MICROLOCAL ANALYSIS ON WONDERFUL VARIETIES. REGULARIZED TRACES AND GLOBAL CHARACTERS

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ABSTRACT. Let $\mathbf{G}$ be a connected reductive complex algebraic group with split real form (G, σ). Consider a strict wonderful $\mathbf{G}$-variety $\mathbf{X}$ equipped with its σ-equivariant real structure, and let X be the corresponding real locus. Further, let E be a real differentiable G-vector bundle over X. In this paper, we introduce a distribution character for the regular representation of G on the space of smooth sections of E, and show that on a certain open subset of G of transversal elements it is locally integrable and given by a sum over fixed points.

CONTENTS

1. INTRODUCTION

Let G be a real reductive group. In classical harmonic analysis a crucial role is played by the global character of an irreducible admissible representation $(\nu, \mathcal{H})$ of G on a Hilbert space $\mathcal{H}$. It is a distribution $\Theta_\nu : \mathscr{S}(G) \ni f \mapsto \operatorname{tr} \nu(f) \in \mathbb{C}$ on the group given in terms of the trace of the convolution operators

$$\nu(f) = \int_G f(g)\nu(g)\, d_G(g),$$

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where $\mathscr{S}(G)$ denotes certain space of rapidly decaying functions on G and d_G a Haar measure on G [20, Chapter 8]. By Harish-Chandra's regularity theorem, Θ_ν is known to be locally integrable, and represents a natural generalization of the character of a finite-dimensional representation. As a consequence of this theorem, Harish-Chandra was able to characterize tempered representations in terms of the growth properties of their global characters, and fully determine the irreducible L^2-integrable representations of G. For a parabolically induced representation, Atiyah and Bott [4] interpreted Θ_ν in terms of a fixed point formula, extending the classical Lefschetz fixed point theorem to geometric endomorphisms on elliptic complexes via pseudodifferential operators.

Let now G be the split real form of a connected reductive complex algebraic group $\mathbf{G}$ and σ the corresponding anti-holomorphic involution of $\mathbf{G}$. Consider further the real locus X of a strict wonderful complex algebraic $\mathbf{G}$-variety $\mathbf{X}$ equipped with its canonical σ-equivariant real structure. The variety X is, in particular, a projective real algebraic G_0-variety, G_0 being the identity component of G. In this paper, we introduce a distribution character Θ_π for the regular representation $(\pi, \mathrm{C}^\infty(X,E))$ of G_0 on the space of smooth sections of a given real differentiable G_0-vector bundle E over X. Since the G-action on X is not transitive, the corresponding convolution operators $\pi(f)$ are not smoothing, so that a trace can no longer be defined by restricting their kernels to the diagonal. Instead, we show that the operators $\pi(f)$ can be characterized as totally characteristic pseudodifferential operators (Theorem 3), based on a certain integral transform which we now describe. As was shown in [1], there is a canonically defined chart on X given by the local structure of X around its unique closed G_0-orbit. More specifically, after fixing a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$, consider the standard parabolic subgroup $P \supset B$ of G such that G_0/P_0 is isomorphic to the closed G_0-orbit of X, and let $P = P^u L$ be its Levi decomposition with $T \subset L$. Then, the aforementioned canonical chart is given by $P^u \cdot Z \simeq P^u \times Z$ where Z is an affine L-subvariety of X isomorphic to $\mathbb{R}^r$, and r denotes the rank of X. We now introduce the mentioned integral transform in terms of the spherical roots of $\mathbf{X}$ as a mapping (Definition 4 and Proposition 3)

$$\mathcal{F}_{\text{spher}} : \mathcal{S}(P_0) \longrightarrow S^{-\infty}(P^u \cdot Z^* \times \mathbb{R}^{s+r})$$

from the Casselman-Wallach space $\mathcal{S}(P_0)$ of rapidly decreasing functions on P_0 to the space of smoothing symbols $S^{-\infty}(P^u \cdot Z^* \times \mathbb{R}^{s+r})$, where $Z^* := \{z \in Z : z_1 \cdots z_r \neq 0\}$ and s denotes the dimension of P^u.

As a consequence, the microlocal description of the convolution operators $\pi(f)$ allows us to characterize the singular nature of their kernels, and to introduce a regularized trace $\mathrm{Tr}_{reg}\, \pi(f)$ for them. This yields a distribution on G_0 which is given by (Theorem 4)

$$\Theta_\pi : \mathrm{C}^\infty_{\mathrm{c}}(G_0) \ni f \mapsto \mathrm{Tr}_{reg}\, \pi(f) \in \mathbb{C},$$

$\mathrm{C}^\infty_{\mathrm{c}}(G_0)$ denoting the space of smooth functions on G_0 with compact support. We call Θ_π the *global character* of the representation π. It should be emphasized that this distribution is given in terms of the spherical roots of $\mathbf{X}$, and therefore encodes a large part of the structure of X. We then prove that on a certain open set of transversal elements $G_0(X) \subset G_0$ the distribution Θ_π is locally integrable and given by (Theorem 5)

$$\Theta_\pi(f) = \int_{G_0(X)} f(g)\, \mathrm{Tr}^\flat\, \pi(g) d_G(g), \qquad f \in \mathrm{C}^\infty_{\mathrm{c}}(G_0(X)),$$

d_G being a Haar measure on G and $\mathrm{Tr}^\flat\,\pi(g)$ the flat trace of $\pi(g)$. Writing $\Phi_g(x) := g^{-1}\cdot x$, the latter can be expressed as a sum over fixed points

$$\mathrm{Tr}^\flat\,\pi(g) = \sum_{x\in \mathrm{Fix}(X,g)} \frac{\mathrm{Tr}\,(g : E_x \to E_x)}{|\det\,(\mathbf{1} - d\Phi_g(x))|}$$

which is manifestly invariant under conjugation.

The global characters Θ_π introduced in this paper are expected to be relevant in the context of harmonic and global analysis on spherical varieties, and the authors intend to study their properties in detail in the future. One of the questions to be dealt with is the invariance under conjugation of the distributions Θ_π on the entire group G_0. Further, it would be interesting to compare the regularized trace introduced in this paper with other existing trace concepts, to compute it for specific representations $(\pi, \mathrm{C}^\infty(X,E))$, and to examine the distributions Θ_π in light of the Harish-Chandra theory of invariant eigendistributions. We also intend to characterize the representations $(\pi, \mathrm{C}^\infty(X,E))$ and their wave front sets more closely for specific vector bundles. In a different direction, it would also be interesting to study the resolvent of a strongly elliptic operator on X together with its meromorphic continuation, and to examine the action of discrete subgroups on X and associated representations.

This paper is based on the local structure theorem for real loci of strict wonderful $\mathbf{G}$-varieties recently proved by Akhiezer and Cupit-Foutou [1], and generalizes results already obtained by Parthasarathy and Ramacher [17] for the Oshima compactification of a Riemannian symmetric space, as well as earlier work of Ramacher [18]. Actually, the mentioned local structure theorem provides a natural framework in which the previous results can be understood in a conceptually simple way.

2. Wonderful varieties

Throughout this article we shall adopt the convention of writing complex objects with boldface letters and the corresponding real objects with ordinary ones. Let G be the split real form of a connected reductive complex algebraic group $\mathbf{G}$ of rank l, and let $\sigma : \mathbf{G} \to \mathbf{G}$ be the anti-holomorphic involution defining the split real form G, so that $G = \mathbf{G}^\sigma = \{g \in \mathbf{G} : \sigma(g) = g\}$. In particular, G is a real reductive group. Since G is not necessarily connected, denote by G_0 the identity component of G. Fix a maximal algebraic torus $\mathbf{T} \simeq (\mathbb{C}^*)^l$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing it. We further assume that $\mathbf{B}$ and $\mathbf{T}$ are both σ-stable. We recall the definition of a wonderful $\mathbf{G}$-variety.

Definition 1. ([14]) *An algebraic* $\mathbf{G}$*-variety* $\mathbf{X}$ *is called* wonderful of rank r *if*

1. $\mathbf{X}$ *is projective and smooth;*
2. $\mathbf{X}$ *admits an open* $\mathbf{G}$*-orbit whose complement consists of a finite union of smooth prime divisors* $\mathbf{X}_1, \dots, \mathbf{X}_r$ *with normal crossings;*
3. *the* $\mathbf{G}$*-orbit closures of* $\mathbf{X}$ *are given by the partial intersections of the* $\mathbf{X}_i$*.*

In particular, notice that $\mathbf{X}$ has a unique closed, hence projective $\mathbf{G}$-orbit, given by the intersection of all prime divisors $\mathbf{X}_i$. Moreover, the $\mathbf{G}$-variety $\mathbf{X}$ is spherical that is, $\mathbf{B}$ has an open orbit in $\mathbf{X}$ [14].

Further, recall that a real structure on $\mathbf{X}$ is an involutive anti-holomorphic map $\mu : \mathbf{X} \to \mathbf{X}$; it is said to be *$\sigma$-equivariant* if $\mu(g \cdot x) = \sigma(g) \cdot \mu(x)$ for all $(g, x) \in \mathbf{G} \times \mathbf{X}$. Crucial for the ensuing analysis is the existence of a unique σ-equivariant real structure (called canonical) on some wonderful varieties. Wonderful varieties whose points have self-normalizing stabilizers are called *strict*. For such varieties one has the following

Theorem 1. [1] *Let $\mathbf{X}$ be a strict wonderful $\mathbf{G}$-variety. Then*

1. *there exists a unique σ-equivariant real structure μ on $\mathbf{X}$;*
2. *the real locus X of $(\mathbf{X}, \mu)$ is not empty, and constitutes a smooth projective real algebraic G_0-variety with finitely many G_0-orbits, finitely many open B-orbits and a unique closed G_0-orbit.*

□

Examples of real loci of strict wonderful varieties include the Oshima-Sekiguchi compactification of a Riemannian symmetric space; such a compactification can be realized as the real locus of the De-Concini-Procesi wonderful compactification of the complexification of the given symmetric space, up to a finite quotient, see [7, Chapter 8, Section II.14].

In what follows, let X be the real locus of a strict wonderful $\mathbf{G}$-variety of rank r equipped with its canonical σ-equivariant real structure. Let Y denote the unique closed G_0-orbit of X, and consider the parabolic subgroup $P = \mathbf{P}^\sigma \supset B = \mathbf{B}^\sigma$ of G such that $Y \simeq G_0/P_0$. Let $P = P^u L$ be the Levi decomposition of P with $T = \mathbf{T}^\sigma \subset L$. Notice that P^u is connected.

The following local structure theorem describes the structure of the real locus X locally around Y, and will be essential for everything that follows. It constitutes the real analogue of Theorem 1.4 in [9].

Theorem 2. [1, Section 5] *There exists a real algebraic L-subvariety Z of X such that*

1. *the natural mapping*
$$P^u \times Z \to P^u \cdot Z$$
is a P^u-equivariant isomorphism;
2. *each G_0-orbit in X contains points of Z;*
3. *the commutator of L acts trivially on Z; furthermore, Z is an affine T-variety isomorphic to $\mathbb{R}^r$ acted upon by linearly independent characters of T.*

□

Let $\gamma_1, \dots, \gamma_r$ be the characters of T mentioned in Theorem 2. These weights are usually called the *spherical roots of* $\mathbf{X}$. The T-action on $Z \simeq \mathbb{R}^r$ is then given explicitly by

$$t \cdot z = (\gamma_1(t) z_1, \dots, \gamma_r(t) z_r) \qquad \text{for all} \quad z = (z_1, \dots, z_r) \in Z \quad \text{and} \quad t \in T. \tag{1}$$

Note that P^u acts on $P^u \cdot Z$ by multiplication from the left on P^u. Further, since L normalizes P^u, L acts on $P^u \cdot Z$ by setting

$$l \cdot (p_u, z) = (l p_u l^{-1}, l z) \in P^u \times Z \simeq P^u \cdot Z, \qquad (p_u, z) \in P^u \times Z, \, l \in L, \tag{2}$$

while (L, L) acts trivially on Z. Consequently, P acts on $P^u \cdot Z$. By Theorem 2, $P^u \cdot Z$ is an open subset of X isomorphic to $\mathbb{R}^s \times \mathbb{R}^r \simeq \mathbb{R}^{s+r}$, P^u being diffeomorphic to $\mathbb{R}^s$ for some s, and $G_0 \, P^u \cdot Z = X$. We can therefore cover X by the G_0-translates

$$U_g := g \cdot U_e, \qquad U_e := P^u \cdot Z, \quad g \in G_0.$$

Consequently, there exist a real-analytic diffeomorphism

$$\varphi : \quad \mathbb{R}^{s+r} \quad \longrightarrow \quad P^u \times Z \simeq P^u \cdot Z$$

and real-analytic diffeomorphisms φ_g

$$\varphi_g : \quad \mathbb{R}^{s+r} \quad \xrightarrow{\varphi} \quad P^u \cdot Z \quad \xrightarrow{g} \quad g\, P^u \cdot Z, \qquad g \in G_0,$$

such that $\left\{(U_g, \varphi_g^{-1})\right\}_{g\in G_0}$ constitutes an atlas of X. More explicitly, if z_j denotes the j-th coordinate function on $Z \simeq \mathbb{R}^r$ and p_j the j-th coordinate function on $P^u \simeq \mathbb{R}^s$, we write

$$\varphi_g^{-1} : U_g \ni x \longmapsto (p_1, \dots, p_s, z_1, \dots, z_r) = y \in \mathbb{R}^{s+r}. \tag{3}$$

Note that U_g is invariant under the subgroups gTg^{-1} and gP^ug^{-1}. In the following, U_e will be called the *canonical chart*.

Next, let $g \in G_0$, $x \in U_g$, and $h \in G_0$ be such that $h \cdot x \in U_g$. From the orbit structure and the analyticity of X one immediately deduces

$$z_j(h \cdot x) = \chi_j(h, x) z_j(x), \tag{4}$$

where $\chi_j(h, x)$ is a function that is real-analytic in h and x that does not vanish. We are interested in a more explicit description of the functions $\chi_j(h, x)$.

Corollary 1. *For any $t \in T$, $u \in P^u$, $x \in U_g$, and $j = 1, \dots, r$ we have*

(a) $z_j(gtg^{-1} \cdot x) = \chi_j(gtg^{-1}, x) z_j(x) = \gamma_j(t) z_j(x)$,
(b) $z_j(gug^{-1} \cdot x) = z_j(x)$.

Proof. Assertion (a) follows readily from (1) and the definition of the open sets U_g. Indeed, let $x = gp \cdot z \in U_g$ and $t \in T$. Then

$$\varphi_g^{-1}(gtg^{-1} \cdot x) = \varphi^{-1}(tp \cdot z) = \varphi^{-1}(tpt^{-1}, t \cdot z),$$

so that the z_j-coordinate of $gtg^{-1} \cdot x$ reads $\gamma_j(t) z_j(x)$. Assertion (b) is a direct consequence of Theorem 2. □

From the classification results of [8] (see precisely the list of Section 5 therein) and [19] (Theorem A), one immediately infers that every wonderful $\mathbf{G}$-variety $\mathbf{X}$ whose $\mathbf{T}$-fixed-points are located on its closed $\mathbf{G}$-orbit is strict. Even if we shall not restrict our attention to these varieties later, we would like to close this section by mentioning that for such varieties one can construct a more refined atlas than the one given above. Indeed, denote by

$$W := N_G(T)/Z_G(T)$$

the Weyl group of G with respect to T, and write $(U_w, \varphi_w^{-1}) := (U_{n_w}, \varphi_{n_w}^{-1})$ for any element $w \in W$, $n_w \in N_G(T)$ being a representative of w. Note that by definition of the Weyl group U_w is independent of the representative n_w. Since $n_w T n_w^{-1} = T$, U_w carries a natural T-action. We then have the following

Proposition 1. *Suppose that $\mathbf{X}$ is a wonderful $\mathbf{G}$-variety such that its $\mathbf{T}$-fixed-points are located on its closed $\mathbf{G}$-orbit. Then*

$$\left\{(U_w, \varphi_w^{-1})\right\}_{w\in W}$$

constitutes a finite atlas of X.

Proof. Let B^- denote the Borel subgroup of G such that $B \cap B^- = T$. The variety X has a unique projective G-orbit and, hence, a unique point fixed by B^- [1]. This fixed point, denoted in the following by y_0, lies in the closed G-orbit by assumption. Next, let $\eta : s \mapsto (s^{a_1}, ..., s^{a_l})$, $a_i > 0$, be a morphism from $\mathbb{C}^*$ to the algebraic torus $\mathbf{T} \simeq (\mathbb{C}^*)^l$, such that the set of $\mathbf{T}$-fixed-points in $\mathbf{X}$ coincides with the set of fixed points of $\{\eta(s)\}_{s \in \mathbb{C}^*}$ in $\mathbf{X}$. By [6], there is a cell decomposition of $\mathbf{X}$ and, consequently, of X in terms of the sets

$$\{x \in X : \lim_{\mathbb{R}^* \ni s \to 0} \eta(s) \cdot x = y\},$$

where y runs over the set of T-fixed-points of X. Furthermore, the open subset $P^u \cdot Z \subset X$ is given by the cell

$$P^u \cdot Z = \{x \in X : \lim_{\mathbb{R}^* \ni s \to 0} \eta(s) \cdot x = y_0\},$$

see [1] for details. By assumption, all T-fixed-points belong to the closed G-orbit of X. On the other side, it is well-known that the T-fixed-points of a projective G-orbit are indexed by the Weyl group W. More specifically, for each such y there exists a $w \in W$ such that $y = n_w y_0$ for any representative $n_w \in N_G(T)$ of w. Noticing that the aforementioned cells are just contained in the W-translates of $P^u \cdot Z$, one finally obtains the proposition. □

Remark 1. The atlas from the previous proposition is a generalization of that constructed by Oshima for the compactification of a Riemannian symmetric space of non-compact type, see [16].

3. Microlocal analysis of integral operators on wonderful varieties

With the notation as in Section 2, let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{C}$ of rank l with split real form (G, σ). Let $\mathbf{X}$ be a strict wonderful $\mathbf{G}$-variety of rank r and X the real locus of $\mathbf{X}$ with respect to the canonical real structure on it. Consider now a real differentiable G-vector bundle E on X of rank d and the corresponding regular representation of G_0 on the space of smooth sections $\mathrm{C}^\infty(X, E)$ of E given by

$$\pi(g)s(x) = g \cdot [s(g^{-1} \cdot x)], \qquad x \in X,\, g \in G_0,\, s \in \mathrm{C}^\infty(X, E).$$

Let $(L, \mathrm{C}^\infty(G_0))$ be the left-regular representation of G_0 and θ a Cartan involution on $\mathfrak{g}$. With respect to the left-invariant Riemannian metric on G_0 given by the modified Cartan-Killing form

$$\langle A, B \rangle_\theta := - \langle A, \theta B \rangle, \qquad A, B \in \mathfrak{g},$$

we denote by $d(g, h)$ the distance between two points $g, h \in G_0$, and set $|g| = d(g, e)$, where e is the identity element of G. A function f on G_0 is said to be of *of moderate growth*, if there exists a $\kappa > 0$ such that $|f(g)| \le Ce^{\kappa |g|}$ for some constant $C > 0$. Let further d_G be a Haar measure on G, and denote by $\mathfrak{U}$ the universal envelopping algebra of the complexification of the Lie algebra $\mathfrak{g}$ of G. We introduce now the *Casselman-Wallach space* of rapidly decreasing functions on G_0 [10, 21].

Definition 2. *A function $f \in \mathrm{C}^\infty(G_0)$ is called* rapidly decreasing *if it satisfies the following condition: For every $\kappa \ge 0$ and $H \in \mathfrak{U}$ there exists a constant $C > 0$ such that*

$$|dL(H)f(g)| \le Ce^{-\kappa |g|}.$$

The space of rapidly decreasing functions on G_0 will be denoted by $\mathcal{S}(G_0)$.

Remark 2. 1) Note that $f \in \mathcal{S}(G_0)$ implies that for every $\kappa \geq 0$ and $H \in \mathfrak{U}$ one has

$$dL(H)f \in \mathrm{L}^1(G_0, e^{\kappa|g|}d_G).$$

Indeed, let $c > 0$ be such that $e^{-c|g|} \in \mathrm{L}^1(G_0, d_G)$, and $\kappa \geq 0$ and $X \in \mathfrak{U}$ be given. Then $|e^{(\kappa+c)|g|}dL(X)f(g)| \leq C$ for all $g \in G_0$ and a suitable constant $C > 0$, so that

$$\left\|dL(X)fe^{\kappa|\cdot|}\right\|_{\mathrm{L}^1(G_0,d_G)} \leq C\left\|e^{-c|\cdot|}\right\|_{\mathrm{L}^1(G_0,d_G)} < \infty.$$

2) If $f \in \mathcal{S}(G_0)$, $dR(X)f \in \mathcal{S}(G_0)$. Furthermore, if one compares the space $\mathcal{S}(G_0)$ with the Fréchet spaces $\mathscr{S}_{a,b}(G_0)$ defined in [20, Section 7.7.1], where a and b are smooth, positive, K-bi-invariant functions on G_0 satisfying certain properties, one easily sees that $a(g) = e^{|g|}$ and $b(g) = 1$ satisfy the selfsame properties, except for the smoothness at $g = e$ and the K-bi-invariance of a. Besides, it should be noticed that the space $\mathcal{S}(G)$ is different from the Schwartz space introduced by Harish-Chandra [10]. In our context, the introduction of the space $\mathcal{S}(G_0)$ was motivated by the study of strongly elliptic operators and the decay properties of the semigroups generated by them [18].

Consider next for each $f \in \mathcal{S}(G_0)$ the linear operator

$$\pi(f) : \mathrm{C}^\infty(X,E) \longrightarrow \mathrm{C}^\infty(X,E), \quad (\pi(f)s)(x) = \int_{G_0} f(g)\, g \cdot [s(g^{-1}\cdot x)]\, d_G(g) \in E_x. \tag{5}$$

It becomes a continuous map when endowing $\mathrm{C}^\infty(X,E)$ with the Fréchet topology of uniform convergence, and its Schwartz kernel is a distribution section $\mathcal{K}_f \in \mathcal{D}'(X \times X, E \boxtimes E')$, where $E' = E^* \times \Omega_X$, and Ω_X denotes the density bundle on X. Observe that the restriction of $\pi(f)s$ to any of the G_0-orbits depends only on the restriction of $s \in \mathrm{C}^\infty(X,E)$ to that orbit. Let X_0 be an open orbit in X. The main goal of this section is to describe the microlocal structure of the operators $\pi(f)$, and characterize them as totally characteristic pseudodifferential operators on the manifold with corners $\overline{X_0}$. Recall that according to Melrose [15] a continuous linear map

$$Q : \mathrm{C}^\infty_{\mathrm{c}}(M) \quad \longrightarrow \quad \mathrm{C}^\infty(M)$$

on a smooth manifold with corners M is called a *totally characteristic pseudodifferential operator or order* $m \in \mathbb{R}$ if it can be written in local charts as an oscillatory integral

$$Q_{loc}u(y) := \int e^{i\langle y,\xi\rangle} q(y,\xi)\hat{u}(\xi)\, đ\xi, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n,k}),$$

where $\hat{u}$ denotes the Fourier transform of u and $\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}$ the standard manifold with corners with $0 \leq k \leq n$ and coordinates $y = (y_1, \dots, y_k, y')$, while $đ\xi = (2\pi)^{-n}\, d\xi$. The amplitude q is supposed to be of the form $q(y,\xi) = \widetilde{q}(y, y_1\xi_1, \dots, y_k\xi_k, \xi')$, where $\widetilde{q}(y,\xi)$ is a symbol of order m satisfying in addition the *lacunary condition*

$$\int e^{i(1-t)\xi_j}\widetilde{q}(y,\xi)\, d\xi_j = 0 \qquad \text{for } t < 0 \text{ and } 1 \leq j \leq k. \tag{6}$$

Conceptually, the algebra of totally characteristic pseudodifferential operators arises from the algebra of totally characteristic differential operators, which is generated by the vector fields tangential to the boundary $\partial\,\mathbb{R}^{n,k}$

$$y_j\frac{\partial}{\partial\, y_j}, \quad 1 \leq j \leq k, \qquad \frac{\partial}{\partial\, y_i}, \quad k+1 \leq i \leq n,$$

compare also [11, Section 18.3]. Similarly, if E and F are vector bundles over M, a continuous linear map

$$Q : \mathrm{C}^\infty_\mathrm{c}(M, E) \quad \longrightarrow \quad \mathrm{C}^\infty(M, F)$$

is called a *totally characteristic pseudodifferential operator of order* $m \in \mathbb{R}$, if for every open subset $U \subset M$ and trivializations

$$\tau_E : E_{|U} \to U \times \mathbb{R}^{d_E}, \qquad \tau_F : F_{|U} \to U \times \mathbb{R}^{d_F},$$

there is a $(d_F \times d_E)$-matrix of totally characteristic pseudodifferential operators Q_{ij} of order m such that

$$(\tau_F \circ (Qs)_{|U})_i = \sum_j Q_{ij}(\tau_E \circ s)_j, \qquad s \in \mathrm{C}^\infty_\mathrm{c}(U; E).$$

In this case, one says that Q is of class L^m_b. For a more detailed exposition on totally characteristic pseudodifferential operators the reader is referred to [17].

3.1. **The toric case.** To make the essential ideas behind our approach as clear as possible, we shall first consider the simplest case, namely the toric one, and restrict ourselves to the left-regular scalar representation. Thus, let $\mathbf{T} = (\mathbb{C}^*)^r$ be an algebraic torus, $\mathbf{Z} = \mathbb{C}^r$, and let $\mathbf{T}$ act effectively on $\mathbf{Z}$ through

$$t \cdot z = (\gamma_1(t)z_1, \dots, \gamma_r(t)z_r) \qquad z = (z_1, \dots, z_r) \in \mathbf{Z}, \quad \mathbf{t} \in \mathbf{T}, \tag{7}$$

where the $\gamma_i(t)$ are linearly independent characters of $\mathbf{T}$, and as such given in terms of monomials with real coefficients. The corresponding action of $T = \mathbf{T}^\sigma$ on the real locus $Z = \mathbb{R}^r$ is given by (1). Next, let $(\nu, \mathrm{C}_0(Z))$ be the continuous left-regular representation of T_0 on the Banach space of continuous functions on Z vanishing at infinity given by

$$(\nu(t)\varphi)(z) = \varphi(t^{-1} \cdot z), \qquad \varphi \in \mathrm{C}_0(Z), \quad t \in T_0.$$

We would like to describe for each $f \in \mathcal{S}(T_0)$ the continuous linear operator

$$\nu(f) : C_0(Z) \supset \mathrm{C}^\infty_\mathrm{c}(Z) \longrightarrow \mathrm{C}^\infty(Z) \subset \mathcal{D}'(Z), \qquad \nu(f) = \int_{T_0} f(t)\, \nu(t)\, d_T(t)$$

as a pseudodifferential operator on $Z = \mathbb{R}^r$ using Fourier analysis, $d_T(t) = (t_1 \dots t_r)^{-1}\, dt$ being Haar measure on T. For this, let $v \in \mathrm{C}^\infty_\mathrm{c}(Z)$. Applying the inverse Fourier transform one computes

$$\begin{aligned}\nu(f)v(z) &= \int_{T_0} f(t)v(t^{-1} \cdot z) d_T(t) = \int_{T_0} \left[\int_{\mathbb{R}^r} e^{i\left\langle t^{-1} \cdot z, \xi \right\rangle} \hat{v}(\xi) đ\xi \right] f(t)\, d_T(t) \\ &= \int_{\mathbb{R}^r} e^{i\langle z, \xi \rangle} q_f(z, \xi) \hat{v}(\xi) đ\xi,\end{aligned}$$

where

$$q_f(z, \xi) := e^{-i\langle z, \xi \rangle} \int_{T_0} f(t) e^{i\left\langle t^{-1} \cdot z, \xi \right\rangle}\, d_T(t)$$

represents the symbol of $\nu(f)$, and constitutes a polynomially growing function in ξ for general z due to the non-transitivity of the T-action on Z. Now, observe that the fundamental vector fields of the T-action on Z are given by linear combinations of the differential operators

$$z_j \frac{\partial}{\partial z_j}, \qquad j = 1, \dots, r,$$

which correspond to vector fields tangential to the divisor $\{z \in Z : z_1 \cdots z_r = 0\}$. Therefore, it is to be expected that $\nu(f)$ constitutes a totally characteristic pseudodifferential operator on each of the 2^r-tants in $\mathbb{R}^r$, for which we have to verify that

$$\widetilde{q}_f(\xi) := q_f(z, \xi_1/z_1, \dots, \xi_r/z_r) = e^{-i\langle(1,\dots,1),\, \xi\rangle} \int_{T_0} f(t) e^{i\left\langle(\gamma_1(t^{-1}),\dots,\gamma_r(t^{-1})),\, \xi\right\rangle}\, d_T(t) \tag{8}$$

defines a lacunary symbol of order $-\infty$. That is, we have to show that $\widetilde{q}_f(\xi)$ satisfies the lacunary condition (6) and that for any $N \in \mathbb{N}$ and arbitrary multi-indices α there exists a constant $C_{N,\alpha}$ such that

$$|(\partial_\xi^\alpha\, \widetilde{q}_f)(\xi)| \leq \frac{1}{(1+|\xi|^2)^N} C_{N,\alpha} \qquad \xi \in \mathbb{R}^r. \tag{9}$$

In this way, we are led to the following

Definition 3. *Let*

$$\mathcal{F}_{\text{spher}} : \mathcal{S}(T_0) \ni f \mapsto \mathcal{F}_{\text{spher}}(f)(\xi) := \int_{T_0} f(t) e^{i\left\langle(\gamma_1(t^{-1}),\dots,\gamma_r(t^{-1})),\, \xi\right\rangle}\, d_T(t) \in \mathrm{C}^\infty(\mathbb{R}^r).$$

Note that $\mathcal{F}_{\text{spher}}$ is invariant under conjugation, that is, for arbitrary $s \in T_0$ we have

$$\mathcal{F}_{\text{spher}}(\iota_s^* f)(\xi) = \int_{T_0} f(t^{-1}) e^{i\left\langle(\gamma_1(s^{-1}t^{-1}s),\dots,\gamma_r(s^{-1}t^{-1}s)),\, \xi\right\rangle}\, d_T(t) = \mathcal{F}_{\text{spher}}(f)(\xi),$$

where $\iota_s : t \mapsto sts^{-1}$ denotes conjugation in T_0. Now, in order to prove that the auxiliary symbol $\widetilde{q}_f(\xi) = e^{-i(\xi_1+\dots+\xi_r)}\mathcal{F}_{\text{spher}}(f)(\xi)$ satisfies (9) we will actually show that

$$\mathcal{F}_{\text{spher}} : \mathcal{S}(T_0) \longrightarrow \mathcal{S}(\mathbb{R}^r), \tag{10}$$

where $\mathcal{S}(\mathbb{R}^r)$ denotes the usual Schwartz space on $\mathbb{R}^r$. In the same way as one verifies that the usual Fourier transform defines a mapping from $\mathcal{S}(\mathbb{R}^r)$ into itself, we shall use partial integration to do so, and exploit the fact that in the definition of $\mathcal{F}_{\text{spher}}(f)$ only the transitive action of T_0 on $Z_+^* := (\mathbb{R}_*^+)^r$ is considered. We shall only outline here the main steps, which will be carried out in detail within the more general context of Section 3.2. Thus, setting $\psi_\xi(t) := \langle(\gamma_1(t),\dots,\gamma_r(t)),\, \xi\rangle$ one computes

$$\frac{\partial}{\partial t_j} e^{i\psi_\xi(t)} = i e^{i\psi_\xi(t)} \sum_{i=1}^r \Gamma_{ji}\, \xi_i, \qquad \Gamma_{ji} = \frac{\partial \gamma_i}{\partial t_j}(t). \tag{11}$$

Since the matrix $\Gamma = \{\Gamma_{ji}\}$ is non-singular for any $t \in T$ due to the fact that the characters γ_i are linearly independent, one can express any polynomial in ξ by a linear combination of t-derivatives of $e^{i\psi_\xi(t)}$. We recall now the following integration formula.

Proposition 2. *Let G be a real reductive group and G_0 the component of the identity. Let $f_1 \in \mathcal{S}(G_0)$, and assume that $f_2 \in \mathrm{C}^\infty(G_0)$, together with all its derivatives, is of* *moderate growth. Further, let* $\{\mathcal{X}_1, \dots, \mathcal{X}_{\dim \mathfrak{g}}\}$ *denote a basis of the Lie algebra of* G*. Then, for arbitrary multiindices* γ*,*

$$\int_{G_0} f_1(g)[dL(\mathcal{X}^\gamma) f_2](g) d_{G_0}(g) = (-1)^{|\gamma|} \int_{G_0} [dL(\mathcal{X}^{\tilde{\gamma}}) f_1](g) f_2(g) d_{G_0}(g), \tag{12}$$

where we wrote $\mathcal{X}^\gamma = \mathcal{X}_{i_1}^{\gamma_1} \dots \mathcal{X}_{i_r}^{\gamma_r}$, $\mathcal{X}^{\tilde{\gamma}} = \mathcal{X}_{i_r}^{\gamma_r} \dots \mathcal{X}_{i_1}^{\gamma_1}$.

Proof. See [12, Lemma 10.22] or [18, Proposition 1]. □

By taking into account that f is rapidly decreasing and integrating $\mathcal{F}_{\mathrm{spher}}(f)$ according to the previous proposition we obtain (10). Since (6) is a direct consequence[1] of the orbit structure of the T-action on Z, it follows that $\nu(f)$ is a totally characteristic pseudodifferential operator of order $-\infty$ on each of the 2^r-tants of $Z = \mathbb{R}^r$.

3.2. **The parabolic case.** Next, let P be the standard parabolic subgroup of G such that G_0/P_0 is isomorphic to the unique closed G_0-orbit, and let $P = P^u L$ be its Levi decomposition with $T \subset L$. Writing $L = S \cdot (L, L)$, the Langlands decomposition of P reads

$$P = MAN, \qquad M := (L, L), \quad A := S, \quad N := P^u \simeq \mathbb{R}^s$$

in standard terminology. Note that $T = AT'$, where T' is the maximal torus of M contained in T, and $\dim A = r$.[2] Recall that P acts on $P^u \cdot Z$ as in (2); in particular, Z is acted upon trivially by the commutator of L, while A acts as in the toric case.

We shall again consider the scalar-valued case, and restrict ourselves to the description of the continuous linear operators

$$\nu(f) : \mathrm{C}_c^\infty(P^u \cdot Z) \longrightarrow \mathrm{C}^\infty(P^u \cdot Z), \quad \nu(f) = \int_{P_0} f(p)\, \nu(p) d_P(p), \quad f \in \mathcal{S}(P_0),$$

in the canonical chart $P^u \cdot Z \subset X$, where $(\nu, C_0(P^u \cdot Z))$ denotes the left-regular representation of P on the Banach space $C_0(P^u \cdot Z)$ of continuous functions on $P^u \cdot Z$ vanishing at infinity. In view of the local structure theorem we identify $P^u \cdot Z$ with $P^u \times Z \simeq \mathbb{R}^{s+r}$ in this subsection. With respect to these isomorphisms the action of $P = MAN$ on $P^u \cdot Z$ is given by

$$\begin{aligned} \varphi^{-1} : (man)^{-1} p_u \cdot z = n^{-1}(ma)^{-1} p_u ma\, a^{-1} \cdot z &\longmapsto (n^{-1}(ma)^{-1} p_u ma, a^{-1} \cdot z) \\ &\longmapsto (p_1(n^{-1}(ma)^{-1} p_u ma), \dots, \gamma_1(a^{-1}) z_1, \dots, \gamma_r(a^{-1}) z_r), \end{aligned} \tag{13}$$

compare (1)-(3). Now, for a function $v \in \mathrm{C}_c^\infty(P^u \cdot Z)$ one has

$$\nu(f) v(p_u \cdot z) = \int_{P_0} f(p) v(p^{-1} p_u \cdot z)\, dp = \int_{P_0} \left[\int_{\mathbb{R}^{s+r}} e^{i \langle \varphi^{-1}(p^{-1} p_u \cdot z), \xi \rangle} \hat{v}(\xi)\, đ\xi \right] f(p)\, dp,$$

where $\hat{v}(\xi)$ denotes the Fourier transform of v as a function on $P^u \cdot Z \simeq P^u \times Z \simeq \mathbb{R}^{s+r}$. Introducing the phase function

$$\begin{aligned} \psi_{p_u \cdot z, \xi}(m, a, n) :&= \langle \varphi^{-1}(nam\, p_u \cdot z), \xi \rangle \\ &= \big\langle \big(p_1(n(am) p_u (am)^{-1}), \dots, \gamma_1(a) z_1, \dots, \gamma_r(a) z_r\big), \xi \big\rangle \end{aligned} \tag{14}$$

[1] See end of Section 3.2 for details.

[2] Notice that in Section 3.1 we assumed $\mathbf{T}$ to act effectively on $\mathbf{Z}$, so that we have $T = A$ there.

and using the integration formulas for real reductive groups we obtain for $P_0 = M_0 A_0 N$

$$\nu(f)v(p_u \cdot z)$$
$$= \int_{M_0\times A_0\times N} \left[\int_{\mathbb{R}^{s+r}} e^{i\psi_{p_u\cdot z,\xi}(m^{-1},a^{-1},n^{-1})}\hat{v}(\xi)\, đ\xi\right] f(man)a^{-2\varrho}\, dm\, da\, dn$$
$$= \int_{\mathbb{R}^{s+r}} e^{i\langle \varphi^{-1}(p_u\cdot z),\xi\rangle} q_f(p_u\cdot z,\xi)\hat{v}(\xi) đ\xi,$$

where we set

$$q_f(p_u\cdot z,\xi) := e^{-i\langle\varphi^{-1}(p_u\cdot z),\xi\rangle}\int_{M_0\times A_0\times N} f(man)e^{i\psi_{p_u\cdot z,\xi}(m^{-1},a^{-1},n^{-1})}a^{-2\varrho}\, dm\, da\, dn.$$

Here $\varrho \in \mathfrak{a}^*$ is given by $\varrho(\mathcal{A}) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}\mathcal{A}_{|\mathfrak{n}})$ and $a^{-2\varrho} = \exp(-2\varrho(\mathcal{A}))$ if $a = \exp\mathcal{A} \in A_0$, $\mathcal{A} \in \mathfrak{a}$, compare [20, Section 2.4]. Next, consider the z-independent auxiliary symbol

$$\begin{aligned}\widetilde{q}_f(p_u,\xi) :&= q_f(p_u\cdot z,(\xi_1,\dots,\xi_s,\xi_{s+1}/z_1,\dots \xi_{s+r}/z_r))\\ &= e^{-i\langle(p_1,\dots,p_s,1,\dots,1),\xi\rangle}\int_{M_0\times A_0\times N} f(man)e^{i\psi_{p_u,\xi}(m^{-1},a^{-1},n^{-1})}a^{-2\varrho}\, dm\, da\, dn,\end{aligned}$$

where

$$\psi_{p_u,\xi}(m,a,n) := \psi_{p_u\cdot(1,\dots,1),\xi}(m,a,n).$$

We now arrive at the following

Definition 4. *Let $\psi_{p_u\cdot z,\xi}$ be defined by (14). We then define the mapping*[3]

$$\mathcal{F}_{\text{spher}} : \mathcal{S}(P_0) \longrightarrow \mathrm{C}^\infty(P^u\cdot Z\times\mathbb{R}^{s+r}),$$
$$f \longmapsto \mathcal{F}_{\text{spher}}(f)(p_u\cdot z,\xi) = \int_{M_0\times A_0\times N} f(man)e^{i\psi_{p_u\cdot z,\xi}(m^{-1},a^{-1},n^{-1})}a^{-2\varrho}\, dm\, da\, dn.$$

Notice that $\mathcal{F}_{\text{spher}}$ is given in terms of the spherical roots of $\mathbf{X}$ which, together with the standard parabolic subgroup $\mathbf{P} \subset \mathbf{G}$, are the combinatorial objects that characterize $\mathbf{X}$. Furthermore,

$$Z^* \simeq \{a \in A : \gamma_i(a) \neq 0 \quad \forall\, 1 \leq i \leq r\}.$$

Of course, $\mathcal{F}_{\text{spher}}$ can be written simply as an integral over P_0, but using the Langlands decomposition of P_0 the spherical roots in $\mathcal{F}_{\text{spher}}$ become manifest. Next, we have the following crucial

Proposition 3. *The transform $\mathcal{F}_{\text{spher}}$ defines a linear map from the Casselman-Wallach space $\mathcal{S}(P_0)$ to the space of symbols $S^{-\infty}(P^u\cdot Z^*\times\mathbb{R}^{s+r})$,*

$$\mathcal{F}_{\text{spher}} : \mathcal{S}(P_0) \longrightarrow S^{-\infty}(P^u\cdot Z^*\times\mathbb{R}^{s+r}),$$

where $Z^* := \{z \in Z : z_1\dots z_r \neq 0\}$.

---

[3]From our point of view, it would be natural to call $\mathcal{F}_{\text{spher}}$ the *spherical Fourier transform of f*. But since this will probably lead to confusion with other transforms in literature that are called similarly, like the *spherical transform*, which is defined for K-bi-invariant functions on a locally compact Lie group, or the *Helgason-Fourier transform*, which is defined for right K-invariant functions on a connected non-compact semisimple Lie group with finite center, K being a maximal compact subgroup, we desisted from doing so.

As in Section 3.1, we would like to use the integration formula of Proposition 2 to prove Proposition 3. But now we have to consider also the action of AN on P^u besides the action of A on Z. Indeed, in analogy to (11) one proves

Lemma 1. *Let $\{\mathcal{N}_1,\dots,\mathcal{N}_s\}$ and $\{\mathcal{A}_1,\dots,\mathcal{A}_r\}$ be bases for the Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$ of N and A, respectively. Further, assume that $p_u\cdot z\in P^u\cdot Z^*$. Then*

$$\begin{pmatrix} dL(\mathcal{N}_1)e^{i\psi_{p_u\cdot z,\xi}} \\ \vdots \\ dL(\mathcal{A}_r)e^{i\psi_{p_u\cdot,\xi}} \end{pmatrix}(m,a,n) = ie^{i\psi_{p_u\cdot z,\xi}(m,a,n)}\,\Gamma(p_u,z,m,a,n)\cdot\xi, \tag{15}$$

where

$$\Gamma(p_u,z,m,a,n)=\left(\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Gamma_3 & \Gamma_4\end{array}\right)=\left(\begin{array}{c|c} dL(\mathcal{N}_i)p_{j,p_u}(m,a,n) & 0 \\ \hline dL(\mathcal{A}_i)p_{j,p_u}(m,a,n) & dL(\mathcal{A}_i)\gamma_j(a)z_j\end{array}\right)$$

belongs to $\mathrm{GL}(s+r,\mathbb{R})$*, and we wrote $p_{j,p_u}(m,a,n)=p_j(n(am)p_u(am)^{-1})$.*

Proof. For $\mathcal{A}\in\mathfrak{a}$ one computes

$$\begin{aligned} dL(\mathcal{A})\psi_{p_u\cdot z,\xi}(m,a,n) &= \frac{d}{d\varepsilon}\psi_{p_u\cdot z,\xi}(m,\,\mathrm{e}^{-\varepsilon\mathcal{A}}\,a,n)_{|\varepsilon=0}\\ &=\sum_{j=1}^{s}\xi_j dL(\mathcal{A})p_{j,x}(m,a,n)+\sum_{j=s+1}^{s+r}\xi_j dL(\mathcal{A})\gamma_j(a)z_j, \end{aligned}$$

and similarly for $\mathcal{N}\in\mathfrak{n}$, showing (15). In particular, Γ_2 is identically zero. To see the invertibility of the matrix $\Gamma=\Gamma(p_u,z,m,a,n)$ note that, like the matrix $\{\Gamma_{ij}\}$ in (11), the matrix Γ_4 is non-singular because of the linear independence of the spherical roots γ_j, provided that $z\in Z^*$. Further, due to the transitivity of the N-action on P^u, the matrix Γ_1 is non-singular, too. Thus, we conclude that Γ is non-singular. □

Proof of Proposition 3. Let $p_u\cdot z\in P^u\cdot Z^*$. As a consequence of the previous lemma one can express any polynomial in ξ as a linear combination of derivatives in $\mathfrak{n}$ and $\mathfrak{a}$ of $e^{i\psi_{p_u\cdot z,\xi}}$. More precisely, consider the extension of $\Gamma=\Gamma(p_u,z,m,a,n)$, regarded as an endomorphism in $\mathbb{C}^1[\mathbb{R}^{s+r}_\xi]$, to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}^{s+r}_\xi])\simeq\mathbb{C}[\mathbb{R}^{s+r}_\xi]$. By Lemma 1 the matrix Γ is invertible, so its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}^{s+r}_\xi])$ is an automorphism, too. We regard the polynomials $\xi_1,\dots,\xi_{s+r}$ as a basis in $\mathbb{C}^1[\mathbb{R}^{s+r}_\xi]$, and denote the image of the basis vector ξ_i under the endomorphism Γ by $\Gamma\xi_i$. Every monomial $\xi_{i_1}\otimes\cdots\otimes\xi_{i_N}\equiv\xi_{i_1}\dots\xi_{i_N}$ can then be written as a linear combination

$$\xi^\alpha=\sum_\beta\Lambda^\alpha_\beta(p_u,z,m,a,n)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_{|\alpha|}},$$

where the $\Lambda^\alpha_\beta(p_u,z,m,a,n)$ are C^∞ functions on $P^u\times Z\times M\times A\times N$ which are of moderate growth in m,a,n. Taking (15) into account, a simple computation yields for arbitrary indices $\beta_1, \dots, \beta_j$ and elements $\mathcal{X}_i \in \mathfrak{a} \oplus \mathfrak{n}$

$$i^j e^{i\psi_{p_u\cdot z,\xi}(m,a,n)} \Gamma\xi_{\beta_1} \cdots \Gamma\xi_{\beta_j} = dL(\mathcal{X}_{\beta_1} \cdots \mathcal{X}_{\beta_j}) e^{i\psi_{p_u\cdot z,\xi}(m,a,n)} + \sum_{j'=1}^{j-1} \sum_{\alpha_1,\dots,\alpha_{j'}} d^{\beta_1,\dots,\beta_j}_{\alpha_1,\dots,\alpha_{j'}}(p_u, z, m, a, n) dL(\mathcal{X}_{\alpha_1} \cdots \mathcal{X}_{\alpha_{j'}}) e^{i\psi_{p_u\cdot z,\xi}(m,a,n)},$$

where the coefficients $d^{\beta_1,\dots,\beta_j}_{\alpha_1,\dots,\alpha_{j'}}$ are smooth and of moderate growth in m, a, n, as well as independent of ξ. Thus, for arbitrary $\widetilde{N} \in \mathbb{N}$ one obtains

$$e^{i\psi_{p_u\cdot z,\xi}(m,a,n)} = (1+|\xi|^2)^{-\widetilde{N}} \sum_{j=0}^{2\widetilde{N}} \sum_{|\alpha|=j} b^N_\alpha(p_u, z, m, a, n) dL(\mathcal{X}^\alpha) e^{i\psi_{p_u\cdot z,\xi}(m,a,n)}, \tag{16}$$

with $\mathcal{X}^\alpha \in \mathfrak{U}(\mathfrak{a} \oplus \mathfrak{n})$ and coefficients $b^N_\alpha(p_u, z, m, a, n)$ that are of moderate growth in m, a, n. Similarly, $a^{-2\varrho}$ is of moderate growth. Since f is rapidly decreasing, integrating $\mathcal{F}_{\text{spher}}(f)(p_u \cdot z, \xi)$ by parts according to Proposition 2 with respect to $N \times A$ yields for any $\widetilde{N} \in \mathbb{N}$, any compact subset $\mathcal{K} \subset P^u \cdot Z^*$, and arbitrary multi-indices α and β the existence of a constant $C_{\alpha,\beta,\mathcal{K}}$ such that

$$|(\partial^\alpha_\xi \, \partial^\beta_{p_u,z} \, \mathcal{F}_{\text{spher}}(f))(p_u \cdot z, \xi)| \leq \frac{1}{(1+|\xi|^2)^{\widetilde{N}}} C_{\alpha,\beta,\mathcal{K}} \qquad p_u \cdot z \in \mathcal{K},\ \xi \in \mathbb{R}^{s+r},$$

thus proving Proposition 3. $\square$

Remark 3. The proof of Proposition 3 is modelled on the proof of [18, Theorem 4] and [17, Theorem 2], where the integral transform $\mathcal{F}_{\text{spher}}$ was not explicitly introduced yet, but is tacitly present.

From Proposition 3 we now infer that

$$\widetilde{q}_f(p_u, \xi) = e^{-i\langle (p_1,\dots,p_s,1,\dots,1),\, \xi\rangle} \mathcal{F}_{\text{spher}}(f)(p_u \cdot (1, \dots, 1), \xi)$$

defines a z-independent symbol in $S^{-\infty}(P_u \cdot Z, \mathbb{R}^{s+r})$. It remains to verify that it satisfies the lacunary condition (6) which, as we will see, is a direct consequence of the orbit structure of the P-action on $P^u \cdot Z$. Indeed, by the previous proposition one clearly has

$$q_f(p_u \cdot z, \xi) = e^{-i\left\langle \varphi^{-1}(p_u\cdot z),\, \xi\right\rangle} \mathcal{F}_{\text{spher}}(f)(p_u \cdot z, \xi) \in \mathrm{S}^{-\infty}(P^u \cdot Z^* \times \mathbb{R}^{s+r}_\xi),$$

so that $\nu(f)_{|P^u\cdot Z^*}$ represents a pseudodifferential operator of class $\mathrm{L}^{-\infty}$ on $P^u \cdot Z^*$. Furthermore, the Schwartz kernel of $\nu(f)_{|P^u\cdot Z^*}$ is given by the oscillatory integral

$$\int e^{i\left\langle \varphi^{-1}(p_u\cdot z) - \varphi^{-1}(p'_u\cdot z'),\cdot\xi\right\rangle} q_f(p_u \cdot z, \xi) \,đ\xi, \qquad z \in Z^*.$$

Now, due to the nature of the P-action on $P^u \cdot Z$, the restriction of $\nu(f)v$ to one of the 2^r-tants $P^u \cdot Z_{\pm\cdots\pm}$, where $Z_{\pm\cdots\pm} := \left\{z \in Z : z_1 \gtreqless 0, \dots, z_r \gtreqless 0\right\}$, only depends on the restriction of v to the selfsame 2^r-tant, so that necessarily

$$\operatorname{supp} K_{\nu(f)} \subset (P^u \cdot Z_{+\cdots+} \times P^u \cdot Z_{+\cdots+}) \cup \cdots \cup (P^u \cdot Z_{-\cdots-} \times P^u \cdot Z_{-\cdots-}),$$

where $K_{\nu(f)} \in \mathcal{D}'(P^u \cdot Z \times P^u \cdot Z)$ denotes the Schwartz kernel of $\nu(f)$ as a continuous linear operator from $\mathrm{C^\infty_c}(P^u \cdot Z)$ to $\mathrm{C}^\infty(P^u \cdot Z)$. Consequently, the integrals

$$\int e^{i(z_j - z_j')\xi_{s+j}} \widetilde{q}_f(p_u, (\xi_1, \dots, \xi_s, z_1\xi_{s+1}, \dots, z_r\xi_{s+r}))\, đ\xi_{s+j}, \qquad 1 \leq j \leq r,$$

which are C^∞-functions on $P^u \cdot Z^* \times P^u \cdot Z^*$, must vanish if z_j and z_j' do have different signs. For $z, z' \in Z^*$ we can perform the substitutions $z_j\xi_{s+j} \mapsto \xi_j$ and write $t = z_j'/z_j - 1$, thus arriving at the conditions

$$\int e^{-it\xi_j} \widetilde{q}_f(p_u, \xi)\, đ\xi_j = 0 \qquad \text{for } 1 \leq j \leq r, \quad t < -1, \quad p_u \in P^u.$$

But these conditions no longer depend on z, meaning that $\widetilde{q}_f(p_u, \xi)$ satisfies the lacunary condition (6) on the whole chart $P^u \cdot Z$. Actually, this condition precisely encodes its orbit structure. Thus, we have shown that $\nu(f)$ is a totally characteristic pseudodifferential operator of order $-\infty$ on the canonical chart $P^u \cdot Z$.

3.3. **The general case.** After these considerations, we are ready to deal with the general case. Thus, let $\mathbf{G}$ be a connected reductive complex algebraic group and $\mathbf{X}$ a strict wonderful $\mathbf{G}$-variety of rank r, and consider the operators (5). Choose for each $x \in X$ open neighbourhoods $\mathcal{U}_x \subset \mathcal{U}_x'$ of x contained in U_g for some $g \in G_0$ depending on x. Since X is compact, we can take a finite sub-covering of the open covering $\{\mathcal{U}_x\}_{x\in X}$ to obtain a finite atlas $\big\{(\mathcal{U}_\varrho, \varphi_\varrho^{-1})\big\}_{\varrho \in R}$ on X, where $\varphi_\varrho = \varphi_{g_\varrho}$ for a suitable $g_\varrho \in G_0$. In addition, assume that the subsets $\mathcal{U}_\varrho'$ have been chosen such that one has the trivializations

$$\tau_E^\varrho : E_{|\mathcal{U}_\varrho'} \longrightarrow \mathcal{U}_\varrho' \times \mathbb{R}^d.$$

One then computes for $s \in \mathrm{C^\infty_c}(\mathcal{U}_\varrho, E)$, $(\tau_E^\varrho \circ s)(x) = (x, e_\varrho(x))$,

$$\begin{aligned}(\tau_E^\varrho \circ (\pi(f)s)_{|\mathcal{U}_\varrho})(x) &= \int_{G_0} f(h)\tau_E^\varrho(h \cdot s(h^{-1}\cdot x))\, d_G(h)\\ &= \Big(x, \int_{G_0} f(h)\, \mathcal{M}_\varrho(h, h^{-1}\cdot x)[e_\varrho(h^{-1}\cdot x)]\, d_G(h)\Big),\end{aligned}$$

where $\mathcal{M}_\varrho(h,x)$ denotes the linear map on $\mathbb{R}^d$ induced by

$$\mathbb{R}^d \simeq \{h^{-1}\cdot x\} \times \mathbb{R}^d \quad \xrightarrow{(\tau_E^\varrho)^{-1}} \quad E_{h^{-1}\cdot x} \quad \xrightarrow{h} \quad E_x \quad \xrightarrow{\tau_E^\varrho} \quad \{x\} \times \mathbb{R}^d \simeq \mathbb{R}^d.$$

We are therefore left with the task of examining the $(d \times d)$-matrix of scalar-valued integrals

$$\int_{G_0} f(h)\, \mathcal{M}_\varrho(h,x)_{ij}[e_\varrho(h^{-1}\cdot x)]_i\, d_G(h), \tag{17}$$

where the components of $\mathcal{M}_\varrho(h,x)$ and of $e_\varrho(h^{-1}\cdot x)$ are given in terms of some fixed basis of $\mathbb{R}^d$. In particular, it is sufficient to consider the scalar case, so that we are left with the description of the convolution operators

$$\nu(f) : C(X) \supset \mathrm{C^\infty_c}(X) \longrightarrow \mathrm{C}^\infty(X) \subset \mathcal{D}'(X), \quad \nu(f) = \int_{G_0} f(h)\, \nu(h) d_G(h), \quad f \in \mathcal{S}(G_0),$$

$(\nu, C(X))$ being the left-regular representation of G on the Banach space $C(X)$ of continuous functions on X. For this, let $v \in \mathrm{C^\infty_c}(g\, P^u \cdot Z)$ be given by $v = u \circ \varphi_g^{-1}$, where $u \in \mathrm{C^\infty_c}(\mathbb{R}^{s+r})$ and $g \in G_0$. By the unimodularity of G_0 one computes for $f \in \mathcal{S}(G_0)$

$$
\begin{aligned}
(\nu(f)v)(g\, p_u \cdot z) &= \int_{G_0} f(h)\, v(h^{-1} g\, p_u \cdot z)\, dh = \int_{G_0} f(ghg^{-1}) v((ghg^{-1})^{-1} g\, p_u \cdot z)\, dh \\
&= \int_{G_0} (L_{g^{-1}} R_{g^{-1}} f)(h) (u \circ \varphi^{-1})(h^{-1} p_u \cdot z)\, dh.
\end{aligned}
$$

Thus, the description of $\nu(f)$ in the chart $U_g = g\, P^u \cdot Z$ is reduced to its study in the canonical chart $U_e = P^u \cdot Z \simeq P^u \times Z \simeq \mathbb{R}^{s+r}$. In the analysis of the integrals (17) we can therefore assume that $\mathcal{U}_\varrho$ is contained in the chart $P^u \cdot Z$. Let

$$
G_0 = K P_0 = K M_0 A_0 N
$$

be the Cartan decomposition of G_0. Besides the action of P, which leaves $P^u \cdot Z$ invariant, we have to consider now also the action of K, which does not leave $P^u \cdot Z$ invariant. In view of (13), for $k \in K$ close to the identity we have

$$
\varphi^{-1} : (kman)^{-1} p_u \cdot z \overset{\simeq}{\longmapsto} (p_1(n^{-1}(ma)^{-1} p_u^k ma), \dots, \gamma_1(a^{-1}) z_1^k, \dots), \tag{18}
$$

where $k^{-1} p_u \cdot z = p_u^k \cdot z^k$ for some $(p_u^k, z^k) \in P^u \times Z$. Let therefore $\alpha_\varrho \in \mathrm{C^\infty_c}(\mathcal{U}_\varrho)$ and $\bar\alpha_\varrho$ be another function satisfying $\bar\alpha_\varrho \in \mathrm{C^\infty_c}(\mathcal{U}'_\varrho)$, $\bar\alpha_{\varrho|\mathcal{U}_\varrho} \equiv 1$. Assume that $v = u \circ \varphi^{-1} \in \mathrm{C^\infty_c}(\mathcal{U}_\varrho)$. With the integration formulas for real reductive groups we obtain for $p_u \cdot z \in \mathcal{U}_\varrho$

$$
\begin{aligned}
\nu(f) v(p_u \cdot z) &= \int_{G_0} f(h) (\bar\alpha_\varrho v)(h^{-1} p_u \cdot z)\, d_G(h) \\
&= \int_{K \times M_0 \times A_0 \times N} \left[\int_{\mathbb{R}^{s+r}} e^{i\psi_{p_u^k \cdot z^k, \xi}(m^{-1}, a^{-1}, n^{-1})} \hat{u}(\xi)\, đ\xi \right] \\
&\quad \cdot \bar\alpha_\varrho((kman)^{-1} p_u \cdot z) f(kman) a^{-2\varrho}\, dk\, dm\, da\, dn \\
&= \int_{\mathbb{R}^{s+r}} e^{i\langle \varphi^{-1}(p_u \cdot z), \xi \rangle} q_f^\varrho(p_u \cdot z, \xi) \hat{u}(\xi) đ\xi,
\end{aligned}
$$

where $\psi_{p_u^k \cdot z^k, \xi}$ was defined in (14) and we set

$$
\begin{aligned}
q_f^\varrho(p_u \cdot z, \xi) :=& e^{-i\langle \varphi^{-1}(p_u \cdot z), \xi \rangle} \int_{K \times M_0 \times A_0 \times N} \bar\alpha_\varrho((kman)^{-1} p_u \cdot z) f(kman) \\
&\cdot e^{i\psi_{p_u^k \cdot z^k, \xi}(m^{-1}, a^{-1}, n^{-1})} a^{-2\varrho}\, dk\, dm\, da\, dn.
\end{aligned}
$$

To characterize $\nu(f)$ as a totally characteristic pseudodifferential operator on $\mathcal{U}_\varrho$ we consider the auxiliary symbol

$$
\widetilde{q}_f^\varrho(p_u \cdot z, \xi) := q_f^\varrho(p_u \cdot z, (\xi_1, \dots, \xi_s, \xi_{s+1}/z_1, \dots \xi_{s+r}/z_r))
$$

and note that in terms of the integral transform introduced in Section 3.2 the symbol $\widetilde{q}_f^\varrho(p_u \cdot z, \xi)$ equals

$$
e^{-i\langle (p_1, \dots, p_s, 1, \dots, 1), \xi \rangle} \int_K \mathcal{F}_{\mathrm{spher}}\big(L_{k^{-1}}(f \overline{A}_{\varrho, p_u \cdot z}) \big) \big(p_u \cdot (\chi_1(k, p_u \cdot z), \dots, \chi_r(k, p_u \cdot z)), \xi \big)\, dk,
$$

where we wrote $\overline{A}_{\varrho,p_u\cdot z}(h) := \overline{\alpha}_\varrho(h^{-1}p_u\cdot z)$, and took into account (4), by which $z_j^k = \chi_j(k,p_u\cdot z)z_j$. If we now apply Proposition 3, we see that

$$\mathcal{F}_{\text{spher}}(L_{k^{-1}}(f\overline{A}_{\varrho,p_u\cdot z}))(p_u\cdot(\chi_1(k,p_u\cdot z),\dots,\chi_r(k,p_u\cdot z)),\xi)\text{ is rapidly decaying in }\xi,$$

since $(\chi_1(k,p_u\cdot z),\dots,\chi_r(k,p_u\cdot z))\in Z^*$. Integrating over K then yields the desired statement $\widetilde{q}^\varrho_f(p_u\cdot z,\xi)\in S^{-\infty}(P_u\cdot Z,\mathbb{R}^{s+r})$, everything being absolutely convergent. Finally, the argument at the end of Section 3.2 that showed that $\widetilde{q}_f(p_u,\xi)$ satisfies the lacunarity condition (6) also proves that $\widetilde{q}^\varrho_f(p_u\cdot z,\xi)$ is lacunary. Thus, we have shown the main result of this section.

Theorem 3. *Let* $\mathbf{G}$ *be a connected reductive algebraic group over* $\mathbb{C}$ *and* (G,σ) *a split real form of* $\mathbf{G}$. *Let* X *be the real locus of a strict wonderful* $\mathbf{G}$*-variety* $\boldsymbol{X}$, E *a smooth real* G*-vector bundle over* X, *and* $(\pi,\mathrm{C}^\infty(X,E))$ *the regular representation of* G_0. *Let* X_0 *be an open* G_0*-orbit in* X *and* $f\in\mathcal{S}(G_0)$. *Then the continuous linear operator*

$$\pi(f)_{|\overline{X_0}}:\mathrm{C}^\infty_{\mathrm{c}}(\overline{X_0},E)\longrightarrow\mathrm{C}^\infty(\overline{X_0},E),$$

is a totally characteristic pseudodifferential operator of class $\mathrm{L}^{-\infty}_b$ *on the manifold with corners* $\overline{X_0}$.

□

Remark 4. Note that if in the previous theorem X_0 is a Riemannian symmetric space, then its closure $\overline{X_0}$ in X is the maximal Satake compactification of X_0, see [7, Remark II.14.10].

For later computations, we will require explicit descriptions of the kernel of $\pi(f)$ in the different charts of X. Thus, let $\{\alpha_\varrho\}_{\varrho\in R}$ be a partition of unity subordinate to the atlas $\{(\mathcal{U}_\varrho,\varphi_\varrho^{-1})\}_{\varrho\in R}$ and let $\{\bar\alpha_\varrho\}_{\varrho\in R}$ be another set of functions satisfying $\bar\alpha_\varrho\in\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{U}'_\varrho)$, $\bar\alpha_{\varrho|\mathcal{U}_\varrho}\equiv 1$. Fix a chart $\mathcal{U}_\varrho\subset g\,P^u\cdot Z$ with $g\in G_0$, and let $v\in\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{U}_\varrho)$ be given by $v=u\circ\varphi_g^{-1}$, where $u\in\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{s+r})$. We now consider the localization of the integrals (17)

$$\begin{aligned}({}^{ij}Q^\varrho_f u)(y) &:= \int_{G_0} f(h)\,\mathcal{M}_\varrho(h,gp_u\cdot z)_{ij}(\bar\alpha_\varrho v)(h^{-1}gp_u\cdot z)\,d_G(h)\\ &= \int_{G_0} f(ghg^{-1})c^{ij}_\varrho(g,h,p_u\cdot z)(u\circ\varphi^{-1})(h^{-1}p_u\cdot z)\,d_G(h),\end{aligned}$$

where we wrote $y=(p,z)=\varphi_g^{-1}(gp_u\cdot z)=\varphi^{-1}(p_u\cdot z)$ and put $c^{ij}_\varrho(g,h,p_u\cdot z):=\mathcal{M}_\varrho(ghg^{-1},gp_u\cdot z)_{ij}\bar\alpha_\varrho(gh^{-1}p_u\cdot z)$. If we now define

$$\begin{aligned}{}^{ij}q^\varrho_f(y,\xi) := e^{-i\langle y,\xi\rangle}\int_{K\times M_0\times A_0\times N} & c^{ij}_\varrho(g,kman,p_u\cdot z)f(gkmang^{-1})\\ &\cdot e^{i\psi_{p_u^k\cdot z^k,\xi}(m^{-1},a^{-1},n^{-1})}a^{-2\varrho}\,dk\,dm\,da\,dn\end{aligned}$$

we obtain

$$({}^{ij}Q^\varrho_f)u(y)=\int_{\mathbb{R}^{s+r}}e^{i\langle y,\xi\rangle}\;{}^{ij}q^\varrho_f(y,\xi)\,\hat u(\xi)đ\xi. \tag{19}$$

By our previous considerations in this subsection,

$$
\begin{aligned}
{}^{ij}\widetilde{q}^{\varrho}_f(y,\xi) := {}&{}^{ij}q^{\varrho}_f(y,(\xi_1,\dots,\xi_s,\xi_{s+1}/z_1,\dots,\xi_{s+r}/z_r))\\
={}&e^{-i\langle (y_1,\dots,y_s,1,\dots,1),\xi\rangle}\int_{K\times M_0\times A_0\times N} c^{ij}_{\varrho}(g,kman,p_u\cdot z)f(gkmang^{-1})\\
&\cdot e^{i\left\langle (p_1(n^{-1}(ma)^{-1}p_u^k(ma)),\dots,\gamma_1(a^{-1})\chi_1(k,p_u\cdot z),\dots),\xi\right\rangle}a^{-2\varrho}\,dk\,dm\,da\,dn
\end{aligned}
\tag{20}
$$

is a lacunary symbol of order $-\infty$. Further, for $f\in\mathcal{S}(G_0)$, the restriction of $\pi(f)$ to $\mathcal{U}_\varrho$ is given by the $(d\times d)$-matrix of operators (19). In particular, the kernel of $\pi(f)$ is determined by its restriction to $\{y=(p,z)\in\mathbb{R}^{s+r}: z_1\cdots z_r\neq 0\}\times\{y=(p,z)\in\mathbb{R}^{s+r}: z_1\cdots z_r\neq 0\}$, and given by the matrix of oscillatory integrals

$$
\begin{pmatrix}
K_{{}^{11}Q^{\varrho}_f}(y,y') & \dots & K_{{}^{1d}Q^{\varrho}_f}(y,y')\\
\vdots & \ddots & \vdots\\
K_{{}^{d1}Q^{\varrho}_f}(y,y') & \dots & K_{{}^{dd}Q^{\varrho}_f}(y,y')
\end{pmatrix}, \qquad y_{s+1}\cdots y_{s+r}\neq 0,
$$

where

$$
\begin{aligned}
K_{{}^{ij}Q^{\varrho}_f}(y,y') &= \int_{\mathbb{R}^{s+r}} e^{i\langle y-y',\xi\rangle}\;{}^{ij}q^{\varrho}_f(y,\xi)\,đ\xi\\
&= \frac{1}{|y_{s+1}\cdots y_{s+r}|}\int_{\mathbb{R}^{s+r}} e^{i\langle y-y',(\xi_1,\dots,\xi_s,\xi_{s+1}/y_{s+1},\dots,\xi_{s+r}/y_{s+r})\rangle}\;{}^{ij}\tilde{q}^{\varrho}_f(y,\xi)\,đ\xi\\
&= \frac{1}{|y_{s+1}\cdots y_{s+r}|}\,{}^{ij}\widetilde{Q}^{\varrho}_f\Big(y,y_1-y_1',\dots,1-\frac{y'_{s+1}}{y_{s+1}},\dots\Big),
\end{aligned}
\tag{21}
$$

and ${}^{ij}\widetilde{Q}^{\varrho}_f(y,\cdot)$ denotes the inverse Fourier transform of the lacunary symbol ${}^{ij}\tilde{q}^{\varrho}_f(y,\cdot)$. In particular, (21) shows that the kernel of $\pi(f)$ is smooth outside any neighborhood of the diagonal. The restriction of the kernel of each of the operators ${}^{ij}Q^{\varrho}_f$ to the diagonal is given by

$$
K_{{}^{ij}Q^{\varrho}_f}(y,y)=\frac{1}{|y_{s+1}\cdots y_{s+r}|}\,{}^{ij}\widetilde{Q}^{\varrho}_f(y,0), \qquad y_{s+1}\cdots y_{s+r}\neq 0.
$$

These restrictions yield a family of smooth functions ${}^{ij}\kappa^{\varrho}_f(x):=K_{{}^{ij}Q^{\varrho}_f}(\varphi_\varrho^{-1}(x),\varphi_\varrho^{-1}(x))$, which define a density ${}^{ij}\kappa_f$ on the union of the open G_0-orbits on X. Nevertheless, the functions ${}^{ij}\kappa^{\varrho}_f(x)$ are not locally integrable on all of X, so that we cannot define a trace of $\pi(f)$ by integrating

$$
\operatorname{Tr}\begin{pmatrix}
{}^{11}\kappa_f & \dots & {}^{1d}\kappa_f\\
\vdots & \ddots & \vdots\\
{}^{d1}\kappa_f & \dots & {}^{dd}\kappa_f
\end{pmatrix}
$$

over the diagonal $\Delta_{X\times X}\simeq X$. Instead, the explicit form of the local kernels (21) suggests a natural regularization for the trace of the integral operators $\pi(f)$, which will be accomplished in the next section.

4. Regularized traces and fixed point formulae

4.1. **Regularized traces.** Let the notation be as in the previous sections. In the following, we shall define a regularized trace for the convolution operators (5), based on the explicit description (21) of their kernels and a classical result of Bernstein-Gelfand on the meromorphic continuation of complex powers.

Proposition 4. *Let $\{\alpha_\varrho\}$ be a partition of unity subordinate to the atlas $\{(\mathcal{U}_\varrho, \varphi_\varrho^{-1})\}_{\varrho\in R}$. Let $f \in \mathcal{S}(G_0)$, $\zeta \in \mathbb{C}$, and define for $\operatorname{Re}\zeta > 0$*

$$\begin{aligned}\operatorname{Tr}_\zeta \pi(f) :&= \sum_{j=1}^{d}\sum_{\varrho}\int_{\mathbb{R}^{s+r}} |y_{s+1}\cdots y_{s+r}|^\zeta (\alpha_\varrho\circ\varphi_\varrho)(y)\, {}^{jj}\widetilde{Q}^\varrho_f(y,0)dy \\ &= \left\langle |y_{s+1}\cdots y_{s+r}|^\zeta, \sum_{j=1}^{d}\sum_{\varrho}(\alpha_\varrho\circ\varphi_\varrho)\, {}^{jj}\widetilde{Q}^\varrho_f(\cdot,0)\right\rangle.\end{aligned}$$

Then $\operatorname{Tr}_\zeta \pi(f)$ can be continued analytically to a meromorphic function in ζ with at most poles at $-1,-3,\dots$. Furthermore, for $\zeta \in \mathbb{C} - \{-1,-3,\dots\}$,

$$\Theta^\zeta_\pi : \mathrm{C}^\infty_\mathrm{c}(G_0) \ni f \mapsto \operatorname{Tr}_\zeta \pi(f) \in \mathbb{C}$$

defines a distribution density on G_0.

Proof. The proof is analogous to the proof of [17, Proposition 4]. In particular, the fact that $\operatorname{Tr}_\zeta \pi(f)$ can be continued meromorphically is a consequence of the analytic continuation of $|y_{s+1}\cdots y_{s+r}|^\zeta$ as a distribution in $\mathbb{R}^{s+r}$ [5].

□

Consider next the Laurent expansion of $\Theta^\zeta_\pi(f)$ at $\zeta = -1$. For this, let $u \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{s+r})$ be a test function, and consider the expansion

$$\left\langle |y_{s+1}\cdots y_{s+r}|^\zeta, u\right\rangle = \sum_{j=-J}^{\infty} S_j(u)(\zeta+1)^j,$$

where $S_j \in \mathcal{D}'(\mathbb{R}^{s+r})$. Since $|y_{s+1}\cdots y_{s+r}|^{\zeta+1}$ has no pole at $\zeta=-1$, we necessarily must have

$$|y_{s+1}\cdots y_{s+r}|\cdot S_j = 0 \quad \text{for } j<0, \qquad |y_{s+1}\cdots y_{s+r}|\cdot S_0 = 1$$

as distributions. In other words, $S_0 \in \mathcal{D}'(\mathbb{R}^{s+r})$ represents a distributional inverse of $|y_{s+1}\cdots y_{s+r}|$. Thus, we arrive at the main result of this paper.

Theorem 4. *For $f \in \mathcal{S}(G_0)$, let the regularized trace of the operator $\pi(f)$ be defined by*

$$\operatorname{Tr}_{reg} \pi(f) := \left\langle S_0, \sum_{j=1}^{d}\sum_{\varrho}(\alpha_\varrho\circ\varphi_\varrho)\, {}^{jj}\widetilde{Q}^\varrho_f(\cdot,0)\right\rangle.$$

Then $\Theta_\pi : \mathrm{C}^\infty_\mathrm{c}(G_0) \ni f \mapsto \operatorname{Tr}_{reg}\pi(f) \in \mathbb{C}$ constitutes a distribution density on G_0 which is given in terms of the spherical roots of $\mathbf{X}$. *It is called the* character of the representation $(\pi, \mathrm{C}^\infty(X,E))$.

□

Remark 5.

(1) The coordinate invariance of the defined regularized trace $\mathrm{Tr}_{reg}\,\pi(f)$ is guaranteed by standard arguments, see [2, Corollary 1].
(2) Alternatively, a similar regularized trace can be defined using the calculus of b-pseudodifferential operators developed by Melrose. For a detailed description, the reader is referred to [13, Section 6].

In what follows, we shall identify distributions with distribution densities on G via the Haar measure d_G.

4.2. **Fixed point formulae.** Our next aim is to understand the distributions Θ_π^ζ and Θ_π in terms of the G_0-action on X. We shall actually show that on a certain open set of transversal elements, they are represented by locally integrable functions given in terms of fixed points. Similar expressions where derived by Atiyah and Bott [4] for the global character of an induced representation of G_0, and we were inspired by these formulae.

Let the notation be as before, and consider for each element $g \in G$ the transformation $\Phi_g : X \to X$, $x \mapsto g^{-1} \cdot x$. Recall that Φ_g is called *transversal* if all its fixed points are *simple*, meaning that one has $\det(\mathbf{1} - (d\Phi_g)_{x_0}) \neq 0$ at each fixed point $x_0 \in X$. Further note that the set $G_0(X) := \{g \in G_0 : \Phi_g \text{ is transversal}\} \subset G_0$ of elements acting transversally on X is open. With the notation as before we then have the following

Theorem 5. *Let $f \in \mathrm{C}_c^\infty(G_0)$ have support in $G_0(X)$, and $\zeta \in \mathbb{C}$ be such that $\mathrm{Re}\,\zeta > -1$. Let further $\mathrm{Fix}(X,h)$ denote the set of fixed points on X of Φ_h, $h \in G$. Then $\mathrm{Tr}_\zeta\,\pi(f)$ is given by the expression*

$$\mathrm{Tr}_\zeta\,\pi(f) = \int_{G_0(X)} f(h) \left(\sum_{x \in \mathrm{Fix}(X,h)} \frac{\mathrm{Tr}(h : E_x \to E_x)}{|\det(\mathbf{1} - d\Phi_h(x))|} \right.$$
$$\left. \cdot \sum_{\varrho} \alpha_\varrho(x) |y_{s+1}({g_\varrho}^{-1} \cdot x) \cdots y_{s+r}({g_\varrho}^{-1} \cdot x)|^{\zeta+1} \right) d_G(h).$$

In particular, $\Theta_\pi^\zeta : \mathrm{C}_c^\infty(G_0) \ni f \to \mathrm{Tr}_\zeta\,\pi(f) \in \mathbb{C}$ is regular on $G_0(X)$.

Proof. The proof is similar to the proof of Theorem 7 in [17]. By Proposition 4,

$$\mathrm{Tr}_\zeta\,\pi(f) = \sum_{j=1}^{d} \sum_{\varrho} \int_{\mathbb{R}^{s+r}} |y_{s+1} \cdots y_{s+r}|^\zeta (\alpha_\varrho \circ \varphi_\varrho)(y)\, {}^{jj}\widetilde{Q}_f^\varrho(y,0) dy$$

is a meromorphic function in ζ with possible poles at $-1, -3, \dots$. Assume that $\mathrm{Re}\,\zeta > -1$. Since ${}^{ij}\widetilde{Q}_f^\varrho(y,0) = \int {}^{ij}\tilde{q}_f^\varrho(y,\xi) đ\xi$, where ${}^{ij}\tilde{q}_f^\varrho(y,\xi) \in \mathrm{S}_{la}^{-\infty}(\mathbb{R}^{s+r} \times \mathbb{R}^{s+r})$ is rapidly decaying in ξ, the order of integration can be interchanged, yielding

$$\mathrm{Tr}_\zeta\,\pi(f) = \sum_{j=1}^{d} \sum_{\varrho} \int_{\mathbb{R}^{s+r}} \int_{\mathbb{R}^{s+r}} |y_{s+1} \cdots y_{s+r}|^\zeta (\alpha_\varrho \circ \varphi_\varrho)(y)\, {}^{jj}\tilde{q}_f^\varrho(y,\xi) dy\, đ\xi.$$

Let $\chi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{s+r}, \mathbb{R}^+)$ be equal 1 in a neighborhood of 0 and $\varepsilon > 0$. Then, by Lebesgue's theorem on bounded convergence,

$$\operatorname{Tr}_s \pi(f) = \lim_{\varepsilon \to 0} I_\varepsilon,$$

where we set

$$I_\varepsilon := \sum_{j=1}^d \sum_\varrho \int_{\mathbb{R}^{s+r}} \int_{\mathbb{R}^{s+r}} |y_{s+1} \cdots y_{s+r}|^\zeta (\alpha_\varrho \circ \varphi_\varrho)(y)\, {}^{jj}\tilde{q}_f^\varrho(y, \xi) \chi(\varepsilon \xi)\, dy\, đ\xi.$$

In what follows, write $\varphi^h(y) := (\varphi^{-1} \circ h^{-1} \circ \varphi)(y)$, and let T_y be the diagonal $(r \times r)$-matrix with entries $y_{s+1}, \dots, y_{s+r}$. Interchanging the order of integration once more one obtains with (20)

$$\begin{aligned} I_\varepsilon = \int_{G_0} \sum_{j=1}^d \sum_\varrho f(g_\varrho h g_\varrho^{-1}) \int_{\mathbb{R}^{s+r}} \int_{\mathbb{R}^{s+r}} & e^{i\langle \Psi(h,y), \xi \rangle} \\ & \cdot c_\varrho^{jj}(g_\varrho, h, \varphi(y)) (\alpha_\varrho \circ \varphi_\varrho)(y) |y_{s+1} \cdots y_{s+r}|^\zeta \chi(\varepsilon \xi) dy\, đ\xi\, d_G(h), \end{aligned}$$

where with $\varphi(y) = p_u \cdot z$ we wrote

$$\Psi(h, y) := [(\mathbf{1}_s \otimes T_y^{-1})(\varphi^h(y) - y)] = \big(y_1(h^{-1} p_u \cdot z) - y_1, \dots, \chi_1(h^{-1}, p_u \cdot z) - 1, \dots, \big),$$

everything being absolutely convergent. Let us now write $I_\varepsilon(h)$ for the integrand of the G_0-integral in I_ε, so that $I_\varepsilon = \int_{G_0} I_\varepsilon(h)\, d_G(h)$. In order to pass to the limit under the integral, we shall show that $\lim_{\varepsilon \to 0} I_\varepsilon(h)$ is an integrable function on G_0. Now, it is not difficult to see that, as $\varepsilon \to 0$, the main contributions to $I_\varepsilon(g)$ originate from the fixed points of Φ_h. Since

$$g \cdot x \in \operatorname{Fix}(X, h) \iff x \in \operatorname{Fix}(X, g^{-1} h g), \tag{22}$$

it is sufficient to examine them in the canonical chart. To compute these contributions, note that due to the fact that all fixed points are simple, $y \mapsto \varphi^h(y) - y$ defines a diffeomorphism near fixed points. Performing successively the changes of variables $y' = y - \varphi^h(y)$ and $y'' = (\mathbf{1}_s \otimes T^{-1}_{y(\varepsilon y')}) y'$ one obtains for $\lim_{\varepsilon \to 0} I_\varepsilon(h)$ the expression

$$\sum_\varrho f(g_\varrho h g_\varrho^{-1}) \sum_{j=1}^d \sum_{x \in \operatorname{Fix}(P^u \cdot Z, h)} \frac{\alpha_\varrho(g_\varrho \cdot x) \mathcal{M}_\varrho(g_\varrho h g_\varrho^{-1}, g_\varrho \cdot x)_{jj} |y_{s+1}(x) \cdots y_{s+r}(x)|^{\zeta+1}}{|\det(\mathbf{1} - d\Phi_h(x))|},$$

where we took into account that for $x = \varphi(y) = p_u \cdot z \in \operatorname{Fix}(P^u \cdot Z, h)$ one has

$$c_\varrho^{ij}(g_\varrho, h, \varphi(y)) (\alpha_\varrho \circ \varphi_\varrho)(y) = \alpha_\varrho(g_\varrho \cdot x) \mathcal{M}_\varrho(g_\varrho h g_\varrho^{-1}, g_\varrho \cdot x)_{ij}.$$

The limit function $\lim_{\varepsilon \to 0} I_\varepsilon(g)$ is therefore clearly integrable on $G(X)$ for $\operatorname{Re} \zeta > -1$. Passing to the limit under the integral and conjugating then yields with (22)

$$\begin{aligned} \operatorname{Tr}_\zeta \pi(f) &= \lim_{\varepsilon \to 0} I_\varepsilon = \lim_{\varepsilon \to 0} \int_{G_0} I_\varepsilon(h)\, d_G(h) \\ &= \int_{G_0} f(h) \sum_{x \in \operatorname{Fix}(X, h)} \sum_{j=1}^d \sum_\varrho \frac{\alpha_\varrho(x) \mathcal{M}_\varrho(h, x)_{jj} |y_{s+1}(g_\varrho^{-1} \cdot x) \cdots y_{s+r}(g_\varrho^{-1} \cdot x)|^{\zeta+1}}{|\det(\mathbf{1} - d\Phi_h(x))|}\, d_G(h). \end{aligned}$$

Since $\sum_{j=1}^{e} \mathcal{M}_{jj}(g,x) = \mathrm{Tr}(g : E_x \to E_x)$, the assertion of the theorem follows. □

From the previous theorem it is now clear that if $f \in \mathrm{C_c^\infty}(G_0(X))$, $\mathrm{Tr}_\zeta \, \pi(f)$ is not singular at $\zeta = -1$. Consequently, we obtain

Corollary 2. *Let $f \in \mathrm{C_c^\infty}(G_0)$ have support in $G_0(X)$. Then*

$$\mathrm{Tr}_{reg}\, \pi(f) = \mathrm{Tr}_{-1}\, \pi(f) = \int_{G_0(X)} f(g) \sum_{x \in \mathrm{Fix}(X,g)} \frac{\mathrm{Tr}(g : E_x \to E_x)}{|\det(\mathbf{1} - d\Phi_g(x))|}\, d_G(g).$$

In particular, the distribution $\Theta_\pi : f \to \mathrm{Tr}_{reg}(f)$ is regular on $G_0(X)$.

Proof. By Theorem 5, $\mathrm{Tr}_\zeta \, \pi(f)$ has no pole at $\zeta = -1$. Therefore, the Laurent expansion of $\Theta^\zeta_\pi(f)$ at $\zeta = -1$ must read

$$\begin{aligned} \mathrm{Tr}_\zeta \, \pi(f) &= \left\langle |y_{s+1} \cdots y_{s+r}|^\zeta, \sum_\varrho \sum_{j=1}^d (\alpha_\varrho \circ \varphi_\varrho)^{jj} \widetilde{Q}^\varrho_f(\cdot, 0) \right\rangle \\ &= \sum_{j=0}^\infty S_j \Big(\sum_\varrho \sum_{j=1}^d (\alpha_\varrho \circ \varphi_\varrho)^{jj} \widetilde{Q}^\varrho_f(\cdot, 0) \Big) (\zeta+1)^j, \end{aligned}$$

where $S_j \in \mathcal{D}'(\mathbb{R}^{s+r})$. Thus,

$$\mathrm{Tr}_{-1}\, \pi(f) = \left\langle S_0, \sum_\varrho \sum_{j=1}^d (\alpha_\varrho \circ \varphi_\varrho)^{jj} \widetilde{Q}^\varrho_f(\cdot, 0) \right\rangle = \mathrm{Tr}_{reg}\, \pi(f),$$

and the assertion follows with the previous theorem. □

Note that from Corollary 2 it is immediate that Θ_π is independent of the chosen atlas of X and invariant under conjugation as a distribution on $G_0(X)$. Furthermore, a flat trace $\mathrm{Tr}^\flat \, \pi(g)$ of $\pi(g)$ can be defined, and as it turns out [3],

$$\mathrm{Tr}^\flat \, \pi(g) = \sum_{x \in \mathrm{Fix}(X,g)} \frac{\mathrm{Tr}(g : E_x \to E_x)}{|\det(\mathbf{1} - d\Phi_g(x))|},$$

so that we finally obtain

$$\Theta_\pi(f) = \mathrm{Tr}_{reg}\, \pi(f) = \int_{G_0(X)} f(g)\, \mathrm{Tr}^\flat \, \pi(g) d_G(g), \qquad f \in \mathrm{C_c^\infty}(G_0(X)).$$

We would like to close by noting that on $G_0(X)$ the distribution Θ_π no longer explicitly depends on the spherical roots of $\mathbf{X}$, but it of course still does on the whole group G_0.

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