MASAS AND BIMODULE DECOMPOSITIONS OF II_1 FACTORS

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Abstract. The *measure-multiplicity-invariant* for masas in II_1 factors was introduced in [10] to distinguish masas that have the same *Pukánszky invariant*. In this paper we study the measure class in the *measure-multiplicity-invariant*. This is equivalent to studying the standard Hilbert space as an associated bimodule. We characterize the type of any masa depending on the *left-right-measure* using Baire category methods (selection principle of Jankov and von Neumann). We present a second proof of Chifan's result [2] and a measure theoretic proof of the equivalence of *weak asymptotic homomorphism property* (*WAHP*) and singularity that appeared in [35].

1. Introduction

This is the first of a series of two papers developed by the author for his Phd thesis. The moral of this paper is : "*The phenomena regularity, semiregularity, singularity, weak asymptotic homomorphism property* (*WAHP*) *and asymptotic homomorphism property* (*AHP*) *of masas in finite von Neumann algebras can all be explained by measure theory*". Throughout the entire paper $\mathcal{M}$ will always denote a separable II_1 factor. Let $A \subset \mathcal{M}$ be a maximal abelian self-adjoint subalgebra henceforth abbreviated as a *masa*. It is a theorem of von Neumann that A is isomorphic to $L^\infty([0,1], dx)$. So the study of masas in type II_1 factors is understanding its position (up to automorphisms) of the ambient von Neumann algebra. For a masa $A \subset \mathcal{M}$, Dixmier in [5] defined the group of *normalizing unitaries* (or *normaliser*) of A to be the set

$$N(A) = \{u \in \mathcal{U}(\mathcal{M}) : uAu^* = A\},$$

where $\mathcal{U}(\mathcal{M})$ denotes the unitary group of $\mathcal{M}$. He called
(i) A to be *regular* (also *Cartan*) if $N(A)'' = \mathcal{M}$,
(ii) A to be *semiregular* if $N(A)''$ is a subfactor of $\mathcal{M}$,
(iii) A to be *singular* if $N(A) \subset A$.
He also exhibited the presence of all three kinds of masas in the hyperfinite II_1 factor.

Two masas A, B of $\mathcal{M}$ are said to be *conjugate* if there is an automorphism θ of $\mathcal{M}$ such that $\theta(A) = B$. If there is an unitary $u \in \mathcal{M}$ such that $uAu^* = B$ then A and B are called *unitarily* (*inner*) *conjugate*.

Feldman and Moore in [11], [12] characterized pairs $A \subset \mathcal{M}$, where A is a Cartan subalgebra, as those coming from r-discrete transitive measured groupoids with a finite measure space X as base. It is a remarkable achievement of Connes, Feldman and Weiss [3] that any countable amenable measured equivalence relation is generated by a single transformation of the underlying space. When translated into the language of operator algebras via the Feldman-Moore construction, this theorem together with a theorem of Krieger [17] says that, *if $\mathcal{M}$ is any injective von Neumann algebra then any two Cartan subalgebras are conjugate by an automorphism of $\mathcal{M}$*. However it follows from their theorem that, there are uncountably many equivalence classes of Cartan

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masas up to unitary conjugacy in the hyperfinite II_1 factor. See [23] for more examples. There exist II_1 factors with non conjugate Cartan masas (see [4]). These masas were distinguished with the presence or absence of nontrivial centralizing sequences. Recently Ozawa and Popa have exhibited examples of II_1 factors with no or at most one Cartan masa up to unitary conjugacy (see [22]).

The absence of Cartan masas in II_1 factors was first due to Voiculescu in [36]. In fact, it was his amazing discovery that, for any diffuse abelian algebra $A \subset L(\mathbb{F}_n)$, the standard Hilbert space $l^2(\mathbb{F}_n)$ as a A, A-bimodule contains a copy of $L^2(A) \otimes L^2(A)$. His result was improved by Dykema in [9] to rule out the presence of masas in free group factors with finite multiplicity.

Getting back to singular masas, in 1960 Pukánszky showed in [28] that there are countable non conjugate singular masas in the hyperfinite II_1 factor by introducing an algebraic invariant for masas in II_1 factors, today known as the *Pukánszky invariant*.

In 1983 Popa [24] succeeded in showing that all separable continuous semifinite von Neumann algebras and all separable factors of type III_λ, $0 \leq \lambda < 1$ have singular masas. Although they exist, citing explicit examples is a very hard job. In this direction, Smith and Sinclair in [33] have given concrete examples of uncountably many non conjugate singular masas in the hyperfinite II_1 factor. White and Sinclair [32] have given explicit examples of a continuous path of non conjugate singular masas (Tauer masas) in the hyperfinite II_1 factor. All the masas in this path have the same algebraic invariant of Pukánszky. Subsequently, White in [37] proved that, any possible value of the Pukánszky invariant can be realized in the hyperfinite II_1 factor, and any McDuff factor which contains a masa of Pukánszky invariant $\{1\}$ contains masas of any arbitrary Pukánszky invariant.

Singularity is often quite hard to check (see [29]). In order to check if a masa is singular analytical properties "*AHP*" and "*WAHP*" were discovered in [30], [31]. Subsequently Smith, Sinclair, White and Wiggins in [35] characterized pairs $A \subset \mathcal{M}$, where A is a singular masa in a II_1 factor $\mathcal{M}$ to be precisely those for which A satisfies "*WAHP*". All the theories that we have outlined have a common theme namely, "*What is the structure of the standard Hilbert space as a w^* A, A-bimodule.*

Although many invariants of masas in II_1 are known the first successful attempt to distinguish masas with a natural invariant, which have the same Pukánszky invariant was due to Dykema, Smith and Sinclair in [10]. We call this the *measure-multiplicity-invariant*. This invariant has two main components, a measure class and a multiplicity function. This invariant is not a new one and has existed in the literature for quite some time. For Cartan masas this invariant has very deep meaning and it is very hard to distinguish Cartan masas with this invariant. The term *multiplicity* in the *measure-multiplicity-invariant* is actually the *Pukánszky* invariant of the masa, making it a stronger invariant. A slightly different invariant was considered by Neshveyev and Størmer in [19].

Our intention is to study singular masas and distinguish them. In order to do so, it is necessary to think of singularity from a different point of view. The theory of Cartan masas and singular masas have so far been viewed from two different angles. While Cartan masas fit to the theory of orbit equivalence on one hand [12], singular masas fit to the intertwining techniques of Popa on the other [35]. But we would like to have an unique approach that explains all these phenomena. This is the primary goal of this paper. In this paper, we characterize masas by studying the structure of the standard Hilbert space as its associated bimodule.

Our second goal is to investigate that, after such a theory is outlined whether it is possible to obtain proofs of important theorems regarding masas that were obtained by a number of researchers by using different ideas. Many old theorems can indeed be proved but we will mainly prove Chifan's result on tensor products [2] and the equivalence of WAHP and singularity [35]. In fact, it seems that studying the bimodule is the most natural way to approach these problems as one can exploit a lot of results from Real Analysis.

In order to distinguish singular masas which have the same multiplicity understanding the measure in the *measure-multiplicity-invariant* is the most important task. So we study this invariant thoroughly throughout this article. The second paper will contain explicit calculations of the invariant and questions related to conjugacy of masas.

We have learned latter that Popa and Shylakhtenko in [26] has results of similar flavor in this direction. However our way of approaching is completely different. We think that what is really involved in understanding the types of masas are the *measurable selection principle* of *Jankov and von Neumann* and some generalized version of Dye's theorem on groupoid normalisers. This is evident from [3], [11] and [12]. We present completely measure theoretic proofs based upon Baire category methods (selection principle). As an outcome of our approach many theorems related to structure theory of masas that were proved by different techniques just follows easily from our technique.

Singular masas are often constructed by considering weakly or strongly mixing actions of infinite abelian groups on finite von Neumann algebras. We will show that the definition of WAHP can be strengthened by considering Haar unitaries and Cesàro sums which exactly resembles the definition of weakly mixing actions. Weakly mixing actions are characterized by null sets of certain measures. The story for singular masas is also similar.

This paper is heavily measure theoretic. Much of the measure theory tools we require are scattered here and there in the literature. This article is organized as follows. In Sec. 2 we present some preliminaries of direct integrals, masas and define the *measure-multiplicity-invariant*. In Sec. 3 we study disintegration of measures and masas. Sec. 4 deals with generalized versions of Dye's theorem. Sec. 5 contains the main result i.e the characterization theorem and a second proof of Chifan's normaliser formula. This is a very technical section. Sec. 6 contains results on calculating certain two-norms and a second proof of the equivalence of WAHP and singularity. Sec. 4 uses the theory of L^1 and L^2 spaces associated to finite von Neumann algebras for which we have cited related results in that section without proofs. Appendix A contains structure theorems of measurable functions satisfying condition (N) of Lusin which is used in Sec. 5.

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2. Preliminaries

The paper relies on the theory of direct integrals. So we have divided this section into three subsections. In the first subsection we give some well known results about direct integrals of Hilbert spaces with respect to an abelian von Neumann algebra. In the second part we give some preliminaries about masas in II_1 factors and in the third subsection we will define the *measure-multiplicity-invariant* of masas in II_1 factors.

Notation: Throughout the entire article $\mathbb{N}_\infty$ will denote the set $\mathbb{N}\cup\{\infty\}$.

2.1. Direct Integrals.

Let a separable Hilbert space $\mathcal{H}$ be the direct integral of a μ-measurable field of

Hilbert spaces $\{\mathcal{H}_x\}_{x\in X}$ over the base space (X,μ) where X is a σ-compact space and μ is a positive, complete Borel measure.

Definition 2.1. An operator $T\in\mathbf{B}(\mathcal{H})$ is said to be *decomposable* relative to the decomposition $\mathcal{H}\cong\int_X^{\oplus}\mathcal{H}_x d\mu(x)$ if there exists a μ-measurable field of operators $T_x\in\mathbf{B}(\mathcal{H}_x)$, such that $x\mapsto\|T_x\|\in L^\infty(X,\mu)$ and $T=\int_X^{\oplus}T_x d\mu(x)$.
If $T_x=c(x)I_{\mathcal{H}_x}$, where $c(x)\in\mathbb{C}$ for almost all x, then T is said to be *diagonalizable.*

It is easy to see that the fibres of a *decomposable* operator are uniquely determined up to an almost sure equivalence. The collection of *diagonalizable* and *decomposable* operators both form von Neumann subalgebras of $\mathbf{B}(\mathcal{H})$, with the later being the commutant of the former. Whenever there is no danger of confusion we will use the term measurable instead of μ-measurable.

Theorem 2.2. *Let $A\subset\mathbf{B}(\mathcal{H})$ be a diffuse abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then there exists a measure space (X,μ), where X is a σ-compact space, μ is a positive, Borel, non-atomic, complete measure on X and a measurable field of Hilbert spaces $\{\mathcal{H}_x\}_{x\in X}$, such that $\mathcal{H}$ is unitarily equivalent to,*

$$\mathcal{H}\cong\int_X^{\oplus}\mathcal{H}_x d\mu(x) \tag{2.1}$$

and A is (unitarily equivalent to) the algebra of diagonalizable operators on $\int_X^{\oplus}\mathcal{H}_x d\mu(x)$ with respect to this decomposition.

The dimension function of the decomposition in Thm. 2.2 is defined as

$$m:X\mapsto\mathbb{N}_\infty \text{ by, } m(x)=dim(\mathcal{H}_x).$$

The dimension function m is μ-measurable. Such results are known in greater generality. For a measure space (X,μ) we denote by $[\mu]$ the equivalence class of measures on X that are mutually absolutely continuous with respect to μ. This decomposition in Thm. 2.2 and hence the *multiplicity function* is unique up to measure equivalence from Thm. 3, 4 of Chapter 6 of [6].

We will be always working with finite measures. Since direct integrals of Hilbert spaces does not change when the measures are scaled, we will most of the time assume that the measures have total mass 1. Details of these facts can be found in [15], [20].

2.2. **Basics on Masas in II_1 factors.**

Definition 2.3. Given a type I von Neumann algebra $\mathcal{B}$ we shall write Type($\mathcal{B}$) for the set of all those $n\in\mathbb{N}_\infty$ such that $\mathcal{B}$ has a nonzero component of type $\mathrm{I_n}$.

Let $\mathcal{M}$ be a separable II_1 factor with the faithful, normal, tracial state τ. This trace induces the two-norm $\|x\|_2=\tau(x^*x)^{1/2}$ on $\mathcal{M}$ and we write $L^2(\mathcal{M})$ for the Hilbert space completion of $\mathcal{M}$ with respect to this norm. Let $\mathcal{M}$ act on $L^2(\mathcal{M})$ via left multiplication. Let J denote the anti-unitary conjugation operator on $L^2(\mathcal{M})$ obtained by extending the densely defined map $J(x)=x^*$. Inclusions of von Neumann algebras will always be assumed to be unital until further notice.

Given a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, let $\mathbb{E}_{\mathcal{N}}$ be the unique trace preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. This conditional expectation is obtained by restricting the orthogonal projection $e_{\mathcal{N}}$ from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})$ to $\mathcal{M}$.

Let $A\subset\mathcal{M}$ be a masa. Then the augmented algebra $\mathcal{A}=(A\cup JAJ)''$ is an abelian algebra, with a type I commutant, the commutant being taken in $\mathbf{B}(L^2(\mathcal{M}))$ and the center of $\mathcal{A}'$ is $\mathcal{A}$. The Jones projection e_A onto $L^2(A)$ lies in $\mathcal{A}$ [34]. Hence, $\mathcal{A}'(1-e_A)$

decomposes as,

$$\mathcal{A}'(1-e_A) = \oplus_{n\in\mathbb{N}_\infty}\mathcal{A}'P_n \tag{2.2}$$

where $P_n \in \mathcal{A}$ are orthogonal projections summing up to $1-e_A$ and $\mathcal{A}'P_n$ is homogenous algebra of type n whenever $P_n \neq 0$.

Lemma 2.4. *If $A \subset \mathcal{M}$ be a masa and $B \subseteq \mathcal{M}$ be any subalgebra, then $(A \cup JBJ)''$ is diffuse.*

Definition 2.5. The *Pukánszky invariant* of a masa A in II$_1$ factor $\mathcal{M}$, denoted by $Puk(A)$ (or $Puk_{\mathcal{M}}(A)$ when the containing factor is ambiguous) is $\{\, n \in \mathbb{N}_\infty : P_n \neq 0\}$ which is precisely Type$(\mathcal{A}'(1-e_A))$.

Definition 2.6. If A is an abelian von Neumann subalgebra of $\mathcal{M}$, let $\mathcal{GN}(A)$ or $\mathcal{GN}(A,\mathcal{M})$ be the *normalising groupoid*, consisting of those partial isometries $v \in \mathcal{M}$ that satisfy $v^*v, vv^* \in A$ and $vAv^* = Avv^* = vv^*A$.

A theorem of Dye [7] says that, a partial isometry $v \in \mathcal{GN}(A)$ if and only if there is an unitary $u \in N(A)$ and a projection $p \in A$ such that $v = up = (upu^*)u$. Thus $\mathcal{GN}(A)'' = N(A)''$. Popa in [25] connected the *Pukánszky invariant* to the type of a masa showing that if $1 \notin Puk(A)$, then A is singular and that the *Pukánszky invariant* of a Cartan masa is $\{1\}$.

Singularity is difficult to verify. The following two conditions were introduced in [31], [30] and [35] as they imply singularity and are often easier to verify in explicit situations.

Definition 2.7. (Smith, Sinclair) Let A be a masa in a II$_1$ factor $\mathcal{M}$.
(i) A is said to have the *asymptotic homomorphism property* (AHP) if there exists an unitary $v \in A$ such that

$$\lim_{|n|\to\infty} \|\mathbb{E}_A(xv^ny) - \mathbb{E}_A(x)v^n\mathbb{E}_A(y)\|_2 = 0 \text{ for all } x, y \in \mathcal{M}.$$

(ii) A has the *weak asymptotic homomorphism property* (WAHP) if, for each $\epsilon > 0$ and each finite subset $x_1, \cdots, x_n \in \mathcal{M}$ there is an unitary $u \in A$ such that

$$\left\|\mathbb{E}_A(x_iux_j^*) - \mathbb{E}_A(x_i)u\mathbb{E}_A(x_j^*)\right\|_2 < \epsilon \text{ for } 1 \leq i, j \leq n.$$

In [35] it was shown that singularity is equivalent to WAHP. We will prove in Sec. 6 that WAHP is indeed the most natural property. The next proposition is well known, we state it for completeness.

Proposition 2.8. *Let $\mathcal{N} \subseteq \mathbf{B}(\mathcal{H})$ be a von Neumann algebra and let $x_{i,j} \in \mathcal{N}$ and $x'_{i,j} \in \mathcal{N}'$ for $i, j = 1, 2, \cdots, n$. Then the following conditions are equivalent:*
(i) $\sum_{k=1}^n x_{i,k}x'_{k,j} = 0$ *for all* $1 \leq i, j \leq n$.
(ii) *There exist elements $z_{i,j} \in \mathbf{Z}(\mathcal{N})$, $i, j = 1, 2, \cdots, n$ such that for all i, j*

$$\sum_{k=1}^n x_{i,k}z_{k,j} = 0, \;\; \sum_{k=1}^n z_{i,k}x'_{k,j} = x'_{i,j}.$$

2.3. **Measure-Multiplicity-Invariant.**

We consider the *conjugacy invariant* for a masa A in a II$_1$ factor $\mathcal{M}$ derived from writing the direct integral decomposition of its left-right action. More precisely, we choose a compact Hausdorff space Y such that $C(Y) \subset A$, is a norm separable unital C^* subalgebra and $C(Y)$ is *w.o.t* dense in A. τ restricted to $C(Y)$ gives rise to a probability measure ν on Y so that A is isomorphic to $L^\infty(Y,\overline{\nu})$, with $\overline{\nu}$ a completion of ν. For simplicity of notation we will use the same symbol ν to denote its completion.

Now $a \otimes b \mapsto aJb^*J$, a, $b \in C(Y)$ extends to an injective $*$-homomorphism π of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Indeed, as $\mathcal{M}$ is a factor so the map,

$$\sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i J b_i^* J$$

is injective by Prop. 2.8. Hence it induces a norm on $C(Y) \otimes_{alg} C(Y)$. Since abelian C^* algebras are nuclear, this norm must be the min norm, and therefore $a \otimes b \mapsto aJb^*J$ extends to an injective representation of $C(Y) \otimes C(Y)$ in $L^2(\mathcal{M})$. Therefore $C(Y \times Y)$ is a *w.o.t* dense unital subalgebra of $\mathcal{A}$, so that $\mathcal{A}$ is isomorphic to $L^\infty(Y \times Y, \eta_{Y \times Y})$ for a complete, positive, Borel measure $\eta_{Y\times Y}$. By Lemma 2.4, $\eta_{Y\times Y}$ is non-atomic.

Remark 2.9. In general, if we allow $\mathcal{M}$ to be a finite von Neumann algebra that is not a factor then the map $\sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i J b_i^* J$ is never injective and the measure will be supported on smaller sets. See Rem. 5.17. This is the reason we consider factors, although most results of this article goes through even for finite von Neumann algebras.

Thus in view of the uniqueness of direct integrals with respect to an abelian algebra (see Thm. 2.2), $L^2(\mathcal{M})$ admits a direct integral decomposition $\{\mathcal{H}_{x,y}\}$ over the base space $(Y \times Y,\ \eta_{Y\times Y})$ so that $\mathcal{A} \cong L^\infty(Y \times Y, \eta_{Y\times Y})$ is the algebra of *diagonalizable operators* with respect to this decomposition. Let m_Y denote the multiplicity function of the above decomposition. It is clear from the direct integral decomposition that, the *Pukánszky invariant* of A is the set of *essential values* of m_Y (also check Cor. 3.2, [19]). We will call $[\eta_{Y\times Y}]$ the *left-right-measure* of A. For reasons that will become clear, we will in most situation use the same terminology for the class of the measure $\eta_{Y\times Y}$ when restricted to the *off diagonal*. This will be clear from the context and will cause no confusion. A related invariant was considered by Neshveyev and Størmer in [19], which was a complete invariant for the pair (A, J).

Although the existence of such a measure is guaranteed we need an algorithm to figure out the *left-right-measure*. In order to do so fix a nonzero vector $\xi \in L^2(\mathcal{M})$. The cyclic projection P_ξ with range $[\mathcal{A}\xi]$ is in $\mathcal{A}'$ and hence *decomposable*. For f, $g \in C(Y)$, there exists a complete positive measure μ_ξ (we complete it if necessary) on $Y \times Y$ such that

$$\langle fJg^*J\xi, \xi\rangle_{L^2(\mathcal{M})} = \int_{Y\times Y} f(t)g(s)d\mu_\xi(t,s). \tag{2.3}$$

$\mathcal{A}P_\xi$ is a diffuse abelian algebra in $\mathbf{B}(P_\xi(L^2(\mathcal{M})))$ with a cyclic vector, so is maximal abelian. Thanks to von Neumann, we have only one. Therefore,

$$P_\xi(L^2(\mathcal{M})) \cong \int_{Y\times Y}^{\oplus} \mathbb{C}_{t,s} d\mu_\xi(t,s) \text{ where } \mathbb{C}_{t,s} = \mathbb{C}. \tag{2.4}$$

Moreover $\mathcal{A}P_\xi$ is the *diagonalizable algebra* with respect to the decomposition in Eq. (2.4).

Two orthogonal cyclic subspaces $[\mathcal{A}\xi_1]$ and $[\mathcal{A}\xi_2]$ with cyclic vectors ξ_1, ξ_2 does not necessarily keep the fibres of its associated projections P_{ξ_1} and P_{ξ_2} orthogonal, neither does assert that they are direct integrals over disjoint subsets of $Y \times Y$. However, using the "gluing lemma" (Lemma 5.7, [10]) we single out a measure μ_{ξ_1,ξ_2} so that $(P_{\xi_1} + P_{\xi_2})(L^2(\mathcal{M}))$ has a direct integral decomposition with respect to $(Y \times Y, \eta_{\xi_1,\xi_2})$ and $\mathcal{A}(P_{\xi_1} + P_{\xi_2})$ is the *diagonalizable algebra* respecting that decomposition. This is the step where one will see the possible updates of the *multiplicity function*. Since we are working on a separable Hilbert space, after at most a countable infinite iterations of this procedure we will finally find a measure μ on $Y \times Y$ so that

$$L^2(\mathcal{M}) \cong \int_{Y\times Y}^{\oplus} \mathcal{H}'_x d\mu(x) \tag{2.5}$$

and $\mathcal{A}$ is *diagonalizable* with respect to the decomposition in Eq. (2.5). Modulo the uniqueness of direct integrals we have found the measure. Needless to say, different choices of cyclic subspaces will produce same measure modulo the uniqueness. However for purpose of explicit computation to distinguish masas one learns, that nice choices of cyclic projections (vectors) is perhaps a little too costly.

For a set X we denote by $\Delta(X)$ the set $\{(x,y) \in X \times X : x = y\}$. The restriction of τ to $C(Y) \subset A$ gives rise to a Borel probability measure whose completion is denoted by ν_Y.

Lemma 2.10. *The measure $\eta_{Y\times Y}$ has the following properties:*
(i) $[\eta_{Y\times Y}]$ is invariant under the flip map θ :$(s,t) \mapsto (t,s)$ on $Y \times Y$.
(ii) If π_1 and π_2 denote the coordinate projections from $Y \times Y$ onto Y then,

$$[(\pi_i)_* \eta_{Y\times Y}] = [\nu_Y] \textit{ for } i = 1, 2. \tag{2.6}$$

(iii) The subspace $\int_{\Delta(Y)}^{\oplus} \mathcal{H}_{t,s} d\eta_{Y\times Y}(t,s)$ is identified with $L^2(A)$ and $m_Y(t,t) = 1$, $\eta_{Y\times Y}$ a.e. on $\Delta(Y)$.
(iv) The topological (closed) support of $\eta_{Y\times Y}$ is $Y \times Y$.
The multiplicity function m_Y has the property that

$$m_Y(s,t) = m_Y(t,s)$$

almost all $\eta_{Y\times Y}$.

Lemma 2.10 is known so we omit its proof. Interested readers can consult [19] or [18]. In fact, it is possible to obtain a choice of $\eta_{Y\times Y}$ such that $\eta_{Y\times Y} = \theta_* \eta_{Y\times Y}$.

We are now almost ready to give the definition of the *measure-multiplicity-invariant* of a masa in a separable II_1 factor. Let A be a masa in $\mathcal{M}$. Let Y be any compact Hausdorff space such that the unital inclusion of $C(Y)$ in A is $w.o.t$ dense and $C(Y)$ is norm separable. To each such Y, we associate a quadruple $(Y, \nu_Y, [\eta_{Y\times Y}], m_Y)$. Define an equivalence relation on the quadruples $(Y, \nu_Y, [\eta_{Y\times Y}], m_Y)$ by
$(Y, \nu_Y, [\eta_{Y\times Y}], m_Y) \sim_{m.m} (Y', \nu_{Y'}, [\eta_{Y'\times Y'}], m_{Y'})$ if and only if there exists a Borel isomorphism $F : Y \mapsto Y'$ such that,

$$\begin{aligned} &F_*\nu_Y = \nu_{Y'},\\ &(F\times F)_*[\eta_{Y\times Y}] = [\eta_{Y'\times Y'}] \text{ and}\\ &m_Y \circ (F\times F)^{-1} = m_{Y'},\ \eta_{Y'\times Y'} \text{ a.e.}\end{aligned} \tag{2.7}$$

We also have, $[\eta_{Y\times Y}] = [\eta_{|\Delta(Y)}] + [\eta_{|\Delta(Y)^c}]$.
Therefore if $(Y, \nu_Y, [\eta_{Y\times Y}], m_Y) \sim_{m.m} (Y', \nu_{Y'}, [\eta_{Y'\times Y'}], m_{Y'})$ then,

$$\begin{aligned} &(F\times F)_*[\eta_{|\Delta(Y)^c}] = [\eta_{|\Delta(Y')^c}],\\ &m_{|\Delta(Y)^c} \circ (F\times F)^{-1} = m_{|\Delta(Y')^c},\ \eta_{|\Delta(Y')^c} \text{ a.e.}\end{aligned} \tag{2.8}$$

Lemma 2.11. *If $C(Y_1) \subseteq C(Y_2) \subset A \subset \mathcal{M}$ be two w.o.t dense, unital, norm separable C^* subalgebras of A then $(Y_1, \nu_{Y_1}, [\eta_{Y_1\times Y_1}], m_{Y_1}) \sim_{m.m} (Y_2, \nu_{Y_2}, [\eta_{Y_2\times Y_2}], m_{Y_2})$.*

Proof. The inclusion $i : C(Y_1) \hookrightarrow C(Y_2)$ results from a continuous surjection $\theta : Y_2 \mapsto Y_1$. Therefore for all $f \in C(Y_1)$,

$$\tau(f) = \int_{Y_1} f d\nu_{Y_1} = \int_{Y_2} i(f) d\nu_{Y_2} = \int_{Y_2} (f\circ\theta) d\nu_{Y_2} = \int_{Y_1} f d(\theta_*\nu_{Y_2}).$$

Therefore, $\theta_* \nu_{Y_2} = \nu_{Y_1}$.
The inclusion i preserves least upper bounds at the level of continuous functions. So i extends to a surjective $*$-homomorphism $\tilde{i}$ between $L^\infty(Y_1, \nu_{Y_1})$ and $L^\infty(Y_2, \nu_{Y_2})$ which is normal (Lemma 10.1.10 [15]). It is easy to see that $\tilde{i}$ is also implemented by θ. That $\tilde{i}$ is injective is obvious. So θ is a Borel isomorphism between the underlying measure spaces.
Arguing similarly it is easy to see that $\theta \times \theta : Y_2 \times Y_2 \mapsto Y_1 \times Y_1$ implements an isomorphism between $L^\infty(Y_1 \times Y_1, \eta_{Y_1 \times Y_1})$ and $L^\infty(Y_2 \times Y_2, \eta_{Y_2 \times Y_2})$. The statements regarding the measure classes now follows easily.
The statement about the multiplicity function is obvious from the uniqueness of direct integrals in Thm. 2.2 and the fact $L^\infty(Y_1 \times Y_1, \eta_{Y_1 \times Y_1}) \cong L^\infty(Y_2 \times Y_2, \eta_{Y_2 \times Y_2}) \cong \mathcal{A}$. $\square$

Proposition 2.12. *Let $A \subset \mathcal{M}$ be a masa. The collection of quadruples $(Y, \nu_Y, [\eta_{Y \times Y}], m_Y)$ for Y a compact Hausdorff space such that $C(Y) \subset A$ is unital, norm separable and w.o.t dense in A, under the equivalence relation $\sim_{m.m}$ has exactly one equivalence class.*

Proof. If $C(Y_1)$, $C(Y_2) \subset A$ be two *w.o.t* dense, unital, norm separable subalgebras of A then $C^*(C(Y_1) \cup C(Y_2)) \cong C(Y_3)$ for a compact Hausdorff space Y_3, and $C(Y_3)$ is unital, norm separable and *w.o.t* dense in A. Therefore by Lemma 2.11, $(Y_3, \nu_{Y_3}, [\eta_{Y_3 \times Y_3}], m_{Y_3}) \sim_{m.m} (Y_i, \nu_{Y_i}, [\eta_{Y_i \times Y_i}], m_{Y_i})$ for i =1, 2. $\square$

Definition 2.13. Let $A \subset \mathcal{M}$ be a masa. We define the *measure-multiplicity-invariant* of A as the *equivalence class* of the quadruples $(Y, \nu_Y, [\eta_{|\Delta(Y)^c}], m_{|\Delta(Y)^c})$ under $\sim_{m.m}$ where,
(i) Y is a compact Hausdorff space such that $C(Y)$ is an unital, norm separable and *w.o.t* dense subalgebra of A.
(ii) ν_Y is the completion of the probability measure obtained from restricting τ on $C(Y)$.
(iii) $[\eta_{|\Delta(Y)^c}]$ is the equivalence class of the measure $\eta_{Y \times Y}$ restricted to $\Delta(Y)^c$,
(iv) $m_{|\Delta(Y)^c}$ is the multiplicity function restricted to $\Delta(Y)^c$,
obtained from the *direct integral decomposition* of $L^2(\mathcal{M})$ over the base space $(Y \times Y, \eta_{Y \times Y})$ so that $\mathcal{A}$ is the algebra of *diagonalizable operators* with respect to this decomposition.

The *measure-multiplicity-invariant* is an *invariant* for masas in the following sense. If $A \subset \mathcal{M}$ and $B \subset \mathcal{N}$ are masas in II_1 factors $\mathcal{M}, \mathcal{N}$ respectively, and there is an unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N})$ such that, $UAU^* = B$ and $UJ_{\mathcal{M}}AJ_{\mathcal{M}}U^* = J_{\mathcal{N}}BJ_{\mathcal{N}}$ then for any choice of compact Hausdorff spaces Y_A, Y_B with
$\overline{C(Y_A)}^{s.o.t} = A$ and $\overline{C(Y_B)}^{s.o.t} = B$, $1_{\mathcal{M}} \in C(Y_A)$, $1_{\mathcal{N}} \in C(Y_B)$ and $C(Y_A), C(Y_B)$ norm separable, there exists a Borel isomorphism

$$F_{Y_A, Y_B} : (Y_A, \nu_{Y_A}) \mapsto (Y_B, \nu_{Y_B}) \text{ such that,}$$

$$\begin{aligned}
&(F_{Y_A,Y_B})_* \nu_{Y_A} = \nu_{Y_B}, \\
&(F_{Y_A,Y_B} \times F_{Y_A,Y_B})_* [\eta_{|\Delta(Y_A)^c}] = [\eta_{|\Delta(Y_B)^c}] \text{ and} \\
&m_{|\Delta(Y_A)^c} \circ (F_{Y_A,Y_B} \times F_{Y_A,Y_B})^{-1} = m_{|\Delta(Y_B)^c}, \ \eta_{|\Delta(Y_B)^c} \text{ a.e.}
\end{aligned} \tag{2.9}$$

We will denote the *measure-multiplicity-invariant* of a masa A by $m.m(A)$ (or $m.m_{\mathcal{M}}(A)$ when the containing factor is ambiguous).

3. Conditional measures and Masas

As we will see latter, the *measure-multiplicity-invariant* contains substantial information of the masa. In order to extract more information we need to establish some house keeping results in measure theory.

Disintegration of measures is a very useful tool in ergodic theory, in the study of conditional probabilities and descriptive set theory. *Measurable selection principle* is a term closely linked with disintegration of measures and has been studied by a number of mathematicians in the last century. A detailed exposition of the existence of disintegration can be found in [1].

For the general definition of disintegration of measures we will restrict to the following set up. Let T be a measurable map from (X,σ_X) to (Y,σ_Y) where σ_X,σ_Y are σ-algebras of subsets of X,Y respectively. Let λ be a σ-finite measure on σ_X and μ a σ-finite measure on σ_Y. Here λ is the measure to be disintegrated and μ is often the push forward measure $T_*\lambda$, although other possibilities for μ is allowed.

Definition 3.1. We say that λ has a disintegration $\{\lambda_t\}_{t\in Y}$ with respect to T and μ or a (T,μ) disintegration if:
(i) λ_t is a σ-finite measure on σ_X concentrated on $\{T=t\}$ (or $T^{-1}\{t\}$), i.e. $\lambda_t(\{T\neq t\})=0$, for μ-almost all t,
and for each nonnegative measurable function f on X
(ii) $t\mapsto\lambda_t(f)$ is measurable.
(iii) $\lambda(f)=\mu^t(\lambda_t(f))\overset{\text{defn}}{=}\int_Y\lambda_t(f)d\mu(t)$.

In probability theory the measures λ_t are called the disintegrating measures and μ is the mixing measure. One also writes $\lambda(\cdot\mid T=t)$ for $\lambda_t(\cdot)$ on occasion.

When λ and almost all λ_t are probability measures one refers to the disintegrating measures as (regular) conditional distributions and $t\mapsto\lambda_t$ is called the transition kernel.

The reader should be cautious that *"measurable"* in Defn. 3.1 $(ii),(iii)$ means measurable with respect to the σ-algebra of completion of λ.

Theorem 3.2. [1](*Existence Theorem*) *Let λ be a σ-finite Radon measure on a metric space X and T be a measurable map into (Y,σ_Y). Let μ be a σ-finite measure on σ_Y such that $T_*\lambda\ll\mu$. If σ_Y is countably generated and contains all singleton sets $\{t\}$, then λ has a (T,μ) disintegration. The measures λ_t are uniquely determined up to an almost sure equivalence: if λ_t^* is another (T,μ) disintegration then $\mu(\{t:\lambda_t\neq\lambda_t^*\})=0$.*

The condition $T_*\lambda\ll\mu$ in Thm. 3.2 is actually necessary for the disintegration to exist. The original version of Thm. 3.2 is due to von Neumann.

Proposition 3.3. *Let λ be a Radon measure on a compact metric space X and T be a measurable map into (Y,σ_Y). Let μ be a σ-finite measure on σ_Y such that $T_*\lambda\ll\mu$. Assume that σ_Y is countably generated and contains all singleton sets. Let $t\mapsto\lambda_t$ denote the (T,μ) disintegration of λ. Let X_a denote the set of atoms of $\{\lambda_t\}_{t\in Y}$ i.e.*

$$X_a=\{x\in X\mid\ \exists\, t\in Y:\lambda_t(\{x\})>0\}\,.$$

Then X_a is a measurable set, measurable with respect to the σ-algebra of the completion of λ.

Proof. There is a measurable set $E\subseteq Y$ with $\mu(E^c)=0$ such that for $t\in E$, λ_t is concentrated on the set $\{T=t\}$. We can assume without loss of generality that $E=Y$. Now for $t\in Y$, the measure λ_t is concentrated on $\{T=t\}$, so

$$\{x\in X\mid\ \exists\, t\in Y:\lambda_t(\{x\})>0\}=\{x\in X\mid\lambda_{Tx}(\{x\})>0\}\,.$$

Let $\mathcal{B}$ be a countable base for the topology on X. Then

$$\{x \in X \mid \lambda_{Tx}(\{x\}) > 0\} = \cup_{n=1}^{\infty} X_a^{(n)} \text{ with}$$

$$\begin{aligned} X_a^{(n)} &= \left\{ x \in X \mid \ \forall U \in \mathcal{B} : x \in U \Rightarrow \lambda_{Tx}(U) \geq \frac{1}{n} \right\} \\ &= \bigcap_{U \in \mathcal{B}} \left((X \setminus U) \cup \left\{ x \in U \mid \lambda_{Tx}(U) \geq \frac{1}{n} \right\} \right). \end{aligned}$$

Therefore, $\{x \in X \mid \ \exists\, t \in Y : \lambda_t(\{x\}) > 0\}$ is a measurable set by property (ii) of disintegration. $\square$

The next few lemmas are undoubtedly known to probablists but we lack the reference. So we record them for convenience. We will omit their proofs. For details check [18].

Lemma 3.4. *Let λ_1, λ_2 be two Radon measures on a compact metric space X and T be a measurable map into (Y, σ_Y). Let μ be a σ-finite measure on σ_Y such that $T_*\lambda_1, T_*\lambda_2 \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let λ_t^1, λ_t^2 be the (T, μ) disintegration of λ_1, λ_2 respectively. Let λ_t^0 be the (T, μ) disintegration of $\lambda_1 + \lambda_2$. Then*

$$\lambda_t^0 = \lambda_t^1 + \lambda_t^2 \text{ -}\mu \text{ a.e.}$$

Lemma 3.5. *Let λ_1, λ_2 be two Radon measures on compact metric spaces X, Y and T, S be measurable maps from X, Y into (Z, σ_Y), (W, σ_W) respectively. Let μ, ν be σ-finite measures on σ_Y, σ_W respectively such that $T_*\lambda_1 \ll \mu, S_*\lambda_2 \ll \nu$.*
Assume σ_Y, σ_W are countably generated and contains all singleton sets $\{t\}, \{s\}$ respectively. Let λ_t^1, λ_s^2 be the $(T, \mu), (S, \nu)$ disintegration of λ_1, λ_2 respectively. Let $\lambda_{t,s}^0$ be the $(T \otimes S, \mu \otimes \nu)$ disintegration of $\lambda_1 \otimes \lambda_2$. Then

$$\lambda_{t,s}^0 = \lambda_t^1 \otimes \lambda_s^2 \text{ -}\mu \otimes \nu \text{ a.e.}$$

Lemma 3.6. *Let λ_1, λ_2 be two Radon measures on a compact metric space X and T be a measurable map into (Y, σ_Y). Let μ be a σ-finite measure on σ_Y such that $T_*\lambda_1 \ll \mu$ and $T_*\lambda_2 \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let λ_t^1, λ_t^2 be the (T, μ) disintegrations of λ_1, λ_2 respectively.*
(i) Assume that $\lambda_1 \ll \lambda_2 \ll \lambda_1$. Then for μ almost all t, $\lambda_t^1 \ll \lambda_t^2 \ll \lambda_t^1$. Moreover, if $g = \frac{d\lambda_1}{d\lambda_2}$ then $\frac{d\lambda_t^1}{d\lambda_t^2} = g_t$ a.e. μ, where

$$g_t = \begin{cases} g_{|\{T=t\}} \text{ on } \{T = t\}, \\ 0 \text{ otherwise.} \end{cases}$$

Conversely if $\lambda_t^1 \ll \lambda_t^2 \ll \lambda_t^1$ for μ almost all t then $\lambda_1 \ll \lambda_2 \ll \lambda_1$.
(ii) If $\lambda_1 \perp \lambda_2$ then $\lambda_t^1 \perp \lambda_t^2$ for μ almost all t.

Lemma 3.7. *Let λ be a Radon measure on $X \times X$ where X is a compact metric space. Let μ be a σ-finite measure on X such that $(\pi_i)_*\lambda \ll \mu$ where π_i, $i = 1, 2$ are coordinate projections onto X.*
Assume that λ is invariant under the flip of coordinates i.e. $\theta_*\lambda \ll \lambda \ll \theta_*\lambda$, where $\theta : X \times X \mapsto X \times X$ by $\theta(x, y) = (y, x)$. Let λ_s^1, λ_t^2 be the $(\pi_1, \mu), (\pi_2, \mu)$ disintegrations of λ respectively. Then for μ almost all t,

$$\lambda_t^1 \ll \theta_*\lambda_t^2 \ll \lambda_t^1.$$

In particular, if for μ almost all t, λ_t^2 has an atom at (s, t), then λ_t^1 has an atom at (t, s) almost everywhere.

Theorem 3.8. *Let $A \subset \mathcal{M}$ and $B \subset \mathcal{N}$ be masas in separable II_1 factors $\mathcal{M},\mathcal{N}$. Let $C(X_1) \subset A$, $C(X_2) \subset B$ be w.o.t dense, norm separable, unital subalgebras of A, B respectively, where X_i are compact metric spaces for $i = 1, 2$. Let ν_{X_i} denote the tracial measures with respect to the w.o.t dense subalgebras on X_i respectively for $i = 1, 2$. Let $[\lambda_1], [\lambda_2]$ denote the left-right-measures of A and B respectively. All the mentioned measures are assumed to be complete. Suppose there is an unitary $U : L^2(\mathcal{M}) \mapsto L^2(\mathcal{N})$ such that $UAU^* = B$ and $UJ_{\mathcal{M}}AJ_{\mathcal{M}}U^* = J_{\mathcal{N}}BJ_{\mathcal{N}}$.*
Then there exists isomorphism of measure spaces $F : X_1 \mapsto X_2$ such that, $F_*\nu_{X_1} = \nu_{X_2}$ and the following is true:
Denoting by $\lambda_t^{1,X_1}, \lambda_s^{2,X_1}$ the $(\pi_1, \nu_{X_1}), (\pi_2, \nu_{X_1})$ disintegrations of λ_1 respectively and $\lambda_{t'}^{1,X_2}, \lambda_{s'}^{2,X_2}$ the (π_1, ν_{X_2}), (π_2, ν_{X_2}) disintegrations of λ_2 respectively, one has

$$[\lambda_{t'}^{1,X_2}] = [(F \times F)_*\lambda_{F^{-1}t'}^{1,X_1}],\ \nu_{X_2} \text{ almost all } t',$$
$$[\lambda_{s'}^{2,X_2}] = [(F \times F)_*\lambda_{F^{-1}s'}^{2,X_1}],\ \nu_{X_2} \text{ almost all } s',$$

where π_1, π_2 denotes the projection onto the first and second coordinates respectively.

If (X, σ) be a measurable space and μ is a signed measure on X then we denote by $\|\mu\|_{t.v}$ to be the total variation norm of μ. The next Lemma is used in this paper but it will be of significant use for computation in the next paper.

Lemma 3.9. *Let $\lambda_n, \lambda, \lambda_0$ be Radon measures on a compact metric space X such that, $\lambda_0 \neq 0$, $\lambda_n \ll \lambda$ for $n = 1, 2, \cdots$, $\lambda_0 \ll \lambda$ and $\lambda_n \to \lambda_0$ in $\|\cdot\|_{t.v}$. Let T be a measurable map into (Y, σ_Y). Let μ be a σ-finite measure on σ_Y such that $T_*\lambda \ll \mu$. Assume σ_Y is countably generated and contains all singleton sets $\{t\}$. Let $\lambda_t^n, \lambda_t^0, \lambda_t$ be the (T, μ) disintegrations of $\lambda_n, \lambda_0, \lambda$ respectively.*
(i) Then there is a μ null set E and a subsequence $\{n_k\}$ ($n_k < n_{k+1}$ for all k) such that for all $t \in E^c$,

$$\sup_{A \subseteq \{T=t\}, A \text{ Borel}} \left|\lambda_t^{n_k}(A) - \lambda_t^0(A)\right| \to 0 \text{ as } k \to \infty.$$

(ii)Moreover, if for μ almost all t one has λ_t^n is completely atomic (or completely non-atomic) for all n, then so is λ_t^0 almost everywhere.

The proof is straight forward. We omit the proof. For details check [18].

For a masa $A \subset \mathcal{M}$, fix a compact Hausdorff space X such that $C(X) \subset A$ is an unital, norm separable and $w.o.t$ dense C^* subalgebra. For $\zeta \in L^2(\mathcal{M})$ let $\kappa_\zeta : C(X) \otimes C(X) \mapsto \mathbb{C}$ be the linear functional defined by

$$\kappa_\zeta(a \otimes b) = \langle a\zeta b, \zeta\rangle.$$

Then κ_ζ induces an unique Radon measure η_ζ on $X \times X$ given by

$$\kappa_\zeta(a \otimes b) = \int_{X\times X} a(t)b(s)d\eta_\zeta(t, s) \tag{3.1}$$

and $\|\eta_\zeta\|_{t.v} = \|\kappa_\zeta\|$.

For $\zeta_1, \zeta_2 \in L^2(\mathcal{M})$ let η_{ζ_1,ζ_2} denote the possibly complex measure on $X \times X$ obtained from the vector functional

$$\langle a\zeta_1 b, \zeta_2\rangle = \int_{X\times X} a(t)b(s)d\eta_{\zeta_1,\zeta_2}(t, s),\ a, b \in C(X). \tag{3.2}$$

We will write $\eta_{\zeta,\zeta} = \eta_\zeta$. Note that η_ζ is a positive measure for all $\zeta \in L^2(\mathcal{M})$. It is easy to see that the following polarization type identity holds:

$$4\eta_{\zeta_1,\zeta_2} = (\eta_{\zeta_1+\zeta_2} - \eta_{\zeta_1-\zeta_2}) + i\,(\eta_{\zeta_1+i\zeta_2} - \eta_{\zeta_1-i\zeta_2})\,. \tag{3.3}$$

Note that the decomposition of η_{ζ_1,ζ_2} in Eq. (3.3) need not be its Hahn decomposition in general, but

$$|\eta_{\zeta_1,\zeta_2}| \leq (\eta_{\zeta_1+\zeta_2} + \eta_{\zeta_1-\zeta_2}) + (\eta_{\zeta_1+i\zeta_2} + \eta_{\zeta_1-i\zeta_2}) = 4(\eta_{\zeta_1} + \eta_{\zeta_2}).$$

So

$$|\eta_{\zeta_1,\zeta_2}| \leq \eta_{\zeta_1} + \eta_{\zeta_2}. \tag{3.4}$$

Lemma 3.10. *If $\zeta_n, \zeta \in L^2(\mathcal{M})$ be such that, $\zeta_n \to \zeta$ in $\|\cdot\|_2$ then*

$$\eta_{\zeta_n} \to \eta_\zeta \text{ in } \|\cdot\|_{t.v}.$$

Proof. Obvious. □

Proposition 3.11. *Let $A \subset \mathcal{M}$ be a masa. Let X be a compact Hausdorff space such that $C(X) \subset A$ is unital, norm separable and w.o.t dense in A and let ν be the tracial measure. Let $0 \neq \zeta \in L^2(N(A)'')$. Then $\eta_{\zeta_t}, \eta_{\zeta_s}$ is completely atomic ν almost all t, s where η_ζ is the measure defined in Eq. (3.1) and $\eta_{\zeta_t}, \eta_{\zeta_s}$ are (π_1, ν) and (π_2, ν) disintegrations of η_ζ respectively.*

Proof. We only prove for the (π_1, ν) disintegration. If $\zeta = u$ where $u \in N(A)$ then the result is obvious as the measure η_u will be concentrated on the automorphism graph. The span of $N(A)$ being $s.o.t$ dense in $N(A)''$ it suffices by Lemma 3.10 and 3.9 to prove the statement when $\zeta = \sum_{i=1}^n c_i u_i$ where $u_i \in N(A)$ and $c_i \in \mathbb{C}$ for $1 \leq i \leq n$. Now for $a, b \in A$

$$\langle a(\sum_{i=1}^n c_i u_i)b, (\sum_{i=1}^n c_i u_i)\rangle = \sum_{i=1}^n |c_i|^2 \langle au_ib, u_i\rangle + \sum_{i\neq j=1}^n c_i\bar{c}_j \langle au_ib, u_j\rangle.$$

The measures given by $a \otimes b \mapsto |c_i|^2 \langle au_ib, u_i\rangle$, $a, b \in C(X)$ are concentrated on the automorphism graphs implemented by u_i and hence definitely disintegrates as atomic measures and so does their sum from Lemma 3.4. The measures given by $a \otimes b \mapsto c_i\bar{c}_j\langle au_ib, u_j\rangle$, $a, b \in C(X)$ for $i \neq j$ are possibly complex measures. However Eq. (3.4) forces that these measures are also concentrated on the union of the automorphism graphs implemented by u_i and u_j. Thus $\eta_{\sum_{i=1}^n c_iu_i}$ is concentrated on the union of the automorphism graphs implemented by u_i, $1 \leq i \leq n$. Hence the result follows. □

4. Fundamental Set and Generalized Dye's Theorem

This section is intended to characterize some operators in the normalizing algebra of a masa. Throughout this section $\mathcal{N}$ will denote a finite von Neumann algebra gifted with a faithful, normal, normalized trace τ. $B \subset \mathcal{N}$ will denote a von Neumann subalgebra of $\mathcal{N}$.

As usual $\mathcal{N}$ will be assumed to be acting on $L^2(\mathcal{N}, \tau)$ by left multipliers. $L^2(\mathcal{N}, \tau)$ is a B-B Hilbert w^*-bimodule for any von Neumann subalgebra $B \subset \mathcal{N}$. We know if $\mathbb{E}_B$ denotes the unique trace preserving conditional expectation onto B, then $\mathbb{E}_B$ is given by the Jones projection e_B associated to B via the formula $\mathbb{E}_B(x)\hat{1} = e_B(x\hat{1})$. For $b_1, b_2 \in B$ and $\zeta \in L^2(\mathcal{N}, \tau)$ one has

$$e_B(b_1\zeta b_2) = b_1 e_B(\zeta) b_2. \tag{4.1}$$

We will interchangeably use the symbols $\mathbb{E}_B$ and e_B.

Definition 4.1. For a subalgebra $B \subset \mathcal{N}$ define the *fundamental set* of B to be

$$N^f(B) = \{x \in \mathcal{N} : Bx = xB\}.$$

Note that $x \in N^f(B)$ implies $x^* \in N^f(B)$.

Definition 4.2. For a subalgebra $B \subset \mathcal{N}$ define the *weak-fundamental set* of B to be

$$N_2^f(B) = \{\zeta \in L^2(\mathcal{N}, \tau) : B\zeta = \zeta B\}.$$

Note that $\zeta \in N_2^f(B)$ implies $\zeta^* \in N_2^f(B)$ and $N^f(B) \subset N_2^f(B)$. When B is a masa, $\zeta \in N_2^f(B)$ implies $a\zeta, \zeta a \in N_2^f(B)$ for all $a \in B$.

To understand the normaliser of a masa the set $N_2^f(B)$ will naturally arise into the scene. However working with vectors in $L^2(\mathcal{N}, \tau)$ is always a technical issue. Polar decomposition of vectors and the theory of L^1 spaces are the tools we need, for which we will give a short exposition. For details check Appendix B of [34] and [18]. To keep it short we will omit most proofs. It is here, where one usually encounters unbounded operators. For results proved in this section we have borrowed ideas from Roger Smith.

The positive cone $L^2(\mathcal{N}, \tau)^+$ in $L^2(\mathcal{N}, \tau)$ is defined to be $\overline{\mathcal{N}^+}^{\|\cdot\|_2}$ i.e. the closure of the positive elements of $\mathcal{N}$ in $L^2(\mathcal{N}, \tau)$. It can be shown that $L^2(\mathcal{N}, \tau)$ is the algebraic span of $L^2(\mathcal{N}, \tau)^+$. For $x \in \mathcal{N}$ the equation $\|x\|_1 = \tau(|x|)$ defines a norm on $\mathcal{N}$. The completion of $\mathcal{N}$ with respect to $\|\cdot\|_1$ is denoted by $L^1(\mathcal{N}, \tau)$. It can be shown that

$$\|x\|_1 = \sup\{|\tau(xy)| : y \in \mathcal{N}, \|y\| \leq 1\}. \tag{4.2}$$

So $|\tau(x)| \leq \|x\|_1$. Thus by density of $\mathcal{N}$ in $L^1(\mathcal{N}, \tau)$, τ extends to a bounded linear functional on $L^1(\mathcal{N}, \tau)$ which will also be denoted by τ. One can analogously define the positive cone of $L^1(\mathcal{N}, \tau)$ which we denote by $L^1(\mathcal{N}, \tau)^+$. Clearly $\|x\|_1 = \|x^*\|_1$. Consequently, the Tomita operator J extends to a surjective anti-linear isometry to $L^1(\mathcal{N}, \tau)$ which will also be denoted by J. Moreover $J^2 = 1$. We will interchangeably use the notations $J\zeta$ and ζ^* for $\zeta \in L^1(\mathcal{N}, \tau)$.

Both the spaces $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ are unitary $\mathcal{N}$-$\mathcal{N}$ bimodules. The space $L^1(\mathcal{N}, \tau)$ can be identified with the predual of $\mathcal{N}$ and $L^2(\mathcal{N}, \tau)$ is dense in $L^1(\mathcal{N}, \tau)$. One also has $\tau(x\zeta) = \tau(\zeta x)$ for $x \in \mathcal{N}$ and $\zeta \in L^1(\mathcal{N}, \tau)$. Note that $\mathbb{E}_B$ is a contraction from $\mathcal{N}$ onto B. It can be shown that for $x \in \mathcal{N}$,

$$\|\mathbb{E}_B(x)\|_1 \leq \|x\|_1 . \tag{4.3}$$

Thus $\mathbb{E}_B$ has an unique bounded extension to a contraction from $L^1(\mathcal{N}, \tau)$ onto $L^1(B, \tau)$, which will as well be denoted by $\mathbb{E}_B$. This extension preserves the extension of the trace τ, is B modular, positive and faithful. The bilinear map $\Psi : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ defined by $\Psi(x, y) = xy$ satisfies

$$\|\Psi(x, y)\|_1 \leq \|x\|_2 \|y\|_2 \tag{4.4}$$

by Cauchy-Schwarz inequality. Therefore Ψ lifts to a jointly continuous map from $L^2(\mathcal{N}, \tau) \times L^2(\mathcal{N}, \tau)$ into $L^1(\mathcal{N}, \tau)$. The extension is actually a surjection. Since Ψ is the product map of operators at the level of von Neumann algebra one calls $\Psi(\zeta_1, \zeta_2)$ to be $\zeta_1\zeta_2$, for $\zeta_1, \zeta_2 \in L^2(\mathcal{N}, \tau)$.

Lemma 4.3. (*B.5.1,* [34]) *Let $a, b \in \mathcal{N}$ be positives. Then*

$$\left\| a^{\frac{1}{2}} - b^{\frac{1}{2}} \right\|_2^2 \leq 2 \|a - b\|_1 . \tag{4.5}$$

Elements of $L^1(\mathcal{N}, \tau)$ and $L^2(\mathcal{N}, \tau)$ can be regarded as unbounded operators on $L^2(\mathcal{N}, \tau)$. By using the unbounded operator theory for operators affiliated to $\mathcal{N}$, for each $\zeta \in L^1(\mathcal{N}, \tau)^+$ there exists an unique $0 \leq \zeta_0 \in L^2(\mathcal{N}, \tau)$ such that $\zeta_0^*\zeta_0 = \zeta_0^2 = \zeta$. In this case, ζ_0 is said to be the *square root* of ζ and one writes $\zeta_0 = \sqrt{\zeta} = \zeta^{\frac{1}{2}}$. For $\zeta \in L^2(\mathcal{N}, \tau)$ one has $\zeta^*\zeta \in L^1(\mathcal{N}, \tau)$. From Eq. 4.4 and Lemma 4.3 it follows that $\zeta^*\zeta \in L^1(\mathcal{N}, \tau)^+$. In particular, $\sqrt{\zeta^*\zeta} \in L^2(\mathcal{N}, \tau)$ for any $\zeta \in L^2(\mathcal{N}, \tau)$ and the square root of any positive in $L^1(\mathcal{N},\tau)$ is an unique element of $L^2(\mathcal{N},\tau)$. One also writes $|\zeta| = \sqrt{\zeta^*\zeta}$ for $\zeta \in L^2(\mathcal{N},\tau)$. If $\zeta \in L^1(\mathcal{N},\tau)$ be self adjoint i.e. $\zeta = \zeta^*$ then $\zeta = \zeta_+ - \zeta_-$ where $\zeta_\pm \in L^1(\mathcal{N},\tau)^+$ and this decomposition is unique by requiring that ${\zeta_+}^{\frac{1}{2}}{\zeta_-}^{\frac{1}{2}} = 0$.

Let $\zeta \in L^2(\mathcal{N},\tau)$. Consider the projections p, q in $\mathbf{B}(L^2(\mathcal{N},\tau))$ whose ranges are $\overline{J\mathcal{N}J\sqrt{\zeta^*\zeta}}$, $\overline{J\mathcal{N}J\zeta}$ respectively. Since the ranges of p, q are invariant subspaces of $J\mathcal{N}J = \mathcal{N}'$ so p, q lies in $\mathcal{N}$. Using unbounded operators one obtains *polar decomposition* of vectors (Eq. (4.7)) which we formalize below.

Theorem 4.4. *There is an unique partial isometry $v \in \mathcal{N}$ with initial projection p and final projection q which satisfy the following condition:*

$$vJx^*J\sqrt{\zeta^*\zeta} = Jx^*J\zeta, \ x \in \mathcal{N}. \tag{4.6}$$

In particular,

$$v\sqrt{\zeta^*\zeta} = \zeta. \tag{4.7}$$

(i) Let $B \subset \mathcal{N}$ be a masa, then $\zeta \in L^2(B,\tau)$ imply $p, q \in B$.
(ii) For $\zeta \in L^2(\mathcal{N},\tau)$ if $\zeta^*\zeta \in \mathcal{N}$ then $\zeta \in \mathcal{N}$.

For $\zeta \in L^2(\mathcal{N},\tau)$ we define the *left and right kernel* of ζ to be respectively $Ker_l(\zeta) = \{x \in \mathcal{N} : \zeta x = 0\}$ and $Ker_r(\zeta) = \{x \in \mathcal{N} : x\zeta = 0\}$. Then $Ker_l(\cdot), Ker_r(\cdot)$ are subspaces of $\mathcal{N}$. $Ker_l(\cdot)$, $Ker_r(\cdot)$ are $w.o.t$ and $s.o.t$ closed.

If $\zeta \in L^1(\mathcal{N},\tau)$ then *the left and the right kernels* of ζ can be defined analogously. We will denote the *kernels* of the L^1 vectors by $Ker_l(\cdot), Ker_r(\cdot)$ as well. This is slight abuse of notation. In this case, they are norm closed subspaces of $\mathcal{N}$.

For $\zeta \in L^2(\mathcal{N},\tau)$ we have

$$Ker_l(\zeta) = Ker_l(\sqrt{\zeta^*\zeta}) = Ker_l(\zeta^*\zeta). \tag{4.8}$$

However the righthand side is defined in L^1 sense. Therefore for $\zeta \in L^2(\mathcal{N},\tau)$, $Ker_l(\zeta^*\zeta)$ (respectively $Ker_r(\zeta\zeta^*)$) are in fact $w.o.t$ closed. Similar statements hold for $Ker_r(\cdot)$ as well.

For $\zeta \in L^2(\mathcal{N},\tau)$ we define the *left and right ranges* of ζ to be respectively $Ran_l(\zeta) = \{\zeta x : x \in \mathcal{N}\}$ and $Ran_r(\zeta) = \{x\zeta : x \in \mathcal{N}\}$.
Note that for $\zeta \in L^2(\mathcal{N},\tau)$,

$$\begin{aligned} \{x \in \mathcal{N} : \zeta x = 0\} &= \{x \in \mathcal{N} : \langle \zeta x, y\rangle = 0 \text{ for all } y \in \mathcal{N}\} \\ &= \{x \in \mathcal{N} : \langle x, \zeta^* y\rangle = 0 \text{ for all } y \in \mathcal{N}\} \end{aligned} \tag{4.9}$$

implies $Ker_l(\zeta) = Ran_l(\zeta^*)^\perp$.

Proposition 4.5. *Let $\zeta \in L^2(\mathcal{N},\tau)$ and let $\zeta = v\sqrt{\zeta^*\zeta}$ be its polar decomposition. Then v^*v is the projection from $L^2(\mathcal{N},\tau)$ onto $Ker_l(\zeta)^\perp$ and vv^* is the projection onto $\overline{Ran_l(\zeta)}$.*

Proposition 4.6. *Let $\zeta \in L^2(\mathcal{N},\tau)$ and let $\zeta = v\,|\zeta|$ be its polar decomposition. Then $|\zeta|^{\frac{1}{2^k}} \to v^*v$ as $k \to \infty$ in $\|\cdot\|_2$.*

The proof of Prop. 4.6 is a direct application of monotone convergence theorem.

Lemma 4.7. *Let $A \subset \mathcal{N}$ be a masa. Let $\zeta \in L^1(\mathcal{N},\tau)$ be a nonzero vector such that $a\zeta = \zeta a$ for all $a \in A$. Then $\zeta \in L^1(A,\tau)$.*

Proof. First assume $\zeta \geq 0$. Then use uniqueness of square roots of L^1 vectors. In, the general case write ζ as a linear combination of four positives. We omit the details. $\square$

Proposition 4.8. *Let $A \subset \mathcal{N}$ be a masa. Let $0 \neq \zeta \in L^1(\mathcal{N},\tau)^+$ be such that $A\zeta = \zeta A$. Then $\zeta \in L^1(A,\tau)^+$.*

Proof. Let $\mathcal{I} = \{a \in A : a\zeta = 0\}$. Then $\mathcal{I}$ is a weakly closed ideal (see Eq. (4.8) and related discussion) in A and so has the form $A(1-p)$ for some projection $p \in A$. Then $p\zeta = \zeta$, so $\zeta = \zeta p$ by operating with extended Tomita's involution operator. Thus $Ap\zeta = A\zeta p = \zeta Ap$.
For $a_1, a_2 \in A$ if $\zeta a_1 p = \zeta a_2 p$ then $\zeta(a_1 - a_2)p = 0$, so $p(a_1^* - a_2^*)\zeta = 0$. Hence $p(a_1^* - a_2^*) \in \mathcal{I}$, but $1-p$ is the identity for $\mathcal{I}$. So $p(a_1^* - a_2^*) = 0$ and hence $a_1 p = a_2 p$. This means there is a well defined map $\psi : Ap \mapsto Ap$ such that

$$ap\zeta = \zeta\psi(ap) \text{ for } a \in A.$$

Taking conditional expectation (see Eq. (4.3) and related discussion) one gets $(ap - \psi(ap))\mathbb{E}_A(\zeta) = 0$ (the left and the right action by elements of A coincides on $L^1(A,\tau)$). Suppose there is an operator $a \in A$ such that $ap - \psi(ap) \neq 0$. Write $ap - \psi(ap) = bp$ for $b \in A$. Then $pb^*bp\mathbb{E}_A(\zeta) = 0$, so $\mathbb{E}_A(pb^*bp\zeta) = 0$. Let $\zeta = \lim_n x_n$ in $\|\cdot\|_1$ where $x_n \in \mathcal{N}^+$. Therefore

$$\lim_n \tau(x_n^{\frac{1}{2}}(bp)^* bp x_n^{\frac{1}{2}}) = \lim_n \tau(pb^*bpx_n) = \lim_n \tau(\mathbb{E}_A(pb^*bpx_n)) = 0.$$

The last statement follows from Eq. (4.2) and Eq. (4.3). So $\lim_n bpx_n^{\frac{1}{2}} = 0$ in $\|\cdot\|_2$ and hence $bp\zeta = \lim_n bpx_n = 0$, in $\|\cdot\|_1$ by Lemma 4.3 and Eq. (4.4). Thus $bp \in \mathcal{I}$ so $bp = bp(1-p) = 0$, a contradiction. Thus $\psi(ap) = ap$ for all $a \in A$.
Now $\zeta \in L^1(p\mathcal{N}p,\tau)$ and Ap is a masa in $p\mathcal{N}p$, thus $\zeta \in L^1(Ap,\tau)$ as $ap\zeta = \zeta\psi(ap) = \zeta ap$ for all $a \in A$, from Lemma 4.7. $\square$

Theorem 4.9. (*Generalized Dye's theorem-L^2 form*) *Let $A \subset \mathcal{N}$ be a masa. Then $\zeta \in N_2^f(A)$ if and only if $\zeta = v\xi$ for some $\xi \in L^2(A,\tau)$ and $v \in \mathcal{GN}(A)$. In particular, $\overline{spanN^f(A)}^{\|\cdot\|_2} = L^2(N(A)'',\tau)$.*

Proof. **Case 1**: Assume $\zeta \in N_2^f(A)$ and $\zeta \geq 0$ i.e. $\zeta \in \overline{\mathcal{N}^+}^{\|\cdot\|_2}$. Then $\zeta \in L^1(\mathcal{N},\tau)^+$ as well. From Prop. 4.8 we get $\zeta \in L^1(A,\tau) \cap L^2(\mathcal{N},\tau) = L^2(A,\tau)$.
Case 2: Let $\zeta \in N_2^f(A)$. We may without loss of generality assume that $\|\zeta\|_2 = 1$. Then as $A\zeta = \zeta A$ we also have $A\zeta^* = \zeta^* A$. So $A\zeta^*\zeta = \zeta^* A\zeta = \zeta^*\zeta A$. From Prop. 4.8,

$$\zeta^*\zeta \in L^1(A,\tau)$$

and similarly we have $\zeta\zeta^* \in L^1(A,\tau)$. Then $\|\zeta^*\zeta\|_1 \leq 1$.
Arguing as in Prop. 4.8, there are projections $p_1, p_2 \in A$ such that $J_1 = \{a \in A : a\zeta = 0\} = A(1-p_1)$ and $J_2 = \{a \in A : \zeta a = 0\} = A(1-p_2)$. Therefore we have $p_1\zeta = \zeta$ and $\zeta p_2 = \zeta$.
Then there is a well defined map (as explained before) $\psi : Ap_1 \mapsto Ap_2$ such that

$$ap_1\zeta = \zeta\psi(ap_1) \text{ for all } a \in A.$$

Let $\zeta = v\sqrt{\zeta^*\zeta}$ be the polar decomposition of ζ from Thm. 4.4. Then v is a partial isometry in $\mathcal{N}$ and the initial space of v is
$\{\sqrt{\zeta^*\zeta}x : x \in \mathcal{N}\}^{-\|\cdot\|_2}$ and the final space is $\{\zeta x : x \in \mathcal{N}\}^{-\|\cdot\|_2}$. Moreover the projections v^*v and vv^* are in A.
Indeed, by Prop. 4.5, v^*v is the projection onto $Ker_l(\zeta)^\perp$ and vv^* onto $\overline{Ran_l(\zeta)}$. By

Prop. 4.6, $v^*v \in A$. Replacing ζ by ζ^* and using $Ker_l(\zeta)^\perp = \overline{Ran_l(\zeta^*)}$ (see Eq. (4.9)), a similar argument will yield $vv^* \in A$. Clearly $v^*v = p_2$ and $vv^* = p_1$. Then

$$ap_1v\sqrt{\zeta^*\zeta} = v\sqrt{\zeta^*\zeta}\psi(ap_1).$$

Now

$$J_0 = \{b \in A : ap_1vb = vb\psi(ap_1) \text{ for all } a \in A\}$$

is a weakly closed ideal in A and its closure in $\|\cdot\|_2$ is precisely the set

$$J_0^{-\|\cdot\|_2} = \{\xi \in L^2(A,\tau) : ap_1v\xi = v\xi\psi(ap_1) \text{ for all } a \in A\}$$

which contains $\sqrt{\zeta^*\zeta}$.

Since the left and right action of A on $L^2(A,\tau)$ agree, so $\xi_0 \in J_0^{-\|\cdot\|_2}$ and $a \in A$ implies that $\xi_0 a, a\xi_0 \in J_0^{-\|\cdot\|_2}$.

Since the $w.o.t$ closed ideal J_0 in A is just a cutdown of A by a projection from A any positive $\zeta_0 \in J_0^{-\|\cdot\|_2}$ is a limit in $\|\cdot\|_2$ of an increasing sequence of positive operators from J_0. Now it follows that $|\zeta|^{\frac{1}{2^k}} \in J_0^{-\|\cdot\|_2}$ for all $k \in \mathbb{N}$. Therefore by Prop. 4.6 it follows that $v^*v = p_2 \in J_0^{-\|\cdot\|_2}$ and hence $p_2 \in J_0 \subseteq A$. Similarly arguing with $\zeta\zeta^*$ one shows $p_1 \in A$. Therefore

$$ap_1vp_2 = vp_2\psi(ap_1) \text{ for all } a \in A.$$

Then

$$v^*av = (vp_2)^*avp_2 = v^*ap_1v = v^*vp_2\psi(ap_1) = \psi(ap_1).$$

Therefore v^* and hence v are groupoid normalisers. So

$$\zeta = v\xi.$$

for $v \in \mathcal{GN}(A)$ and $\xi = |\zeta| \in L^2(A,\tau)^+$. □

5. Characterization by Baire Category Methods

The study of Cartan masas in II_1 factors has received special attention by many experts. Our approach of studying *measure-multiplicity-invariant* was also considered implicitly by Popa and Shlyakhtenko in [27]. In this section we will use an alternative approach to characterize masas by their *left-right-measure*. As it turns out, many known theorems related to structure and normalisers of masas that were solved using different techniques can be solved by a single technique.

Let $A = L^\infty(X,\nu_X), B = L^\infty(Y,\nu_Y)$ be two diffuse commutative von Neumann algebras, where ν_X,ν_Y are probability measures. Let $C(A,B)$ denote the set of all A,B-bimodules. This set $C(A,B)$ contains three distinguished subsets.

We will use the variable s to denote the first variable and t to denote the second variable. Following [27] we define:

Definition 5.1. A *discrete* (respectively, *diffuse*, *mixed*) A,B-bimodule is a Hilbert space $\mathcal{H}$ so that $\mathcal{H} \cong \bigoplus_{i\in I} L^2(X\times Y,\mu_i)$ where for all i, μ_i disintegrates as $\mu_i(s,t) = \mu_t^{(i)}(s)\nu_Y(t)$ with $\mu_t^{(i)}$ atomic (respectively non-atomic, a combination of both nonzero atomic part and nonzero non-atomic part) for ν_Y almost all t.

It is to be noted that in view of Lemma 3.6, the definition above only cares about the equivalence class of the measures μ_i and not a particular member of the class. The definition forces μ_i to be a non-atomic measure, and the existence of such a disintegration actually forces the push forward of μ_i's on the space Y to be dominated by ν_Y. We will restrict ourselves to the case I is countable. Let $C_d(A,B), C_{n.a}(A,B), C_m(A,B)$ denote the set of all discrete, diffuse, mixed A,B-bimodules respectively.

Denote by $C_d(A) \subset C_d(A,A) \subset C(A,A)$ the set of those bimodules $\mathcal{H} \in C_d(A,A)$

for which $\bar{\mathcal{H}} \in C_d(A, A)$. Here $\bar{\mathcal{H}}$ is the opposite Hilbert space of $\mathcal{H}$ with left and right actions interchanged. Bimodules in $C_d(A)$ are precisely those for which the associated measures μ_i's in Defn. 5.1 also have a completely atomic ν_X disintegration. Similarly define $C_{n.a}(A), C_m(A)$. Note that the spaces $C_d(A), C_{n.a}(A), C_m(A)$ are all closed with respect to taking sub bimodules.

When A, B are masas in a II_1 factor $\mathcal{M}$ the standard Hilbert space $L^2(\mathcal{M})$ is naturally a w^*-continuous A, B bimodule, meaning it carries a pair of mutually commuting normal representations of A and B.

Note that when we deal with the *left-right-measure* of a masa, knowing the disintegration along the second variable enables us to know the disintegration along the first variable as well, by pushing forward the former with the flip map (see Lemma 3.7).

Before we proceed to the characterization of masas we will have to make few definitions and statements that are very valuable tools yet not appear in standard measure theory courses. For details see [14], [21].

Definition 5.2. Let X be a Polish space. A subset B of X is said to have *Baire property* if there is an open set $\mathcal{O} \subset X$ and a comeager set $A \subset X$ such that $A \cap \mathcal{O} = A \cap B$.

The collection of sets with *Baire property* forms a σ-algebra which includes the *Borel σ-algebra.*

Definition 5.3. Let X and Y be Polish spaces. A function $f : X \mapsto Y$ is said to be *Baire measurable* if the inverse image of any *open set* has *Baire property.* The function f is said to be *universally Baire measurable* if given any Borel function g into X the function $f \circ g$ is Baire measurable.

Note that in particular every Borel function is *Baire measurable.*

Definition 5.4. A subset E of a Polish space is said to be *universally measurable* if it is measurable with respect to any *complete Borel probability measure.*

Definition 5.5. A subset E of a Polish space X is said to be $\underset{\sim}{\Sigma}^1_1$ or *analytic*, if there is a Polish space Y, a Borel subset B of Y and a Borel function $f : Y \mapsto X$ such that $f(B) = E$. In other words, $\underset{\sim}{\Sigma}^1_1$ sets are Borel images of Borel sets.

Remark 5.6. The above definition of analytic sets is as per [14]. However in, [15] continuous images rather than Borel images are used. The two definitions are in fact equivalent.

A very nontrivial theorem of Lusin says the following.

Theorem 5.7. (*Lusin*) *Every $\underset{\sim}{\Sigma}^1_1$ set has Baire property. Every $\underset{\sim}{\Sigma}^1_1$ set is universally measurable.*

For a function $f : Y \mapsto X$, the graph of f will be denoted by $\Gamma(f) = \{(f(y), y) : y \in Y\}$. The next theorem is very crucial in all our analysis.

Theorem 5.8. (*Selection Principle - Jankov, von Neumann*) *Let X, Y be Polish spaces and let $E \subset X \times Y$ be in $\underset{\sim}{\Sigma}^1_1$. Then E can be uniformized by a function that is both Baire and universally measurable, in the sense that for some $h : Y \mapsto X$ we have*

$$\Gamma(h_{|\pi_Y(E)}) \subseteq E$$

with the property that $h^{-1}(U)$ has the Baire property and is measurable with respect to any Borel probability measure for all open $U \subseteq X$.

Remark 5.9. Let ν_X and ν_Y be any two Borel probability measures on X, Y respectively. Let σ_{ν_X} and σ_{ν_Y} be the σ-algebras associated to the measures ν_X, ν_Y respectively. If h is the function in Thm. 5.8, then the inverse image of any Borel set in X

under h will lie in σ_{ν_Y}, because the collection of subsets of X whose inverse images fall in σ_{ν_Y} is a σ-algebra and contains all open sets. If in addition, h satisfies the property that $\nu_X(h(F)) = 0$ if and only if $\nu_Y(F) = 0$, then h is $(\sigma_{\nu_Y}, \sigma_{\nu_X})$ measurable.

Let $A \subset \mathcal{M}$ be a masa. Without loss of generality we assume that $A = L^\infty([0,1], \lambda)$ where λ is the Lebesgue measure on $[0,1]$. Let $[\eta_{[0,1]\times[0,1]}]$ denote the *left-right-measure* of A. We are including the diagonal. Fix any member $\eta_{[0,1]\times[0,1]}$ from the equivalence class. Since our base space is now fixed we will rename $\eta_{[0,1]\times[0,1]}$ by η to reduce the notation. We assume that η is a finite measure.

Consider the set $S_a = ([0,1] \times [0,1])_a$ as defined in Prop. 3.3 with respect to the disintegration along the y-axis i.e. the t variable. Then by Prop. 3.3, S_a is a $[\eta]$-measurable set, i.e. measurable with respect to the completion σ-algebra associated to η. Define measures

$$\eta_a = \eta_{|(S_a \setminus \Delta([0,1]))} \text{ and } \eta_{n.a} = \eta_{|(S_a^c \setminus \Delta([0,1]))}.$$

Then
(i) $\eta_{|\Delta([0,1])^c} = \eta_a + \eta_{n.a},\ \eta_a \perp \eta_{n.a}$.
(ii) Both $\eta_a, \eta_{n.a}$ have disintegrations along the x, y axes with respect to λ.

Note that the disintegration of the measure η_a along the x and y-axes must have at most countably many atoms almost all fibres (see Lemma 3.7), otherwise η is an infinite measure. Since changing the measure η_a or η on a set of measure 0 does not change the measure class of η_a or η, we can as well assume without loss of generality that, the disintegration of the measure η_a along y-axis (second variable) has at most countable number of atoms for all fibres. With this as set up we formalize the main theorem of this manuscript. Thm. 5.10 will be proved latter in this section.

Theorem 5.10. (*Classification of Types*) *A masa* $A \subset \mathcal{M}$ *is*

(i) *Cartan if and only if* $\eta_{n.a} = 0$ *equivalently* $L^2(A)^\perp \in C_d(A)$,
(ii) *singular if and only if* $\eta_a = 0$ *equivalently* $L^2(A)^\perp \in C_{n.a}(A)$,
(iii) $A \subsetneq N(A)'' \subsetneq \mathcal{M}$ *if and only if* $\eta_a \neq 0, \eta_{n.a} \neq 0$ *equivalently* $L^2(A)^\perp \in C_m(A)$.
(iv) A *is semiregular if and only if the closed support of* η_a *is* $[0,1] \times [0,1]$.

Remark 5.11. First of all, in view of Lemma 3.4 and 3.6, the characterization does not depend on any particular member of the *left-right-measure*.
Secondly, $L^2(A)$ is always included in $C_d(A)$, the disintegration having one atom at each point of the diagonal. That is the reason one excludes $L^2(A)$ from statements in Thm. 5.10.
Finally, from our discussion on direct integrals in Sec. 2, it follows that $L^2(A)^\perp$ is the direct integral over $[0,1] \times [0,1]$ with respect to the measure $\eta_{|(\Delta[0,1])^c}$, the measurable field of Hilbert spaces depending on $m_{[0,1]}$ or the *Pukánszky invariant*. So the equivalent statements regarding the type of bimodules and measure in Thm. 5.10 are obvious statements.

The next technical lemma is the key to characterization of masas. There are several measures involved in its statement and proof. Since there is danger of confusion with measurability of objects involved we will always use phrases like "μ-measurable".

Lemma 5.12. *Let* $\eta_a \neq 0$. *Let* $Y \subseteq (\Delta[0,1])^c$ *be a* η*-measurable set of strictly positive* η_a*-measure. There exists a* λ*-measurable set* $E^Y \subseteq [0,1]$ *with* $\lambda(E^Y) > 0$ *and a function* $h_Y : [0,1] \mapsto [0,1]$ *such that*
(i) h_Y *is* λ*-measurable,*

(ii) $\Gamma(h_Y)$ *is a* η*-measurable set,*
(iii) $\eta(\Gamma(h_Y)) > 0$ *and* $(h_Y(t), t) \in Y \cap S_a$ *for* $t \in E^Y$*,*
(iv) for $E \subset [0,1]$*,* $\lambda(E) = 0$ *if and only if* $\lambda(h_Y(E)) = 0$*.*

Proof. We have
$$\eta(S_a \cap Y) = \eta_a(S_a \cap Y) = \eta_a(Y) > 0.$$
Consider the disintegration of $\eta_{|Y}$ along the y-axis. There is a set $F^Y \subseteq [0,1]$ such that $\lambda(F^Y) > 0$ and for each $t \in F^Y$ the measure $(\eta_{|Y})_t$ has atoms with at most countable number of atoms and for $t \notin F^Y$ the same disintegration has no atoms. This is true because η is a finite measure, the set F^Y being $\pi_y(S_a \cap Y)$, π_y denoting the projection on to the y-axis. The set $S_a \cap Y$ is η-measurable, so $S_a \cap Y = B \cup N$ where B is a Borel set in $[0,1] \times [0,1]$ and N is a η-null set. The set B is a continuous image of a Polish space by Thm. 14.3.5 of [15] and so is $\pi_y(B)$. By Defn 3.1, $\lambda(\pi_y(N)) = 0$. So F^Y is λ-measurable set by Thm. 5.7. Throwing off another λ-null set from F^Y if necessary we can as well assume without loss of generality that F^Y is a Borel set.
Let $F_a^Y = \big((Y \cap S_a) \cap ([0,1] \times F^Y)\big)$ which is η-measurable. Write $F_a^Y = E_a^Y \cup N_1$ where N_1 is a η-null set and E_a^Y is a Borel set. Then by Thm. 14.3.5 of [15], E_a^Y is in $\underset{\sim}{\Sigma}_1^1$, in fact it is the continuous image of a Polish space. The hypothesis guarantees $\eta(E_a^Y) > 0$.
Let $E^Y = \pi_y(E_a^Y)$. Then E^Y is in $\underset{\sim}{\Sigma}_1^1$ and hence E^Y is λ-measurable by Thm. 5.7. Therefore by Def 3.1, $\lambda(E^Y) > 0$. By Thm. 5.8 applied to E_a^Y, there exists a function $h_Y : [0,1] \mapsto [0,1]$ that is both *Baire* and *universally measurable* in the sense of Thm. 5.8, such that $\Gamma(h_{Y|E^Y}) \subseteq E_a^Y$.
The inverse image under h_Y of any Borel subset of $[0,1]$ belongs to σ_λ. Therefore given $\epsilon > 0$, by Lusin's theorem there is a closed subset $G^Y \subseteq E^Y$ such that $\lambda(E^Y \setminus G^Y) < \epsilon$ and $h_{Y|G^Y}$ is continuous. Then $h_{Y|G^Y}$ is Borel measurable. So by Cor. 2.11 of [20], $\Gamma(h_{Y|G^Y})$ is Borel measurable and hence η-measurable.
The disintegration along the y-axis of the measure $\eta_{|\Gamma(h_{Y|G^Y})}$ is precisely the atom at the point $(h_Y(t), t)$ for each $t \in G^Y$ of the measure η_t. Outside G^Y we don't care. If $\eta(\Gamma(h_{Y|G^Y})) = 0$ then by definition of disintegration
$$0 = \int_{G^Y} \eta_t(\Gamma(h_Y)) d\lambda(t)$$
which implies that for λ almost all $t \in G^Y$ the point $(h_Y(t), t)$ is not an atom of η_t and hence cannot be in S_a. So $\eta(\Gamma(h_{Y|G^Y})) > 0$.
Clearly, $h_{Y|G^Y}$ satisfies the property that for any $E \subset G^Y$, $\lambda(E) = 0$ if and only if $\lambda(h_Y(E)) = 0$. Therefore by Thm. A.2, extend $h_{Y|G^Y}$ to a continuous function $\tilde{h}_Y$ which satisfies the property that for any $E \subset [0,1]$, $\lambda(E) = 0$ if and only if $\lambda(\tilde{h}_Y(E)) = 0$. So by Rem 5.9, $\tilde{h}_Y$ is $(\sigma_\lambda, \sigma_\lambda)$ measurable. Rename $\tilde{h}_Y$ to h_Y and G^Y to E^Y. The rest is clear from construction. □

Lemma 5.13. *Let* $\eta_a \neq 0$*. Let* $Y \subseteq (\Delta[0,1])^c$ *be a* η*-measurable set of strictly positive* η_a*-measure. Then* $\mathcal{U}(A) \subsetneq N(A)$*, where* $\mathcal{U}(A)$ *denotes the unitary group of* A*.*
More precisely, there exists a subset F^Y *of* $[0,1]$ *such that* $\lambda(F^Y) > 0$*, a invertible map* $h_Y : F^Y \mapsto h_Y(F^Y)$ *and a nonzero vector* $\zeta_Y \in L^2(N(A)'') \ominus L^2(A)$ *such that*

(i) $\zeta_Y = v_Y \rho_Y$ *with* $v_Y \in \mathcal{GN}(A), \rho_Y \in L^2(A)^+$

(ii) $\overline{A\zeta_Y A}^{\|\cdot\|_2} \cong \int_{\Gamma(h_Y)}^{\oplus} \mathbb{C}_{s,t} d\eta(s,t),$ *where* $\mathbb{C}_{s,t} = \mathbb{C}$,
(iii) $\Gamma(h_Y) \subseteq Y \cap S_a$,
(iv) $\eta(\Gamma(h_Y)) > 0$,
$(v) 1 \in Puk(A)$.

Proof. Using Lemma 5.12, choose the function h_Y that satisfies the conclusion of that Lemma. Note that h_Y satisfies the conditions of Prop. A.4. Apply Prop. A.4 to the function h_Y and the set E^Y to extract a set $F^Y \subseteq E^Y$ such that $\lambda(F^Y) > 0$ and h_Y is one to one on F^Y. So

$$h_Y : F^Y \mapsto h_Y(F^Y) \text{ is invertible.}$$

Note that as $\lambda(F^Y) > 0$ so $\eta(\Gamma(h_{Y|F^Y})) > 0$. There is no information of the *Pukánszky invariant* yet. So assume that $Puk(A) = \{n_i : n_i \in \mathbb{N}_\infty, i \in I\}$, where the indexing set I could be finite or countable. Let

$$E_{n_i} = \{(s,t) \in \Delta([0,1])^c : m_{[0,1]}(s,t) = n_i\},$$

where $m_{[0,1]}$ denotes the multiplicity function of the direct integral decomposition of $L^2(\mathcal{M})$ over $[0,1] \times [0,1]$ with respect to the measure η. Then for each $i \in I$ it is well known that E_{n_i} are η-measurable sets. Also

$$\int_{E_{n_i}}^{\oplus} \mathbb{C}^{n_i}_{s,t} d\eta(s,t) \cong L^2(E_{n_i}, \eta_{|E_{n_i}}) \otimes \mathbb{C}^{n_i} \text{ where } \mathbb{C}^{n_i}_{s,t} = \mathbb{C}^{n_i}, \text{ and}$$
$$\bigoplus_{i\in I} L^2(E_{n_i}, \eta_{|E_{n_i}}) \otimes \mathbb{C}^{n_i} \cong L^2(\mathcal{M}) \ominus L^2(A).$$

In the above equation $\mathbb{C}^\infty$ stands for $l^2(\mathbb{N})$. Fix orthonormal bases $\{e^{(n_i)}_j\}_{1\leq j\leq n_i}$ of $\mathbb{C}^{n_i}$ for all $i \in I$.
Then

$$\sum_{i\in I} \chi_{\Gamma(h_{Y|F^Y}) \cap E_{n_i}} \otimes e^{(n_i)}_1$$

where χ denotes the indicator function, can be identified with a vector $\zeta_Y \in (1 - e_A)(L^2(\mathcal{M}))$ such that

$$A\zeta_Y A = A\zeta_Y = \zeta_Y A. \tag{5.1}$$

Eq. (5.1) is easy to check, in fact one only uses that fact that h_Y is locally one to one and onto. That $\zeta_Y \neq 0$ is due to the fact $\eta(\Gamma(h_{Y|F^Y})) > 0$. Then from Theorem 4.9, it follows that $\zeta_Y = v_Y \rho_Y$ where $\rho_Y = (\zeta_Y^* \zeta_Y)^{\frac{1}{2}} \in L^2(A)^+$ and $v_Y \in \mathcal{GN}(A)$. Clearly, $v_Y \notin A$, as otherwise $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(A)$ would become the direct integral of complex numbers over some subset of the diagonal with respect to the measure $\Delta_*\lambda$, where $\Delta : [0,1] \mapsto [0,1] \times [0,1]$ is the map $\Delta(x) = (x,x)$.
Thus $\zeta_Y \in L^2(N(A)'')$ and hence $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(N(A)'')$. Clearly,

$$\overline{A\zeta_Y A}^{\|\cdot\|_2} \cong \int_{\Gamma(h_{Y|F^Y})}^{\oplus} \mathbb{C}_{s,t} d\eta(s,t), \text{ where } \mathbb{C}_{s,t} = \mathbb{C}. \tag{5.2}$$

So $\overline{A\zeta_Y A}^{\|\cdot\|_2} \perp L^2(A)$ and $\overline{A\zeta_Y A}^{\|\cdot\|_2} \in C_d(A)$. Since $\overline{A\zeta_Y A}^{\|\cdot\|_2} \subseteq L^2(N(A)'')$ so $\eta(\Gamma(h_{Y|F^Y}) \cap E_{n_i}) = 0$ if $n_i \geq 2$ from a result of Popa [25]. Thus $1 \in Puk(A)$. $\square$

Each partial isometry $0 \neq v \in \mathcal{GN}(A)$ implements a measure preserving local isomorphism $T : ([0,1],\lambda) \mapsto ([0,1],\lambda)$ such that $vav^* = a \circ T^{-1}$ for all $a \in A$. With abuse of notation we will write $v = T$. Then $\Gamma(v) = \{(T(t), t) : t \in Dom(T)\}$, $Dom(T)$ denoting the domain of T.

Lemma 5.14. *Let $\eta_a \neq 0$. Let $Y \subseteq (\Delta[0,1])^c$ be a η-measurable set of strictly positive η_a-measure. Then there is a nonzero partial isometry $v \in \mathcal{GN}(A)$ such that $\Gamma(v) \subseteq Y$.*

Proof. By Lemma 5.13, there exists a subset F^Y of $[0,1]$ such that $\lambda(F^Y) > 0$, a invertible map $h_Y : F^Y \mapsto h_Y(F^Y)$ and a nonzero vector $\zeta_Y \in L^2(N(A)'') \ominus L^2(A)$ such that $\zeta_Y = v_Y\rho_Y$ with $v_Y \in \mathcal{GN}(A)$, $\rho_Y \in L^2(A)^+$ and satisfying property (ii), (iii), (iv) of Lemma 5.13.
Let $\eta_{\zeta_Y}, \eta_{v_Y}$ be the measures on $[0,1]\times[0,1]$ defined in Eq. (3.1). Let $q_Y = v_Y v_Y^* \in A$. With abuse of notation we will regard q_Y as a measurable subset of $[0,1]$ as well. We claim that, $\eta_{\zeta_Y} \ll \eta_{v_Y} \ll \eta_{\zeta_Y}$. Indeed for $a, b \in C[0,1]$,

$$\begin{aligned}
\int_{[0,1]\times[0,1]} a(s)b(t)d\eta_{\zeta_Y}(s,t) &= \int_{\Gamma(h_Y)} a(s)b(t)d\eta_{\zeta_Y}(s,t)\\
&= \tau(\rho_Y^* v_Y^* a v_Y \rho_Y b)\\
&= \tau(\rho_Y^* v_Y^* a v_Y b \rho_Y) \text{ (as } \rho_Y b = b\rho_Y)\\
&= \tau(v_Y^* a v_Y b \rho_Y \rho_Y^*)\\
&= \tau(v_Y^* a v_Y b \rho_Y^* \rho_Y)\\
&= \tau(v_Y^* a v_Y \rho_Y^* \rho_Y b)\\
&= \int_{q_Y} a(v_Y(t))b(t)\left|\rho_Y(t)\right|^2 d\lambda(t)\\
&= \int_{\Gamma(v_Y)} a(s)b(t)\left|\rho_Y(t)\right|^2 d\eta_{v_Y}(s,t)\\
&= \int_{[0,1]\times[0,1]} a(s)b(t)\left|\rho_Y(t)\right|^2 d\eta_{v_Y}(s,t).
\end{aligned}$$

In the above string of equalities we have used the facts that τ extends to a trace like functional on $L^1(A)$ and the left and right actions of A on $L^2(A)$, $L^1(A)$ coincides. Using Thm. 4.9, by standard arguments it follows that $\eta_{\zeta_Y} \ll \eta_{v_Y} \ll \eta_{\zeta_Y}$. Thus the result follows with $v = v_Y$. □

Suppose $\{v_j\}_{j\in J}$ is a family of partial isometries in $\mathcal{GN}(A)$ such that $Av_j \perp Av_{j'}$ whenever $j \neq j'$. Denote by [25]

$$\sum_{j\in J} Av_j = \left\{ x \in \mathcal{M} : x = \sum_{j\in J} a_j v_j, \text{ for } a_j \in A \text{ with } \sum_{j\in J} \|a_j v_j\|_2^2 < \infty \right\}.$$

Theorem 5.15. (*Compare Cor.* 2.5 [25]) *Let $\eta_a \neq 0$. Then $A \subsetneq N(A)''$. Moreover, there is a sequence $\{v_n\}_{n=0}^{\infty} \subset \mathcal{GN}(A)$ of nonzero partial isometries (with possibility that the sequence could be finite) with $v_0 = 1$ such that,*

(i) $\Gamma(v_n) \cap \Gamma(v_m) = \emptyset$ *for* $n \neq m$,

(ii) $\eta_a([0,1]\times[0,1]) = \sum_{n=1}^{\infty} \eta_a(\Gamma(v_n))$,

(iii) $\oplus_{n=0}^{\infty} \overline{Av_n}^{\|\cdot\|_2} \cong \int_{\cup_{n=0}^{\infty}\Gamma(v_n)}^{\oplus} \mathbb{C}_{s,t} d(\eta_a + \Delta_*\lambda)(s,t) \cong L^2(N(A)'')$,

(*where* $\mathbb{C}_{s,t} = \mathbb{C}$ *and* $\Delta : [0,1] \mapsto [0,1]\times[0,1]$ *by* $\Delta(x) = (x,x)$)

$$(iv)\ N(A)'' = \sum_{n=0}^{\infty} Av_n,$$

and $\mathcal{A}$ restricted to $\oplus_{n=0}^{\infty}\overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to the decomposition in (iii).

Proof. First of all assuming that (i) in the statement is true it follows that $Av_n \perp Av_m$ whenever $n \neq m$. Indeed, $\overline{Av_n}^{\|\cdot\|_2} \subseteq L^2(N(A)'')$. Now $\mathcal{A}$ restricted to $\overline{Av_n}^{\|\cdot\|_2}$ is an abelian algebra with a cyclic vector, so it is maximal abelian. The projection e_{Av_n} onto $\overline{Av_n}^{\|\cdot\|_2}$ is in $\mathcal{A}$. So $\overline{Av_n}^{\|\cdot\|_2}$ is the direct integral of complex numbers over a subset X_n of $[0,1]\times[0,1]$ with respect to the measure η and $\mathcal{A}$ restricted to $\overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to this decomposition. But $\eta(X_n\Delta\Gamma(v_n)) = 0$. Again $\eta_{n.a}(\Gamma(v_n)) = 0$. So the direct integral as stated above is actually with respect to the measure $\eta_a + \Delta_*\lambda$. The graphs being disjoint for $n \neq m$ forces the orthogonality of Av_n and Av_m whenever $n \neq m$. The sum in (iii) therefore makes sense.
Using Lemma 5.14, choose a maximal family $\{v_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{GN}(A)$, for some indexing set Λ, such that $\Gamma(v_\alpha) \subset \Delta([0,1])^c$ for all $\alpha \in \Lambda$ and $\Gamma(v_\alpha)\cap\Gamma(v_\beta) = \emptyset$ whenever $\alpha \neq \beta$. Since $Av_\alpha \perp Av_\beta$ whenever $\alpha \neq \beta$ (by similar argument as above) so the indexing set must be countable by separability assumption of $L^2(\mathcal{M})$. So we index this maximal family by $\{v_n\}_{n=1}^{\infty}$. Let $v_0 = 1$. So (i) follows by construction.
If $\eta_a([0,1]\times[0,1]) > \sum_{n=1}^{\infty}\eta_a(\Gamma(v_n))$ then $S_a \setminus \cup_{n=1}^{\infty}\Gamma(v_n)$ is a set of strictly positive η_a measure. A further application of Lemma 5.14 violates the maximality of $\{v_n\}_{n=1}^{\infty}$. This proves (ii).
By the argument of the first paragraph and Lemma 5.7 [10],

$$\oplus_{n=1}^{\infty}\overline{Av_n}^{\|\cdot\|_2} \cong \int_{\cup_{n=1}^{\infty}\Gamma(v_n)}^{\oplus} \mathbb{C}_{s,t}\, d\eta_a(s,t) \subseteq L^2(N(A)'') \ominus L^2(A) \tag{5.3}$$

and $\mathcal{A}$ restricted to $\oplus_{n=1}^{\infty}\overline{Av_n}^{\|\cdot\|_2}$ is diagonalizable with respect to the decomposition in Eq. (5.3).
If $0 \neq \zeta = \zeta^* \in L^2(N(A)'')\ominus L^2(A)$ is such that $\zeta \perp Av_n$ for all $n \geq 1$ then $A\zeta A \perp Av_n$ for all $n \geq 0$. By arguments similar to the first paragraph, $\overline{A\zeta A}^{\|\cdot\|_2}$ is the direct integral over a η-measurable set X_ζ, of complex numbers with respect to the measure η and $\mathcal{A}$ restricted to $\overline{A\zeta A}^{\|\cdot\|_2}$ is diagonalizable respecting this decomposition. If ζ as a L^2 function stays nonzero on a set of positive $\Delta_*\lambda$-measure then ζ cannot be perpendicular to $L^2(A)$. By Prop. 3.11, $\overline{A\zeta A}^{\|\cdot\|_2} \in C_d(A)$ and hence by Theorem 3.8 and Lemma 5.7 [10], we can assume $X_\zeta \subset S_a \setminus \Delta([0,1])$. Since $\zeta \neq 0$ so $\eta(X_\zeta) = \eta_a(X_\zeta) > 0$. Since η_a is concentrated on $\cup_{n=1}^{\infty}\Gamma(v_n)$, so $X_\zeta \cap \Gamma(v_n)$ has strictly positive η_a and hence η measure for some $n \geq 1$. Note that $e_{N(A)''} \in \mathcal{A}$ and $\mathcal{A}e_{N(A)''} = \mathcal{A}'e_{N(A)''}$ from [25]. On the other hand, by Lemma 5.7 [10], $L^2(N(A)'') \ominus L^2(A)$ will be expressed as a direct integral over some subset of $[0,1]\times[0,1]$ with respect to η, with multiplicity strictly bigger than 1 on a set of positive η-measure. This contradicts $\mathcal{A}e_{N(A)''}$ is maximal abelian. Thus

$$\oplus_{n=0}^{\infty}\overline{Av_n}^{\|\cdot\|_2} \cong \int_{\cup_{n=0}^{\infty}\Gamma(v_n)}^{\oplus} \mathbb{C}_{s,t}\, d(\eta_a + \Delta_*\lambda)(s,t) \cong L^2(N(A)''),$$

with associated statements about diagonalizability of $\mathcal{A}$. Finally

$$\sum_{n=0}^{\infty} Av_n = \overline{\sum_{n=0}^{\infty} Av_n}^{\|\cdot\|_2} \cap \mathcal{M} = \left(\oplus_{n=0}^{\infty}\overline{Av_n}^{\|\cdot\|_2}\right) \cap \mathcal{M} = L^2(N(A)'') \cap \mathcal{M} = N(A)''.$$

□

Remark 5.16. Thm. 5.15 generalizes Cor. 2.5 of [25]. In general we cannot hope to find unitaries as was the case in Cor. 2.5 [25]. The situation in Cor. 2.5 of [25] was completely different, where the assumption was that, the masa is Cartan. Assuming the masa is Cartan, forces the disintegration of the measure η_a to have at least one atom off the diagonal in almost every fibre. Such an assumption cannot be made for a general masa. For example consider the following situation. Let $C \subset \mathcal{R}$ be a Cartan masa and let $S \subset \mathcal{R}$ be a singular masa, where $\mathcal{R}$ denotes the hyperfinite II$_1$ factor. Then $C \oplus S \subset \mathcal{R} \oplus \mathcal{R}$ is a masa, where the trace on $\mathcal{R} \oplus \mathcal{R}$ is $\frac{1}{2}\tau_{\mathcal{R}} \oplus \frac{1}{2}\tau_{\mathcal{R}}$, $\tau_{\mathcal{R}}$ denoting the unique, normal, faithful tracial state of $\mathcal{R}$. Then $C \oplus S \subset (\mathcal{R} \oplus \mathcal{R}) * \mathcal{R} \cong L(\mathbb{F}_2)$ (from [8]) is a masa for which such an assumption will fail from Prop. 5.10 [10].

We will now present the proof of Thm 5.10.

Proof of 5.10. Case (i). The necessary and sufficient condition for Cartan masas follows directly from Thm. 5.15.
Case (ii). The result for singular masas also follows from Thm. 5.15.
Case (iii). Let A $\subsetneq N(A)'' \subsetneq \mathcal{M}$. If $\eta_a = 0$ then, by conclusion of (ii), A would become singular. Therefore $\eta_a \neq 0$. If $\eta_{n.a} = 0$ then by conclusion of part (i), A would be Cartan. Therefore $\eta_{n.a} \neq 0$ as well.
Conversely, if $\eta_{n.a} \neq 0$ and $\eta_a \neq 0$, then by Theorem 5.15, $A \subsetneq N(A)'' \subsetneq \mathcal{M}$.
Case (iv). First assume that $N(A)''$ is a factor. From Thm. 5.15 it follows that

$$L^2(N(A)'') \cong \int_{[0,1]\times[0,1]}^{\oplus} \mathbb{C}_{s,t}\, d(\eta_a + \tilde{\Delta}_*\lambda)(s,t), \text{ where } \mathbb{C}_{s,t} = \mathbb{C}$$

and $\mathcal{A}$ restricted to this subspace is diagonalizable, where $\tilde{\Delta} : [0,1] \mapsto [0,1] \times [0,1]$ is defined by $\tilde{\Delta}(x) = (x,x)$. Therefore $[\eta_a + \tilde{\Delta}_*\lambda]$ is the *left-right-measure* for the inclusion $A \subset N(A)''$. If the closed support of η_a is strictly contained in $[0,1] \times [0,1]$ then, there must be a open set $U \subset [0,1] \times [0,1]$ such that $\eta_a(U) = 0$. It follows that the map $a \otimes b \mapsto aJ_{N(A)''}b^*J_{N(A)''}$, where $a, b \in C([0,1])$ was not an injection. On the other hand, as $N(A)''$ is a factor the above map must be an injection by Prop. 2.8. This contradiction proves that the closed support of η_a is $[0,1] \times [0,1]$.
Conversely assume $N(A)''$ has a nontrivial center. Let $p \in \mathbf{Z}(N(A)'')$ be a projection which is different from 0 and 1. Then

$$N(A)'' = N(A)''p \oplus N(A)''(1-p).$$

So $p \in A' \cap N(A)''$ and hence $p \in A$. So

$$A = Ap \oplus A(1-p). \tag{5.4}$$

It follows that are exists λ-measurable sets $F_1, F_2 \subset [0,1]$ such that, $F_1 \cup F_2 = [0,1]$, $F_1 \cap F_2 = \emptyset$, $C(F_1)$, $C(F_2)$ are *w.o.t* dense unital subalgebras of Ap and $A(1-p)$ respectively and $C(F_1) \oplus C(F_2)$ is *w.o.t* dense in A. With respect to the Eq. (5.4) let $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ be the decompositions of $a, b \in C([0,1])$, where $a_i, b_i \in C(F_i)$ for $i = 1,2$. For $\zeta \in L^2(N(A)'')$ one has an analogous decomposition $\zeta = \zeta_1 \oplus \zeta_2$ with $\zeta_1 = p\zeta$ and $\zeta_2 = (1-p)\zeta$. The equation

$$\langle a\zeta b, \zeta\rangle = \langle (a_1 \oplus a_2)(\zeta_1 \oplus \zeta_2)(b_1 \oplus b_2), (\zeta_1 \oplus \zeta_2)\rangle = \langle a_1\zeta_1 b_1, \zeta_1\rangle + \langle a_2\zeta_2 b_2, \zeta_2\rangle$$

shows that the *left-right-measure* for the inclusion $A \subset N(A)''$ will be concentrated on $F_1 \times F_1 \cup F_2 \times F_2$. It follows that closed support of η_a is strictly contained in $[0,1] \times [0,1]$. This completes the proof. $\square$

Remark 5.17. The proof of Case (iv) actually shows that if $A \subset \mathcal{N}$ is a masa where $\mathcal{N}$ is a finite type II algebra with a nontrivial center then for any choice of compact Hausdorff space X such that $C(X)$ is $w.o.t$ dense, unital and norm separable subalgebra of A, the map

$$\sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i J b_i^* J$$

from $C(X) \otimes_{alg} C(X) \mapsto \mathbf{B}(L^2(\mathcal{N}))$ is not an injection for any choice of trace.

The following results about masas that were proved by experts in different ways are just easy consequences of the *measurable selection principle* as we have described in this section.

Corollary 5.18. *If $A \subset \mathcal{M}$ is a Cartan masa then $A \subset B$ is a Cartan masa for all von Neumann subalgebra $A \subsetneq B \subsetneq \mathcal{M}$.*

Proof. By Lemma 5.7 of [10] the *left-right-measure* of the inclusion $A \subset \mathcal{M}$ is $[\eta_B + \eta_{B^\perp}]$ where $[\eta_B]$ is the *left-right-measure* of the inclusion $A \subset B$ and $\eta_B \perp \eta_{B^\perp}$. It follows that η_B has atomic disintegration along both axes. The result is then immediate from Thm. 5.10 and Thm 5.15. $\square$

Corollary 5.19. *Let $A \subset \mathcal{M}$ be a masa and let Q be a finite von Neumann algebra such that $dim(Q) \geq 2$. Then $N_{\mathcal{M}*Q}(A) = N_{\mathcal{M}}(A)$.*

Proof. In this proof we consider *left-right-measures* restricted to the off diagonal. First of all it well known that $A \subset \mathcal{M} * Q$ is a masa. Let $[\eta_{\mathcal{M}}]$ denote the *left-right-measure* of the inclusion $A \subset \mathcal{M}$. Write $\eta_{\mathcal{M}} = \eta_1 + \eta_2$ where $\eta_1 \ll \lambda \otimes \lambda$ and $\eta_2 \perp \lambda \otimes \lambda$. Using Prop. 5.10 and Lemma 5.7 [10] it follows that the *left-right-measure* $[\eta_{\mathcal{M}*Q}]$ of the inclusion $A \subset \mathcal{M} * Q$ is given by

$$\eta_{\mathcal{M}*Q} = \begin{cases} \eta_{\mathcal{M}} + \lambda \otimes \lambda & \text{if } \eta_1 = 0, \\ \eta_2 + \lambda \otimes \lambda & \text{if } \eta_1 \neq 0. \end{cases}$$

The rest is obvious from Thm. 5.10 and Thm. 5.15. $\square$

Corollary 5.20. *Let $A \subset \mathcal{M}$ be a Cartan masa and let $A \subset B \subsetneq \mathcal{M}$ be an intermediate subalgebra. Then there is a $v \in \mathcal{GN}(A)$ such that $v \perp B$.*

Proof. By Lemma 5.7 [10], the *left-right-measure* of the inclusion $A \subset \mathcal{M}$ is $[\eta_B + \eta_{B^\perp}]$ where $\eta_B \perp \eta_{B^\perp}$ and $[\eta_B]$ is the *left-right-measure* of the inclusion $A \subset B$. Note $\eta_{B^\perp} \neq 0$. Apply Lemma 5.14. $\square$

We prove the next theorem in the context of II_1 factors. But it can be easily generalized to finite von Neumann algebras. Let $\mathcal{M}_i$, $i = 1, 2$ be separably acting II_1 factors with normal, faithful tracial states τ_i respectively. Let $\mathcal{M}_i$ act on $L^2(\mathcal{M}_i, \tau_i)$ by left multiplication. Let $A \subset \mathcal{M}_1$ and $B \subset \mathcal{M}_2$ be masas. Fix compact Polish spaces X, Y such that $C(X) \subset A$ and $C(Y) \subset B$ are unital, norm separable and $w.o.t$ dense. Let ν_X and ν_Y denote the tracial measures for A, B respectively, which will be assumed to be complete. Let the *left-right-measure* of A on $X \times X$ be $[\sigma_1]$ and that of B on $Y \times Y$ be $[\sigma_2]$. Here we are allowing the diagonals, i.e. we are assuming $\sigma_{1|\Delta(X)} = (\tilde{\Delta}_X)_* \nu_X$ and $\sigma_{2|\Delta(Y)} = (\tilde{\Delta}_Y)_* \nu_Y$ where $\tilde{\Delta}_X : X \mapsto X \times X$ by $\tilde{\Delta}_X(x) = (x, x)$ and $\tilde{\Delta}_Y : Y \mapsto Y \times Y$ by $\tilde{\Delta}_Y(y) = (y, y)$.

By Tomita's theorem on commutants $A\overline{\otimes}B$ is a masa in $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$. $X \times Y$ is compact and Polish, and $C(X \times Y)$ is unital, norm separable and *w.o.t* dense in $A\overline{\otimes}B$. The standard Hilbert space and the Tomita's involution operator for $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$ is $L^2(\mathcal{M}_1,\tau_1)\otimes L^2(\mathcal{M}_2,\tau_2)$ and $J_{\mathcal{M}_1}\otimes J_{\mathcal{M}_2}$ respectively. The tracial measure for $A\overline{\otimes}B$ on $X\times Y$ is clearly $\nu_X\otimes\nu_Y$. With this as set up we formulate the next theorem which appeared in [2]. The same proof actually generalizes to infinite tensor products.

Theorem 5.21. (*Chifan's Normaliser Formula*) *Let $A\subset\mathcal{M}_1$ and $B\subset\mathcal{M}_2$ be masas in separably acting* II_1 *factors $\mathcal{M}_1$ and $\mathcal{M}_2$. Then*

$$N(A\overline{\otimes}B)''=N(A)''\overline{\otimes}N(B)''.$$

Proof. Fix σ_1 and σ_2 from the aforesaid class of *left-right-measures*. The *left-right-measure* of $A\overline{\otimes}B$ on $(X\times Y)\times(X\times Y)$ which is denoted by $[\beta]$ is given by

$$d\beta(s_X,s_Y,t_X,t_Y)=d\sigma_1(s_X,t_X)d\sigma_2(s_Y,t_Y)$$

from Prop. 5.2 [10]. Here s is the variable running along the first coordinate (horizontal direction) and t along the second coordinate (vertical direction). Then from Lemma 3.5 it follows that the disintegration of β along the t variable (vertical direction) is given by

$$\beta_{t_{X\times Y}}=\sigma_{1t_X}\otimes\sigma_{2t_Y},\ (t_X,t_Y)\text{ - a.e }\nu_X\otimes\nu_Y,\ \text{where } t_{X\times Y}=(t_X,t_Y).$$

For fixed $t_{X\times Y}=(t_X,t_Y)\in X\times Y$ the measure $\beta_{t_{X\times Y}}$ has an atom at the point (s_X,s_Y,t_X,t_Y) if and only if σ_{1t_X} has an atom at (s_X,t_X) and σ_{2t_Y} has an atom at (s_Y,t_Y). Therefore

$$((X\times Y)\times(X\times Y))_a=S_{2,3}((X\times X)_a\times(Y\times Y)_a),$$

where $S_{2,3}$ denotes the permutation $(2,3)$ on four symbols (see Prop. 3.3). Therefore,

$$\beta_{|((X\times Y)\times(X\times Y))_a}=\sigma_{1|(X\times X)_a}\otimes\sigma_{2|(Y\times Y)_a}.$$

Hence denoting $\mathbb{C}_{s_X,s_Y,t_X,t_Y}=\mathbb{C}$, $\mathbb{C}_{s_X,t_X}=\mathbb{C}=\mathbb{C}_{s_Y,t_Y}$ we have

$$\begin{aligned}L^2(N(A\overline{\otimes}B)'')&\cong\int^{\oplus}_{((X\times Y)\times(X\times Y))_a}\mathbb{C}_{s_X,s_Y,t_X,t_Y}d\beta(s_X,s_Y,t_X,t_Y)\\&\cong\int^{\oplus}_{(X\times X)_a}\mathbb{C}_{s_X,t_X}d\sigma_1(s_X,t_X)\otimes\int^{\oplus}_{(Y\times Y)_a}\mathbb{C}_{s_Y,t_Y}d\sigma_2(s_Y,t_Y)\\&\cong L^2(N(A)'')\otimes L^2(N(B)'')\text{ from Thm. 5.15.}\end{aligned}$$

Since the containment $N(A)''\overline{\otimes}N(B)''\subseteq N(A\overline{\otimes}B)''$ is obvious we are done. □

Corollary 5.22. [35] *Let $A\subset\mathcal{M}_1$ and $B\subset\mathcal{M}_2$ be singular masas in separably acting* II_1 *factors $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $A\overline{\otimes}B$ is singular in $\mathcal{M}_1\overline{\otimes}\mathcal{M}_2$.*

6. Asymptotic Homomorphism and Measure Theory

The equivalence of WAHP and singularity is a nontrivial theorem [35]. In this section we will give a direct proof of the equivalence of WAHP and singularity by using measure theoretic tools. We will also present partial results about AHP. In order to do so we will first have to relate certain norms to the *left-right-measure*. The measure theoretic tools described in this section will be used in a future paper for explicit calculation of *left-right-measures*.

Let $A\subset\mathcal{M}$ be a masa. Let λ denote the Lebesgue measure on $[0,1]$ so that $A\cong L^\infty([0,1],\lambda)$. Then λ is the tracial measure. Let $[\eta]$ denote the *left-right-measure* for A. We assume that η is a probability measure on $[0,1]\times[0,1]$ and $\eta(\Delta([0,1]))=0$. Let $B[0,1]$ denote the collection of all bounded measurable functions on $[0,1]$.

Notation: The disintegrated measures are usually written with a subscript $t \mapsto \eta_t$ in the literature. But in this section we will use the superscript notation $t \mapsto \eta^t$ to denote them. The (π_1, λ) disintegration of measures will be indexed by the variable t and the (π_2, λ) disintegration will be indexed by the variable s.

In all the following results that uses disintegration of measures we will only state or prove the result with respect to the (π_1, λ) disintegration. Statements about the (π_2, λ) disintegration are analogous.

Lemma 6.1. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let η_x denote the measure on $[0,1] \times [0,1]$ defined in Eq. (3.1). Then η_x admits (π_i, λ) disintegrations $[0,1] \ni t \mapsto \eta_x^t$ and $[0,1] \ni s \mapsto \eta_x^s$, where π_i, $i = 1,2$ denotes the coordinate projections. Moreover,*

$$\eta_x^t([0,1] \times [0,1]) = \mathbb{E}_A(xx^*)(t), \ \lambda \ a.e.$$

Proof. From Lemma 5.7 of [10] it follows that there is a measure η_0 such that (i) $\eta_0 \perp \eta_x$, (ii) $[\eta_x + \eta_0]$ is the *left-right-measure* for A. Therefore $[(\pi_i)_*(\eta_0 + \eta_x)] = [\lambda]$ by Lemma 2.10 and hence $(\pi_i)_*(\eta_x) \ll \lambda$ for $i = 1,2$. Consequently from Thm. 3.2, η_x admits (π_i, λ) disintegrations for $i = 1,2$.
Note that $\eta_x([0,1] \times [0,1]) = \tau(xx^*) = \tau(\mathbb{E}_A(xx^*))$. From (ii) of Defn. 3.1 it follows that $[0,1] \ni t \mapsto \eta_x^t([0,1] \times [0,1])$ is measurable. Let $E \subseteq [0,1]$ be any Borel set. Then there exists a sequence of functions $f_n \in C[0,1]$ such that $0 \leq f_n \leq 1$ and $f_n \to \chi_E$ pointwise. By dominated convergence theorem we have $\eta_x(f_n \otimes 1) \to \eta_x(\chi_E \otimes 1)$. On the other hand,

$$\begin{aligned}
\eta_x(f_n \otimes 1) = \langle f_n x, x \rangle = \tau(f_n xx^*) = \tau(f_n \mathbb{E}_A(xx^*)) &= \int_0^1 f_n(t) \mathbb{E}_A(xx^*)(t) d\lambda(t) \\
&\to \int_0^1 \chi_E(t) \mathbb{E}_A(xx^*)(t) d\lambda(t), \text{ as } n \to \infty, \\
&= \int_E \mathbb{E}_A(xx^*)(t) d\lambda(t).
\end{aligned}$$

From Defn. 3.1 again we have

$$\eta_x(\chi_E \otimes 1) = \int_0^1 \eta_x^t(\chi_E \otimes 1) d\lambda(t) = \int_E \eta_x^t([0,1] \times [0,1]) d\lambda(t).$$

Therefore for all Borel sets $E \subseteq [0,1]$ we have

$$\int_E \eta_x^t([0,1] \times [0,1]) d\lambda(t) = \int_E \mathbb{E}_A(xx^*)(t) d\lambda(t).$$

Thus, $\eta_x^t([0,1] \times [0,1]) = \mathbb{E}_A(xx^*)(t)$ for λ almost all t. $\square$

Lemma 6.2. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let $f \in B[0,1]$. Then the functions $[0,1] \ni t \mapsto \eta_x^t(1 \otimes f), [0,1] \ni s \mapsto \eta_x^s(f \otimes 1)$ are in $L^\infty([0,1], \lambda)$.*

Proof. We will only prove for the (π_1, λ) disintegration. From Lemma 6.1 we know that η_x admits a (π_1, λ) disintegration. From Defn. 3.1 we also know that $[0,1] \ni t \mapsto \eta_x^t(1 \otimes f)$ is measurable. Now of $0 \leq t \leq 1$

$$\left|\eta_x^t(1 \otimes f)\right| \leq \|f\| \, \eta_x^t([0,1] \times [0,1]).$$

Now use Lemma 6.1. $\square$

Lemma 6.3. *Let $x \in \mathcal{M}$ be such that $\mathbb{E}_A(x) = 0$. Let $b, w \in B[0,1]$. Then*

$$\|\mathbb{E}_A(bxwx^*)\|_2^2 = \int_0^1 |b(t)|^2 \left|\eta_x^t(1 \otimes w)\right|^2 d\lambda(t).$$

Proof. We have noted before that η_x admits (π_i, λ) disintegrations for $i = 1, 2$. Secondly, as $b, w \in B[0,1]$, so $[0,1] \ni t \mapsto b(t)\eta_x^t(1 \otimes w)$ is in $L^\infty([0,1], \lambda)$ from Lemma 6.2. Now

$$\begin{aligned}
\|\mathbb{E}_A(bxwx^*)\|_2^2 &= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} |\langle a, \mathbb{E}_A(bxwx^*)\rangle|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} |\tau(a\mathbb{E}_A(bxwx^*))|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} |\tau(\mathbb{E}_A(abxwx^*))|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} |\tau(abxwx^*)|^2 \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} \left| \int_{[0,1]\times[0,1]} a(t)b(t)w(s)d\eta_x(t,s) \right|^2 \text{ (from Eq. (3.1))} \\
&= \sup_{\substack{a \in C[0,1] \\ \|a\|_2 \le 1}} \left| \int_0^1 a(t)b(t)\eta_x^t(1 \otimes w)d\lambda(t) \right|^2 \text{ (from Defn. 3.1)} \\
&= \int_0^1 |b(t)|^2 \left|\eta_x^t(1 \otimes w)\right|^2 d\lambda(t) \text{ (from Lemma 6.2).}
\end{aligned}$$

□

The following facts are well known, we just record them for completeness. For details we refer the reader to [16]. Recall that a subset $S \subseteq \mathbb{Z}$ is said to be of *full density* if

$$\lim_n \frac{\#(S \cap [-n, n])}{2n+1} = 1.$$

Definition 6.4. A measure μ on $[0,1]$ is called *mixing* (or sometimes *Rajchman*) if its Fourier coefficients $\hat{\mu}_n = \int_0^1 e^{2\pi int} d\mu(t)$ converge to 0 as $|n| \to \infty$.

By the Riemann-Lebesgue lemma any absolutely continuous measure is mixing. However there are many mixing singular measures as well. Atomic measures can never be mixing. The next proposition justifies why non-atomic measures are called *weak* (or *weakly*) *mixing measures*.

Proposition 6.5. (*Wiener*) *A measure μ on $[0,1]$ is non-atomic* (*diffuse*) *if and only if for a set $S \subseteq \mathbb{Z}$ of full density*

$$\lim_{n \in S, |n| \to \infty} \hat{\mu}_n = 0.$$

From Prop. 2.5 and Prop. 2.19 of [16], *mixing* and *weakly mixing* are just not properties of measures, they are in fact properties of equivalence class of measures.

We need the following fact from the calculus course. A bounded sequence of complex numbers $\{a_n\}_{n\in\mathbb{Z}}$ converges to 0 strongly in the sense of Cesàro i.e.

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|a_n|=0 \tag{6.1}$$

if and only if there is a set $S\subseteq\mathbb{Z}$ of full density such that

$$\lim_{n\in S,|n|\to\infty}|a_n|=0. \tag{6.2}$$

Let $x,y\in\mathcal{M}$ be such that $\mathbb{E}_A(x)=\mathbb{E}_A(y)=0$. Let $a\in A$. Then the following polarization identity holds:

$$\begin{aligned} 4\,\mathbb{E}_A(xay^*) &= \mathbb{E}_A((x+y)a(x+y)^*)-\mathbb{E}_A((x-y)a(x-y)^*)\\ &\quad +i\,\mathbb{E}_A((x+iy)a(x+iy)^*)-i\,\mathbb{E}_A((x-iy)a(x-iy)^*). \end{aligned} \tag{6.3}$$

Thus WAHP for a masa is equivalent to the following. For each finite set $\{x_i\}_{i=1}^n\subset\mathcal{M}$ with $\mathbb{E}_A(x_i)=0$ for all $1\leq i\leq n$ and $\epsilon>0$, there exists an unitary $u\in A$ such that

$$\|\mathbb{E}_A(x_iux_i^*)\|_2\leq\epsilon \text{ for all } 1\leq i\leq n.$$

We will only prove the harder part of the equivalence of singularity and WAHP.

Theorem 6.6. *Let $A\subset\mathcal{M}$ be a masa such that $L^2(A)^{\perp}\in C_{n.a}(A)$. Then A has WAHP.*

Proof. Suppose to the contrary A does not have WAHP. Then there is a $\epsilon>0$ and operators $0\neq x_i\in\mathcal{M}$, $1\leq i\leq n$ with $\mathbb{E}_A(x_i)=0$ for all i, such that

$$\inf_{u\in\mathcal{U}(A)}\sum_{i=1}^{n}\|\mathbb{E}_A(x_iux_i^*)\|_2^2\geq\epsilon,$$

where $\mathcal{U}(A)$ denotes the unitary group of A. Note that for all $1\leq i\leq n$, $\overline{Ax_iA}^{\|\cdot\|_2}\in C_{n.a}(A)$ by Lemma 5.7 of [10] and Thm. 5.10. Equivalently, if $t\mapsto\eta_{x_i}^t$ and $s\mapsto\eta_{x_i}^s$ denote the (π_1,λ) and (π_2,λ) disintegrations respectively of η_{x_i}, then for λ almost all t, the measure $\eta_{x_i}^t$ is completely non-atomic and similar statements hold for $\eta_{x_i}^s$.
Let $v\in A$ be the Haar unitary corresponding to the function $t\mapsto e^{2\pi it}$. Then v generates A. Now from Lemma 6.3 we have

$$\sum_{i=1}^{n}\left\|\mathbb{E}_A(x_iv^kx_i^*)\right\|_2^2=\int_0^1\sum_{i=1}^{n}\left|\eta_{x_i}^t(1\otimes v^k)\right|^2d\lambda(t)\geq\epsilon \text{ for all } k\in\mathbb{Z}. \tag{6.4}$$

Throwing off a λ-null set F we assume that for $t\in F^c$ the measures $\eta_{x_i}^t$ are completely non-atomic, finite, concentrated on $\{t\}\times[0,1]$ and $\eta_{x_i}^t([0,1]\times[0,1])=\mathbb{E}_A(x_ix_i^*)(t)$ for all $1\leq i\leq n$ (see Lemma 6.1). Let

$$a_k(t)=\sum_{i=1}^{n}\left|\eta_{x_i}^t(1\otimes v^k)\right|^2,\ k\in\mathbb{Z},t\in[0,1].$$

Then a_k is measurable for all $k\in\mathbb{Z}$. For $k\in\mathbb{Z}$ and $t\in F^c$ we have

$$a_k(t)=\sum_{i=1}^{n}\left|\int_{[0,1]\times[0,1]}e^{2\pi iks}d\eta_{x_i}^t(t',s)\right|^2\leq\sum_{i=1}^{n}\left(\eta_{x_i}^t([0,1]\times[0,1])\right)^2.$$

Then by Lemma 6.1, $a_k(t) \leq \sum_{i=1}^{n} |\mathbb{E}_A(x_i x_i^*)(t)|^2 < \infty$, for all $t \in F^c$ and for all $k \in \mathbb{Z}$. Define

$$s_N(t) = \frac{1}{2N+1} \sum_{k=-N}^{N} a_k(t), N \in \mathbb{N}.$$

Then s_N is measurable for all $N \in \mathbb{N}$. Since $\eta_{x_i}^t$ is completely non-atomic for all $1 \leq i \leq n$ and $t \in F^c$ so

$$s_N(t) \to 0 \text{ as } N \to \infty \text{ for all } t \in F^c \text{ from Eq. (6.1), (6.2) and Prop 6.5.}$$

Again since $s_N(t) \leq \sum_{i=1}^{n} |\mathbb{E}_A(x_i x_i^*)(t)|^2$ for $t \in F^c$ (from Lemma 6.1), so by dominated convergence theorem

$$\int_0^1 s_N(t) d\lambda(t) \to 0 \text{ as } N \to \infty.$$

Therefore,

$$\begin{aligned}\int_0^1 s_N(t) d\lambda(t) &= \frac{1}{2N+1} \sum_{k=-N}^{N} \int_0^1 \sum_{i=1}^{n} \left|\eta_{x_i}^t (1 \otimes v^k)\right|^2 d\lambda(t) \\ &= \frac{1}{2N+1} \sum_{k=-N}^{N} \left(\sum_{i=1}^{n} \left\|\mathbb{E}_A(x_i v^k x_i^*)\right\|_2^2 \right) \to 0 \text{ as } N \to \infty.\end{aligned}$$

Consequently from Eq. (6.2) there is a set $S \subseteq \mathbb{Z}$ of full density such that

$$\lim_{k \in S, |k| \to \infty} \sum_{i=1}^{n} \left\|\mathbb{E}_A(x_i v^k x_i^*)\right\|_2^2 = 0.$$

This is a contradiction to Eq. (6.4). So A must have WAHP. □

The proof of Thm. 6.6 yields the following result.

Theorem 6.7. *Let $A \subset \mathcal{M}$ be a singular masa. Then given any finite set $\{x_i\}_{i=1}^n \subset \mathcal{M}$ with $\mathbb{E}_A(x_i) = 0$ for all i,*

$$\frac{1}{2N+1} \sum_{k=-N}^{N} \left(\sum_{i=1}^{n} \left\|\mathbb{E}_A(x_i v^k x_i^*)\right\|_2^2 \right) \to 0 \text{ as } N \to \infty. \tag{6.5}$$

where v is a Haar unitary generator of A.

Remark 6.8. Thus the unitary in the definition of WAHP (Defn. **??**) can always be chosen to be v^k where k is a large integer and v is a Haar unitary generator of the masa. This strengthens the definition of WAHP. Note that Eq. (6.5) is very closely related to definition of weakly mixing actions of abelian groups on finite von Neumann algebras.

The measures η_x^t, η^t are concentrated on $\{t\} \times [0,1]$ for λ almost all t. We will denote by $\tilde{\eta}_x^t, \tilde{\eta}^t$ the restriction of the measures η_x^t and η^t respectively on $\{t\} \times [0,1]$. Thus $\tilde{\eta}_x^t, \tilde{\eta}^t$ can be regarded as measures on $[0,1]$.

Theorem 6.9. *Let $A \subset \mathcal{M}$ be a masa. Let $[\eta]$ denote the left-right-measure for A. If for λ almost all t the measures $\tilde{\eta}^t$ are mixing, then A has AHP with respect to a Haar unitary generator of A.*

Proof. From Prop. 2.5 of [16] it follows that for λ almost all t, any measure in the equivalence class $[\tilde{\eta}^t]$ is mixing. In view of Eq. (6.3), it is enough to show that for all $x \in \mathcal{M}$ with $\mathbb{E}_A(x) = 0$,

$$\|\mathbb{E}_A(xv^nx^*)\|_2 \to 0 \text{ as } |n| \to \infty,$$

where $v \in A$ is a Haar unitary generator of A. Let $v \in A$ correspond to the function $s \mapsto e^{2\pi is}$. By Lemma 6.3

$$\|\mathbb{E}_A(xv^nx^*)\|_2^2 = \int_0^1 \left|\eta_x^t(1 \otimes v^n)\right|^2 d\lambda(t).$$

From Lemma 5.7 [10] we know that $\eta_x \ll \eta$ and hence for λ almost all t, $\eta_x^t \ll \eta^t$ from Lemma 3.6. So $\tilde{\eta}_x^t \ll \tilde{\eta}^t$ for λ almost all t. Thus $\tilde{\eta}_x^t$ is mixing measure from Prop. 2.5 of [16] for λ almost all t. Also from Lemma 6.1, the measures η_x^t are finite for λ almost all t. Use Lemma 6.1 and apply dominated convergence theorem to finish the proof. □

Appendix A. Structure of measurable functions

Making a measurable selection as we attempted in Lemma 5.12 is not enough. One likes to make a measurable selection so that the graph of the selection is an automorphism graph of the masa, the automorphism being implemented by an unitary in the factor. But this is a very delicate issue. We are not aware of such selection theorems. We can overcome this obstacle though. Structure theorems of continuous and measurable functions are what comes into play.

Definition A.1. Let $f : [0,1] \mapsto \mathbb{R}$ be a function and E be a subset of [0,1]. Then f is said to satisfy condition (N) or null condition of Lusin relative to E if $f(A)$ is a set of measure 0 whenever $A \subset E$ is a set of measure 0.

The definition implicitly assumes that there are two measures on $[0,1]$ and $\mathbb{R}$. For our purpose these measures will always be the Lebesgue measure, which we will denote by λ.

Proposition A.2. (*Tietze's Extension Type*) *Let $E \subsetneq [0,1]$ be closed and let $f : E \mapsto [0,1]$ be a continuous function that satisfy the property that for a measurable set $A \subset E$, $\lambda(A) = 0$ if and only if $\lambda(f(A)) = 0$. Then there exists a continuous function $F : [0,1] \mapsto [0,1]$ such that*
(i) $F_{|E} = f$,
(ii) F satisfies the property that for a measurable set $A \subset [0,1]$, $\lambda(A) = 0$ if and only if $\lambda(F(A)) = 0$.

Proof. Since E is closed it is a compact subset of $[0,1]$. Therefore E has greatest and least members m and M respectively. If $m \neq 0$ or $M \neq 1$ then extend f to a function h on $E_1 = E \cup \{0\} \cup \{1\}$ by assigning the values $f(m)$ and $f(M)$ at the points 0 and 1 respectively. The function h is continuous on E_1 and satisfies the same condition as f relative to E_1. So without loss of generality we can assume $0, 1 \in E$.
The complement of E is a open set in $[0,1]$ and $E^c \subset (0,1)$. Then E^c can be written as a countable disjoint union of intervals $\bigcup\limits_{i=1}^{\infty}(a_i, b_i)$. Then note that $a_i, b_i \in E$ for all i.

So we only have to define an extension on (a_i, b_i). Define

$$F(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \lambda f(a_i) + (1-\lambda) f(b_i) & \text{if } x = \lambda a_i + (1-\lambda) b_i \in (a_i, b_i), \\ & \quad 0 < \lambda < 1 \text{ and } f(a_i) \neq f(b_i), \\ \frac{2(1-f(a_i))}{b_i - a_i}(x - a_i) + f(a_i) & \text{if } a_i < x \leq \frac{a_i+b_i}{2} \text{ and} \\ & f(a_i) = f(b_i) < 1, \\ \frac{2(1-f(b_i))}{a_i - b_i}(x - b_i) + f(b_i) & \text{if } \frac{a_i+b_i}{2} \leq x < b_i \text{ and} \\ & f(a_i) = f(b_i) < 1, \\ \frac{2(x-a_i)}{a_i - b_i} + 1 & \text{if } a_i < x \leq \frac{a_i+b_i}{2} \text{ and} \\ & f(a_i) = f(b_i) = 1, \\ \frac{2(x-b_i)}{b_i - a_i} + 1 & \text{if } \frac{a_i+b_i}{2} \leq x < b_i \text{ and} \\ & f(a_i) = f(b_i) = 1. \end{cases}$$

The function F is now continuous, as it is a linear interpolation obtained from f and the construction satisfy the required conditions. $\square$

Theorem A.3. (*Foran,* [13]) *A necessary and sufficient condition for a continuous function $F : [0,1] \mapsto [0,1]$ to satisfy condition (N) relative to $[0,1]$ is that there exists a sequence of measurable sets $E_n \subseteq [0,1]$, $n = 0, 1, \cdots$, such that the following properties are true:*
(i) $[0,1] = \bigcup_{n=0}^{\infty} E_n$,
(ii) $\lambda(F(E_n)) \leq n\lambda(E_n)$ *for all* $n \geq 0$,
(iii) for each $n > 0$, F is one to one on E_n.

Proposition A.4. *Let $F : [0,1] \mapsto [0,1]$ be a measurable function such that for any measurable set $A \subset [0,1]$, $\lambda(A) = 0$ if and only if $\lambda(F(A)) = 0$. Then there exists a measurable set $E \subseteq [0,1]$ such that $\lambda(E) > 0$ and F is one to one on E.*
Moreover, if $Y_0 \subseteq [0,1]$ is such that $\lambda(Y_0) > 0$, then there exists $Y_1 \subseteq Y_0$ with $\lambda(Y_1) > 0$ such that F is one to one on Y_1.

Proof. Let $\epsilon > 0$. By Lusin's theorem, choose a closed set $H \subset [0,1]$ such that $\lambda([0,1] \setminus H) < \epsilon$ and $F_{|H}$ is continuous relative to H. Clearly, $F_{|H}$ satisfy the property that $A \subset H$, $\lambda(A) = 0$ if and only if $\lambda(F_{|H}(A)) = 0$. By Prop. A.2, extend F to a continuous function $\tilde{F} : [0,1] \mapsto [0,1]$ such that $\tilde{F}$ has the property that for $A \subset [0,1]$, $\lambda(A) = 0$ if and only if $\lambda(\tilde{F}(A)) = 0$.

Now by Thm. A.3, choose measurable subsets $E_n \subseteq [0,1]$ such that $[0,1] = \bigcup_{n=0}^{\infty} E_n$, $\lambda(\tilde{F}(E_n)) \leq n\lambda(E_n)$ for all $n = 0, 1, \cdots$, and for each $n > 0$, $\tilde{F}$ is one to one on E_n. Since $\lambda(\tilde{F}(E_0)) = 0$ so $\lambda(E_0) = 0$. If $\lambda(E_n \cap H) = 0$ for all $n > 0$ then $\lambda(H) = 0$, which is not the case. Therefore there is a $n_0 > 0$ such that $\lambda(E_{n_0} \cap H) > 0$. But $\tilde{F}_{|E_{n_0} \cap H} = F_{|E_{n_0} \cap H}$ and clearly F is one to one on $E_{n_0} \cap H$. Rename $E = E_{n_0} \cap H$.
Suppose $\lambda(Y_0) > 0$. By choosing $\epsilon > 0$ small enough one can make sure that the closed set H in the first part of the proof satisfies $\lambda(Y_0 \cap H) > 0$. The same argument as the first part applies, and there exists a $n_0 > 0$ such that F is one to one on $Y_1 = Y_0 \cap H \cap E_{n_0}$ and $\lambda(Y_1) > 0$. $\square$

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