version of the CFP.** As before, let $\Lambda \subset \mathcal{F}_d$ be a finite initial segment. Departing from the previous subsection, let $\mathcal{U}$ be a separable Hilbert space and let the polynomial $p = \sum_{w\in\Lambda} p_w w$, where $\{p_w\}_{w\in\Lambda} \subset B(\mathcal{U})$ be given.

Theorem 5. *There exists a formal power series $\tilde{x} = \sum_{w\in\mathcal{F}_d} \tilde{x}_w w$ such that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and $\|\tilde{x}\| = \sup\{\|p(X)\| : X \in \mathcal{K}, X$ is $\Lambda-$nilpotent$\}$.*

Proof. Let $\{u_1, u_2, ...\}$ denote an orthonormal basis for the separable Hilbert space $\mathcal{U}$ and $\mathcal{U}_m$ be the subspace of $\mathcal{U}$ spanned by the vectors $\{u_j\}_{j=1}^m$. For notation ease, let $C = \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda-\text{nilpotent}\}$.

For $w \in \Lambda$, define $M_m \ni (p_m)_w = V_m^* p_w V_m$ where $V_m : \mathcal{U}_m \to \mathcal{U}$ is the inclusion map. Let p_m denote the formal power series

$$p_m = \sum_{w\in\Lambda} (p_m)_w w$$

For each $X \in \mathcal{K}$, Observe that $\|p_m(X)\| \leq \|p(X)\|$. Thus $\|p_m\| \leq \|p\|$ and $p_m \in M_m(\mathcal{A}(\mathcal{K})^\infty)$ for all $m \in \mathbb{N}$. From Theorem 4, there exists $f_m \in M_m(\mathcal{I}(\mathcal{K}))$ such that $x_m = p_m + f_m \in M_m(\mathcal{A}(\mathcal{K})^\infty)$ and

$$\|x_m\| = \sup\{\|p_m(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda-\text{nilpotent}\}.$$

For $w \in \mathcal{F}_d$, define $B(\mathcal{U}) \ni (\tilde{x}_m)_w = V_m (x_m)_w V_m^*$. Let $\tilde{x}_m$ denote the formal power series $\sum_{w\in\mathcal{F}_d} (\tilde{x}_m)_w w$. For $X \in \mathcal{K}$ and $j = 0, 1, 2, ...$, it follows from From Lemma 1 that there exists $0 \leq \rho < 1$ such that

$$\begin{aligned} \|\sum_{|w|=j} (\tilde{x}_m)_w \otimes X^w\| &\leq \|\sum_{|w|=j} (x_m)_w \otimes X^w\| \\ &\leq \rho^j \|x_m\| \\ &\leq C\rho^j \end{aligned} \tag{23}$$

This implies that series for $\tilde{x}_m(X)$ converges for each $X \in \mathcal{K}$ and moreover we have

$$\|\tilde{x}_m\| \leq \|x_m\| \leq C. \tag{24}$$

Recall γ and $S(\ell)$ from Subsection 3.2. Let $u \in \mathcal{U}$ be an arbitrary unit vector. For each $0 \le j \le \ell$ and $X \in \mathcal{K}$, it follows that

$$\begin{aligned} C^2 &\ge \| \sum_{|w|=j} (\tilde{x}_m)_w \otimes S(\ell)^w (u \otimes \emptyset) \|^2 \\ &\ge \| \sum_{|w|=j} (\tilde{x}_m)_w u \otimes w \|^2 \\ &\ge \sum_{|w|=j} \|(\tilde{x}_m)_w u\|^2 \end{aligned}$$

Thus $\|(\tilde{x}_m)_w\| \le C$ for all $w \in \mathcal{F}_d$ and $m \in \mathbb{N}$.

Since $\mathcal{U}$ is a separable Hilbert space and the sequence $\{(\tilde{x}_m)_w\}_{m=1}^\infty$ is bounded (by C), for each $w \in \mathcal{F}_d$, there exists a subsequence of $\{(\tilde{x}_m)_w\}_{m=1}^\infty$ that converges with respect to the WOT on $B(\mathcal{U})$. By a diagonal argument similar to the one in Lemma 3, it follows that there exists a subsequence $\{\tilde{x}_{m_k}\}$ of $\{\tilde{x}_m\}$ and $\{\tilde{x}_w\}_{w \in \mathcal{F}_d} \subset B(\mathcal{U})$ such that for each $w \in \mathcal{F}_d$

$$(\tilde{x}_{m_k})_w \to \tilde{x}_w$$

with respect to the WOT on $B(\mathcal{U})$.

Let $\tilde{x}$ denote the formal power series $\sum_{w \in \mathcal{F}_d} \tilde{x}_w w$. The proof of the Theorem is completed by showing that $\|\tilde{x}\| \le C$ and that noting that $\tilde{x}_w = \lim (\tilde{x}_{m_k})_w = \lim V_{m_k} (p_{m_k})_w V_{m_k}^* = p_w$ (WOT limits) for $w \in \Lambda$. The details are omitted. □

8. Examples and the case of infinite intial segments Λ

Of course the results of this paper apply to the examples in Subsection 2.4.

In the case of the non-commutative matrix polydisc, the operators obtained by applying the representation of the quotient algebra to the generators $[g_j] = g_j + \mathcal{I}(\mathcal{K})$; $1 \le j \le d$, are automatically contractions and thus certain technical details of the proof of Theorem 1 are absent. Consequently, the argument easily extends to handle infinite intial segments, provided the underlying domain is expanded to include operators on separable Hilbert space.

Fix a separable infinite dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{C}^d$ denote the *operator non-commutative polydisc*

$$\mathcal{C}^d = \{(T_1, T_2, ..., T_d) : T_j \in B(\mathcal{H}) \ \&\ \|T_j\| < 1\}.$$

The following variant of Theorem 5 holds.

Theorem 6. *Let $\mathcal{U}$ be a separable Hilbert space, $\Lambda \subset \mathcal{F}_d$ be an infinite initial segment and $p = \sum_{w \in \Lambda} p_w w$ be a formal power series with coefficients $p_w \in B(\mathcal{U})$ such that $\|p\| = \sup\{\|p(X)\| : X \in \mathcal{C}^d\} < \infty$. There exists operators $\tilde{x}_w \in B(\mathcal{U})$ and a formal power series $\tilde{x} = \sum_{w \in \mathcal{F}_d} \tilde{x}_w w$ such that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and $\|\tilde{x}\| = \sup\{\|p(T)\| : T \in \mathcal{C}^d,\ T$ is $\Lambda-$nilpotent$\}$.*

Similarly, consider the dd'-dimensional *operator non-commutative mixed ball*,

$$\mathcal{D}^{dd'} = \{T = (T_{11}, T_{12}, ..., T_{dd'}) \,:\, T_{ij} \in B(\mathcal{H}) \,\&\, \|T\|_{op} < 1\}$$

where $\|T\|_{op}$ is the norm of the operator $(T_{ij})_{i,j=1}^{d,d'} : B(\mathcal{H}^{d'}) \to B(\mathcal{H}^{d})$.

A variant of Theorem 5 holds in this case as well, the statement of which can be obtained by replacing $\mathcal{C}^d$ in the statement of Theorem 6 by $\mathcal{D}^{dd'}$.

Acknowledgements

I would like to thank my advisor Scott McCullough for his guidance in the preparation of this article.

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