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Line-bundle-valued ternary quadratic forms over schemes

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Abstract

We study degenerations of rank 3 quadratic forms and of rank 4 Azumaya algebras, and extend what is known for good forms and Azumaya algebras. By considering line-bundle-valued forms, we extend the theorem of Max-Albert Knus that the Witt-invariant—the even Clifford algebra of a form—suffices for classification. An algebra Zariski-locally the even Clifford algebra of a ternary form is so globally up to twisting by square roots of line bundles. The general, usual and special orthogonal groups of a form are determined in terms of automorphism groups of its Witt-invariant. Martin Kneser's characteristic-free notion of semiregular form is used.

MSC: 11E; 14F05; 15A75; 15A78; 16H05; 16S80; 16W20; 20G35

1. Introduction

It is a Theorem of Max-Albert Knus that a rank 4 Azumaya algebra arises as the even Clifford algebra of a good ternary quadratic form, which is unique up to tensoring by square roots of the structure sheaf (2.9).[1] This can also be interpreted as a bijection in terms of suitable cohomologies (Section 2.7). The central result of this work, 3.1, extends this result to the limit, i.e., to bad quadratic forms and degenerations or specialisations of rank 4 Azumaya algebras.

The formulation of specialisations of rank 4 Azumaya algebras was done in Part A of [9] where it was shown that the scheme of such specialisations (over a fixed underlying bundle) is relatively smooth (2.11). This was applied to obtain desingularisations of moduli spaces extending the work of Seshadri and Madhav Nori in [7]. It turns out that the specialised algebras are precisely those that are Zariski-locally isomorphic to even Clifford algebras of ternary quadratic bundles (3.2).

The technical issue of what a good quadratic form in characteristic 2 should be is resolved by Martin Kneser's notion of semiregularity and reduces to the usual regularity in other characteristics (Section 2.3).

A rank 4 Azumaya algebra comes with a standard involution with which are associated a norm and a trace. The norm is a quadratic form which when restricted to the rank 3 subbundle of trace zero elements gives a ternary quadratic

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[1] Each two-digit number refers to a definition or a result. Here it is Theorem 2.9.

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bundle whose even Clifford algebra is the Azumaya algebra we began with. But this method fails for specialised algebras when the trace is not surjective. However, by using explicit computations and scheme-theoretic methods, we are able to obtain a ternary quadratic bundle whose even Clifford algebra is a given specialised algebra A (the surjectivity part of 3.1). In doing so, we need to consider forms with values in possibly nontrivial line bundles, one such being $\det^{-1}(A)$ itself (3.7, (a)). Fortunately for such forms, there is a notion of even Clifford algebra defined by Bichsel and Knus (Section 2.4 and [1]). A specialised algebra arises from an honest quadratic form iff its determinant has a square root and arises from a bilinear form iff the line subbundle generated by 1 is a direct summand (3.7, (b)). The last condition is fulfilled for example if the base is affine, or if 2 is invertible. Hence over such a base, the line subbundle generated by 1 is always a direct summand (3.8).

Working over an algebraically closed field of characteristic not 2, Seshadri in [7] considers the map that associates a rank 3 quadratic form with its even Clifford algebra. In the same paper, he mentions the remark of S. Ramanan that "the form can be read off from the multiplication table of the algebra". This may be considered as the prime motivation for the present work.

Seshadri's method has problems in characteristic 2 since he works with the associated symmetric bilinear forms; working with the quadratic forms themselves, the existence of semiregular forms and the use of the notion of specialised algebra allows the correct generalisation over an arbitrary base scheme (3.11). Ramanan later pointed out to the author (Section 2.3 of [8]) that the right objects to be considered locally should be bilinear forms, including the non-symmetric ones. Combined with the notion of specialised algebra and the Bourbaki Tensor Operations (2.2), this generalises over an arbitrary base scheme (3.9). These points of view are crucial to the computations that form the backbone of the present work. They lead to the explicit determination of the isomorphisms between ternary quadratic bundles (3.5 and 3.6). Moreover, 3.5 leads to the injectivity of 3.1 by the construction of the "differing twisted discriminant bundle" via the Bourbaki Tensor Operations. 3.5 also plays a central role in the surjectivity of 3.1 since it provides an explicit cocycle defining a quadratic bundle for a given specialised algebra.

In 3.6, the general, usual and special orthogonal groups of a quadratic bundle are canonically determined in terms of the automorphisms of its even Clifford algebra. Any automorphism of the latter arises from a similarity, and in fact from an orthogonal transformation if its determinant is a square. The special orthogonal group is thus identified with the subgroup of automorphisms with trivial determinant. If the base is integral and the quadratic form is semiregular at some point, then every automorphism of the even Clifford algebra has determinant 1 and is thus induced from a self-isometry; the orthogonal group is also seen to be a semidirect product in this case.

Section 2 fixes notation, and recalls definitions and results for future use. The formulation of the statements of the main results follows in Section 3. The proofs will occupy Section 4 through Section 6. A more elaborate account may be found in [10] where 3.1 is also applied to indicate examples of vector bundles which do not admit rank 4 Azumaya structures and rank 4 regular/ rank 3 semiregular quadratic forms; the full details of the proof appear in [11].

2. Notation and preliminaries

2.1. Quadratic and bilinear forms with values in a line bundle

Let V be a vector bundle (=coherent $\mathcal{O}_X$-module locally free of constant positive rank) and I a line bundle on a scheme X. Then a quadratic form q on V with values in I is defined to be a section of the quotient $\mathrm{Quad}_{(V,I)}$ of the bundle $\mathrm{Bil}_{(V,I)}$ of I-valued bilinear forms on V by the subbundle $\mathrm{Alt}^2_{(V,I)}$ of alternating 2-forms. We denote such a datum by a triple (V, q, I). Given a quadratic form $q \in \Gamma(U, \mathrm{Quad}_{(V,I)})$, recall that the usual 'associated' symmetric bilinear form $b_q \in \Gamma(U, \mathrm{Bil}_{(V,I)})$ is given on sections (over open subsets of U) by $v \otimes v' \mapsto q(v+v') - q(v) - q(v')$. Also recall that, given a (not necessarily symmetric!) bilinear form b, we have the induced quadratic form q_b given on sections by $v \mapsto b(v \otimes v)$.

2.2. Sets of similarities of quadratic bundles

Let (V, q, I) and (V', q', I') be quadratic bundles on the scheme X. We denote by $\mathrm{Sim}[(V, q, I), (V', q', I')]$ the set of generalised similarities from (V, q, I) to (V', q', I'). These consist of pairs (g, m) such that $g : V \cong V'$ and $m : I \cong I'$ are linear isomorphisms and $q'g = mq$ (here q and q' are considered as morphisms of sheaves of *sets*). When $I = I'$, since an $m \in \mathrm{Aut}(I)$ may be thought of as multiplication by a scalar $l \in \Gamma(X, \mathcal{O}_X^*) \equiv \mathrm{Aut}(I)$, we may call the isomorphism (g, m) an I-similarity with multiplier l. In such a case we may as well denote (g, m) by the pair (g, l) and we often write $g : (V, q, I) \cong_l (V', q', I)$. Let $\mathrm{Iso}[(V, q, I), (V', q', I)]$ be the subset of isometries (i.e., those pairs (g, m) with m = Identity or I-similarities with trivial multipliers). When $V = V'$, the subset of isometries with trivial determinant is denoted as $\text{S-Iso}[(V, q, I), (V, q', I)]$. On taking $q = q'$ these sets naturally become subgroups of $\mathrm{Aut}(V) \times \Gamma(X, \mathcal{O}_X^*) = \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$ and we define the general, the usual, and the special orthogonal groups respectively by $\mathrm{GO}(V, q, I) := \mathrm{Sim}[(V, q, I), (V, q, I)]$, $\mathrm{O}(V, q, I) := \mathrm{Iso}[(V, q, I), (V, q, I)]$, and $\mathrm{SO}(V, q, I) := \text{S-Iso}[(V, q, I), (V, q, I)]$. Of course, $\mathrm{O}(V, q, I)$ and $\mathrm{SO}(V, q, I)$ may be thought of as subgroups of $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ respectively.

2.3. *Semiregular bilinear and quadratic forms*

Basic problems in dealing with quadratic forms over arbitrary schemes arise essentially from two abnormalities in characteristic two: firstly, the mapping that associates a quadratic form with its symmetric bilinear form is not bijective and secondly, there do not exist regular quadratic forms on any odd-rank bundle. The remedy for this is to consider semiregular quadratic forms, a concept due to Kneser [5] and elaborated upon by Knus in [6], which in fact works over an arbitrary base scheme (and hence in a characteristic-free way) and further reduces to the usual notion of regular form in characteristics $\neq 2$.

Let $\mathrm{Spec}(R) = U \hookrightarrow X$ be an open affine subscheme of X such that $V \mid U$ is trivial. Let $\mathrm{Quad}_V := \mathrm{Quad}_{(V, \mathcal{O}_X)}$. Consider a quadratic form $q \in \Gamma(U, \mathrm{Quad}_V)$ on $V \mid U$ and its associated symmetric bilinear form b_q. The matrix of this bilinear form relative to any fixed basis is a symmetric matrix of odd rank and in particular, if R is a ring of characteristic two, then this matrix is also alternating and is hence singular, immediately implying that q cannot be regular. However, computing the determinant of such a matrix in *formal variables* $\{\zeta_i, \zeta_{ij}\}$ over the integers shows that it is twice the following polynomial: $P_3(\zeta_i, \zeta_{ij}) = 4\zeta_1\zeta_2\zeta_3 + \zeta_{12}\zeta_{13}\zeta_{23} - (\zeta_1\zeta_{23}^2 + \zeta_2\zeta_{13}^2 + \zeta_3\zeta_{12}^2)$. The value $P_3(q(e_i), b_q(e_i, e_j))$ corresponding to a basis $\{e_1, e_2, e_3\}$ of $V \mid U$ is called the *half-discriminant* of q relative to that basis, and q is said to be semiregular if its half-discriminant is a unit. It turns out that this definition is independent of the basis chosen (Section 3, Chapter IV, [6]).

Even if $V \mid U$ is not free (but only locally free), the semiregularity of q may be defined as the semiregularity of $q \otimes_R R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$, and it turns out that with this definition, the notion of a quadratic form being semiregular is local and is well behaved under base-change (Proposition 3.1.5, Chapter IV, [6]). We may thus define the subfunctor of $\mathrm{Quad}_V := \mathrm{Quad}_{(V, \mathcal{O}_X)}$ of semiregular quadratic forms. This subfunctor is represented by a GL_V-invariant open subscheme $i : \mathrm{Quad}_V^{sr} \hookrightarrow \mathrm{Quad}_V$ because, over each affine open subscheme $U \hookrightarrow X$ which trivialises V, it corresponds to localisation by the non-zero-divisor P_3. Note that this canonical open immersion is affine and schematically dominant as well. We next turn to semiregular bilinear forms. We define a bilinear form b to be semiregular iff its induced quadratic form q_b is semiregular. Thus by definition, Bil_V^{sr} is the fiber product

$$\begin{array}{ccc} \mathrm{Bil}_V & \xrightarrow{\;p\;} & \mathrm{Quad}_V \\ {\scriptstyle i'}\uparrow & & \uparrow{\scriptstyle i} \\ \mathrm{Bil}_V^{sr} & \xrightarrow[\;p'\;]{} & \mathrm{Quad}_V^{sr} \end{array}$$

Since p is a Zariski-locally trivial principal Alt_V^2-bundle, it is smooth and surjective (in particular faithfully flat). It therefore follows that the affineness and schematic dominance of i imply those of i'. We record these facts below.

Proposition 2.1. *The open immersion* $\mathrm{Bil}_V^{sr} \hookrightarrow \mathrm{Bil}_V$ *is a* GL_V*-equivariant schematically dominant affine morphism. Further this open immersion behaves well under base-change (relative to* X*).*

2.4. *The generalised Clifford algebra of Bichsel–Knus*

Let R be a commutative ring (with 1), I an invertible R-module and V a projective R-module. Consider the Laurent–Rees algebra of I defined by $L[I] := R \oplus (\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1})))$ and define the $\mathbb{Z}$-gradation on the tensor product of algebras $TV \otimes L[I]$ by requiring elements of V (resp. of I) to be of degree one (resp. of degree two). Let $q : V \longrightarrow I$ be an I-valued quadratic form on V. Following the definition of Bichsel and Knus [1], let

$J(q, I)$ be the two-sided ideal of $TV \otimes L[I]$ generated by the set $\{(x \otimes_{TV} x) \otimes 1_{L[I]} - 1_{TV} \otimes q(x) \mid x \in V\}$ and let the generalised Clifford algebra of q be defined by

$$\widetilde{C}(V, q, I) := TV \otimes L[I]/J(q, I).$$

This is a $\mathbb{Z}$-graded algebra by definition. Let C_n be the submodule of elements of degree n. Then C_0 is a subalgebra, playing the role of the even Clifford algebra in the classical situation (i.e., $I = R$) and C_1 is a C_0-bimodule. Bichsel and Knus call C_0 and C_1 respectively as the *even Clifford algebra* and the *Clifford module* associated with the triple (V, q, I). The generalised Clifford algebra satisfies an appropriate universal property which ensures it behaves well functorially. Since V is projective, the canonical maps $V \longrightarrow \widetilde{C}(V, q, I)$ and $L[I] \longrightarrow \widetilde{C}(V, q, I)$ are injective. For proofs of these facts, see Section 3 of [1]. If (V, q, I) is an I-valued quadratic form on the vector bundle V over a scheme X, with I a line bundle, then the above construction may be globalised to define the generalised Clifford algebra bundle $\widetilde{C}(V, q, I)$ which is a $\mathbb{Z}$-graded algebra bundle on X. Its degree zero subalgebra bundle is denoted as $C_0(V, q, I)$ and is called the even Clifford algebra bundle of (V, q, I).

2.5. Bourbaki's tensor operations with values in a line bundle

Let R be a commutative ring and $L[I] := R \oplus (\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1})))$ as above. We denote by $\otimes_T$ (resp. by $\otimes_L$) the tensor product and by 1_T (resp. 1_L) the unit element in the algebra TV (resp. in $L[I]$).

Theorem 2.2 (*With the Above Notation*).

(1) *Let $q : V \longrightarrow I$ be an I-valued quadratic form on V and $f \in \mathrm{Hom}_R(V, I)$. Then there exists a unique R-linear endomorphism t_f of the algebra $TV \otimes L[I]$ such that:*(a) *for each $\lambda \in L[I]$ we have $t_f(1_T \otimes \lambda) = 0$;*(b) *for any $x \in V$, $y \in TV$, and $\lambda \in L[I]$ we have $t_f((x \otimes_T y) \otimes \lambda) = y \otimes (f(x) \otimes_L \lambda) - (x \otimes_T 1_T)t_f(y \otimes \lambda)$;*(c) *if $J(q, I)$ is the two-sided ideal of $TV \otimes L[I]$ as defined in Section 2.4 above, then we have $t_f(J(q, I)) \subset J(q, I)$.*

(2) *Let $q, q' : V \longrightarrow I$ be two I-valued quadratic forms whose difference is the quadratic form q_b induced by an I-valued bilinear form $b \in \mathrm{Bil}_R(V, I) := \mathrm{Hom}_R(V \otimes_R V, I)$, i.e., $q'(x) - q(x) = q_b(x) := b(x, x) \forall x \in V$. Further, for any $x \in V$ denote by b_x the element of $\mathrm{Hom}_R(V, I)$ given by $y \mapsto b(x, y)$. Then there exists an R-linear automorphism Ψ_b of $TV \otimes L[I]$ which is unique with respect to the first three of the following properties it satisfies:*(a) *for any $\lambda \in L[I]$ we have $\Psi_b(1_T \otimes \lambda) = (1_T \otimes \lambda)$;*(b) *for any $x \in V$, $y \in TV$ and $\lambda \in L[I]$ we have $\Psi_b((x \otimes_T y) \otimes \lambda) = (x \otimes 1_L).\Psi_b(y \otimes \lambda) + t_{b_x}(\Psi_b(y \otimes \lambda))$;*(c) *$\Psi_b(J(q', I)) \subset J(q, I)$;*(d) *by the previous property, Ψ_b induces an isomorphism of $\mathbb{Z}$-graded R-modules $\psi_b : \widetilde{C}(V, q', I) \cong \widetilde{C}(V, q, I)$; in particular, given a quadratic form $q_1 : V \longrightarrow I$, since there always exists an I-valued bilinear form b_1 that induces q_1 (i.e., such that $q_1 = q_{b_1}$), setting $q' = q_1$, $q = 0$ and $b = b_1$ in the above gives a $\mathbb{Z}$-graded linear isomorphism*

$$\psi_{b_1} : \widetilde{C}(V, q_1, I) \cong \widetilde{C}(V, 0, I) = \Lambda(V) \otimes L[I];$$

(e1) *$\Psi_b(T^{2n}V \otimes L[I]) \subset \oplus_{(i \le n)}(T^{2i}V \otimes L[I])$;* (e2) *$\Psi_b(T^{2n+1}V \otimes L[I]) \subset \oplus_{(\text{odd } i \le 2n+1)}(T^iV \otimes L[I])$;*(e3) *$\Psi_b(T^{2n}V \otimes I^{-n}) \subset \oplus_{(i \le n)}(T^{2i}V \otimes I^{-i})$;*(e4) *$\Psi_b(T^{2n+1}V \otimes I^{-n}) \subset \oplus_{(\text{odd } i \le 2n+1)}(T^iV \otimes I^{\frac{1-i}{2}})$;*(f) *in particular, for $x, x' \in V$, $\Psi_b((x \otimes_T x') \otimes 1_L) = (x \otimes_T x') \otimes 1_L + 1_T \otimes b(x, x')$ so that for $\psi_b : C_0(V, q_b, I) \cong C_0(V, 0, I) = \oplus_{n \ge 0}(\Lambda^{2n}(V) \otimes I^{-n})$ we have*

$$\psi_b(((x \otimes_T x') \otimes \zeta) \bmod J(q_b, I)) = (x \wedge x') \otimes \zeta + \zeta(b(x, x')).1$$

for any $x, x' \in V$ and $\zeta \in I^{-1} \equiv \mathrm{Hom}_R(I, R)$;(g) *if $f \in \mathrm{Hom}_R(V, I)$, and t_f is given by* (1) *above, then $\Psi_b \circ t_f = t_f \circ \Psi_b$;*(h) *for I-valued bilinear forms b_i on V, $\Psi_{b_1+b_2} = \Psi_{b_1} \circ \Psi_{b_2}$ and $\Psi_0 =$ Identity on $TV \otimes L[I]$;*(i) *the map $b \mapsto \Psi_b$ is a $\mathrm{GL}_R(V)$-equivariant group homomorphism (the group $\mathrm{GL}_R(V)$ acts on $\mathrm{Bil}_R(V, I)$ by $g.b : (x, x') \mapsto b(g^{-1}(x), g^{-1}(x'))$ and on $\mathrm{Aut}_R(TV \otimes L[I])$ by $g.\Phi := (T(g) \otimes \mathrm{Id}_{L[I]}) \circ \Phi \circ (T(g^{-1}) \otimes \mathrm{Id}_{L[I]})$).*

(3) *The constructions in*(2) *above behave well under base-change.*

2.6. Tensoring by bilinear and twisted discriminant bundles

Let V, M be vector bundles on a scheme X and let I, J be line bundles on X. Let q be a quadratic form on V with values in I and let b be a symmetric bilinear form on M with values in J. By abuse of notation, we also use b to denote the corresponding J-valued linear form on $M \otimes M$.

Proposition 2.3 (*With the Above Notation*). (1) $(V, q, I) \otimes (M, b, J) := (V \otimes M, q \otimes b, I \otimes J)$ *where the quadratic form $q \otimes b$ on $V \otimes M$ is given on sections by $v \otimes m \mapsto q(v) \otimes b(m \otimes m)$ and has the associated bilinear form*

$b_{q\otimes b} = b_q \otimes b$.(2) *When M is a line bundle, (M, b, J) is regular (=nonsingular) iff (M, q_b, J) is semiregular iff $b : M \otimes M \cong J$ is an isomorphism.*(3) *Let V be of odd rank and M a line bundle such that (M, b, J) is regular. Then (V, q, I) is semiregular iff $(V, q, I) \otimes (M, b, J) = (V \otimes M, q \otimes b, I \otimes J)$ is semiregular.*

Definition 2.4. A triple (L, h, J), consisting of a linear isomorphism $h : L \otimes L \cong J$, with L, J line bundles, is called a *twisted discriminant bundle* on X.

A twisted discriminant bundle (L, h, J) specifies L as a square root of the line bundle J via h. The terminology is motivated by the following: when J is the trivial line bundle, such a datum is referred to as a *discriminant bundle* in Section 3, Chapter III, of Knus' book [6]. By part (2) of the preceding proposition, h is a regular bilinear form on L (necessarily symmetric) with values in J, so that we may speak of an isometry between two twisted discriminant bundles (L, h, J) and (L', h', J'): it is a pair (ζ, η) consisting of linear isomorphisms $\zeta : L \cong L'$ and $\eta : J \cong J'$ such that $\eta h = h'(\zeta \otimes \zeta)$.

Lemma 2.5. *On the set* T-Disc(X) *of isometry classes of twisted discriminant bundles on X, we have a natural group structure induced by the tensor product.* T-Disc(X) *is functorial in X. If we consider only isometry classes of discriminant bundles, i.e., of triples $(L, h, \mathcal{O}_X)$, then we obtain a subgroup* Disc$(X) \subset$ T-Disc(X) *which is of exponent* 2.

Proposition 2.6. *Let V and V' be vector bundles of the same rank on the scheme X, (L, h, J) a twisted discriminant bundle on X and $\alpha : V' \cong V \otimes L$ an isomorphism of bundles.*

(1) *Over any open subset $U \hookrightarrow X$, given a bilinear form $b' \in \Gamma(U, \mathrm{Bil}_{(V',I)})$, we can define a bilinear form $b \in \Gamma(U, \mathrm{Bil}_{(V,I\otimes J^{-1})})$ using α and h as follows: we let $b := (b' \otimes J^{-1}) \circ (\zeta_{(\alpha,h)})^{-1}$ where $\zeta_{(\alpha,h)} : V' \otimes V' \otimes J^{-1} \cong V \otimes V$ is the linear isomorphism given by the composition of the following natural morphisms:*

$$V' \otimes V' \otimes J^{-1} \overset{\alpha\otimes\alpha\otimes\mathrm{Id}(\cong)}{\longrightarrow} V \otimes L \otimes V \otimes L \otimes J^{-1} \overset{\mathrm{SWAP}(2,3)(\equiv)}{\longrightarrow}$$
$$V \otimes V \otimes L^2 \otimes J^{-1} \overset{\mathrm{Id}\otimes h\otimes\mathrm{Id}(\cong)}{\longrightarrow}$$
$$\overset{\mathrm{Id}\otimes h\otimes\mathrm{Id}(\cong)}{\longrightarrow} V \otimes V \otimes J \otimes J^{-1} \overset{\mathrm{CANON}(\equiv)}{\longrightarrow} V \otimes V.$$

Then the association $b' \mapsto b$ induces linear isomorphisms shown by vertical upward arrows in the following commutative diagram of vector bundle morphisms:

$$\begin{array}{ccccc} \mathrm{Alt}^2_{(V,I\otimes J^{-1})} & \xrightarrow[\text{immersion}]{\text{closed}} & \mathrm{Bil}_{(V,I\otimes J^{-1})} & \xrightarrow[\text{trivial}]{\text{locally}} & \mathrm{Quad}_{(V,I\otimes J^{-1})} \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathrm{Alt}^2_{(V',I)} & \xrightarrow[\text{immersion}]{\text{closed}} & \mathrm{Bil}_{(V',I)} & \xrightarrow[\text{trivial}]{\text{locally}} & \mathrm{Quad}_{(V',I)} \end{array}$$

(2) *Let $b' \in \Gamma(X, \mathrm{Bil}_{(V',I)})$ be a global bilinear form and let it induce $b \in \Gamma(X, \mathrm{Bil}_{(V,I\otimes J^{-1})})$ via α and h as defined in*(1) *above. Let*

$$\Psi_{b'} \in \mathrm{Aut}_{\mathcal{O}_X}(TV' \otimes L[I])(\text{resp. } \Psi_b \in \mathrm{Aut}_{\mathcal{O}_X}(TV \otimes L[I \otimes J^{-1}]))$$

be the $\mathbb{Z}$-graded linear isomorphism induced by b' (resp. by b) defined locally (and hence globally) as in(2), 2.2 *above. Let*

$$Z_{(\alpha,h)} : \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) \cong \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n)$$

be the $\mathcal{O}_X$-algebra isomorphism induced via the isomorphism $\zeta_{(\alpha,h)}$ defined in(1) *above. Then, taking into account*(2e), 2.2*, the following diagram commutes:*

$$\begin{array}{ccc} \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) & \xrightarrow[\cong]{Z_{(\alpha,h)}} & \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n) \\ \Psi_{b'}\downarrow\cong & & \cong\downarrow\Psi_b \\ \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) & \xrightarrow[Z_{(\alpha,h)}]{\cong} & \oplus_{n\geq 0}(T^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n) \end{array}$$

thereby inducing by(2d), 2.2 *the following commutative diagram of $\mathcal{O}_X$-linear isomorphisms:*

$$\begin{array}{ccc} C_0(V', q_{b'}, I) & \xrightarrow[\cong]{\text{via } Z_{(\alpha,h)}} & C_0(V, q_b, I \otimes J^{-1}) \\ \psi_{b'} \downarrow \cong & & \cong \downarrow \psi_b \\ \oplus_{n\geq 0}(\Lambda^{2n}_{\mathcal{O}_X}(V') \otimes I^{-n}) & \xrightarrow[\text{via } Z_{(\alpha,h)}]{\cong} & \oplus_{n\geq 0}(\Lambda^{2n}_{\mathcal{O}_X}(V) \otimes I^{-n} \otimes J^n) \end{array}$$

(3) *Let b and b' be as in*(2) *above. Then $\alpha : V' \cong V \otimes L$ induces an isometry of bilinear form bundles $\alpha : (V', b', I) \cong (V, b, I \otimes J^{-1}) \otimes (L, h, J)$ and also an isometry of the induced quadratic bundles $\alpha : (V', q_{b'}, I) \cong (V, q_b, I \otimes J^{-1}) \otimes (L, h, J)$. Moreover, if we are just given a global $I \otimes J^{-1}$-valued quadratic form q on V (resp. an I-valued q' on V'), then we may define the global I-valued quadratic form q' on V' (resp. $I \otimes J^{-1}$-valued q on V) via $q' := (q \otimes h) \circ \alpha$ (resp. via $q := (q' \circ \alpha^{-1}) \otimes (h^\vee)^{-1}$) and again we have an isometry of quadratic bundles $\alpha : (V', q', I) \cong (V, q, I \otimes J^{-1}) \otimes (L, h, J)$.*

Proposition 2.7. *Let $g : (V, q, I) \cong_l (V', q', I)$ be an I-similarity with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$.*

(1) *There exists a unique isomorphism of $\mathcal{O}_X$-algebra bundles $C_0(g, l, I) : C_0(V, q, I) \cong C_0(V', q', I)$ such that for sections v, v' of V and s of I^{-1} we have $C_0(g, l, I)(v.v'.s) = g(v).g(v').l^{-1}s$.*
(2) *There exists a unique vector bundle isomorphism $C_1(g, l, I) : C_1(V, q, I) \cong C_1(V', q', I)$ such that the following hold for any section v of V and any section c of $C_0(V, q)$:*(a) *$C_1(g, l, I)(v.c) = g(v).C_0(g, l, I)(c)$ and*(b) *$C_1(g, l, I)(c.v) = C_0(g, l, I)(c).g(v)$. Thus $C_1(g, l, I)$ is $C_0(g, l, I)$-semilinear.*
(3) *If $g_1 : (V', q', I) \cong_{l_1} (V'', q'', I)$ is another similarity with multiplier l_1, then the composition $g_1 \circ g : (V, q, I) \cong_{ll_1} (V'', q'', I)$ is also a similarity with multiplier given by the product of the multipliers. Further $C_i(g_1 \circ g, ll_1, I) = C_i(g_1, l_1, I) \circ C_i(g, l, I)$* for $i = 0, 1$.

A local computation shows that tensoring by a twisted discriminant bundle is locally the same as applying a similarity. In this case also one gets a global isomorphism of even Clifford algebras:

Proposition 2.8. *Let (V, q, I) be a quadratic bundle on a scheme X and (L, h, J) be a twisted discriminant bundle. There exists a unique isomorphism of algebra bundles $\gamma_{(L,h,J)} : C_0((V, q, I) \otimes (L, h, J)) \cong C_0(V, q, I)$ such that for any sections v, v' of V, λ, λ' of L, s of $I^{-1} \equiv I^\vee$ and t of $J^{-1} \equiv J^\vee$ we have $\gamma_{(L,h,J)}((v \otimes \lambda).(v' \otimes \lambda').(s \otimes t)) = t(h(\lambda \otimes \lambda'))v.v'.s$.*

2.7. The theorem of Max-Albert Knus

For a scheme X, denote by $\mathcal{Q}_3^{sr}(X; \mathcal{O}_X)$ the set of isomorphism classes of *semiregular* ternary quadratic bundles with values in the trivial line bundle on X. Here isomorphism stands for isometry as defined in Section 2.2. Recall the group $\text{Disc}(X)$ of discriminant bundles on X (2.5). By assertions (2) and (3) of 2.3, this group acts on $\mathcal{Q}_3^{sr}(X; \mathcal{O}_X)$.

For a ternary quadratic form $q : V \to \mathcal{O}_X$, recall from Section 2.4 the Bichsel–Knus even-Clifford algebra $C_0(V, q, \mathcal{O}_X)$, namely the degree zero subalgebra of the generalised Clifford algebra $\widetilde{C}(V, q, \mathcal{O}_X)$. Since the values of the form are in the trivial line bundle, this even-Clifford algebra is the same as the classically defined even-Clifford algebra.

Now the even-Clifford algebra of a semiregular form is Azumaya (see, for instance, Proposition 3.2.4, Section 3, Chapter IV, [6]). So if we let $\mathcal{AZU}_4(X)$ denote the set of algebra isomorphism classes of rank 4 Azumaya bundles over X, then by 2.8, the association $(V, q, \mathcal{O}_X) \rightsquigarrow C_0(V, q, \mathcal{O}_X)$ induces a map

$$\text{Witt-Invariant}^{sr}_{(X;\mathcal{O}_X)} : \mathcal{Q}_3^{sr}(X; \mathcal{O}_X)/\text{Disc}(X) \to \mathcal{AZU}_4(X),$$

where the left side represents the set of orbits.

Theorem 2.9 (*Max-Albert Knus, Section 3, Chapter V, [6]*). *For any scheme X, the map $\text{Witt-Invariant}^{sr}_{(X;\mathcal{O}_X)}$ defined above is a bijection.*

By Proposition 3.2.2, Section 3, Chapter III, [6], the group $\mathrm{Disc}(X)$ is naturally isomorphic to the cohomology (abelian group) $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mu_2)$. Further, by Lemma 3.2.1, Section 3, Chapter IV, [6], the cohomology $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)$ classifies the set of isomorphism classes of semiregular rank 3 quadratic bundles with values in the trivial line bundle, so it is the same as $\mathcal{Q}_3(X; \mathcal{O}_X)$. On the other hand, the set of isomorphism classes of rank 4 Azumaya algebras on X may be interpreted as the cohomology $\check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$ (see page 145, Section 5, Chapter III, [6]). Thus the bijection of 2.9 can be thought of as a statement in cohomology: $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mathsf{O}_3)/\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, \mu_2) \cong \check{\mathrm{H}}^1_{\text{étale}}(X, \mathsf{PGL}_2)$.

2.8. *The notion of schematic image*

Let $f : X \longrightarrow Y$ be a morphism of schemes. If there exists a smallest closed subscheme $Y' \hookrightarrow Y$ such that the inverse image scheme $f^{-1}(Y') := Y' \times_Y ({}_f X)$ is equal to X, one calls Y' the *schematic image* of f (or of X in Y under f) (Definitions. 6.10.1–2, Chapter I, EGA I [3]). If X is a subscheme of Y and f the canonical immersion, and if f has a schematic image Y', then Y' is called the *schematic limit* or the *limiting scheme* of the subscheme $X \overset{f}{\hookrightarrow} Y$. By Proposition 6.10.5, Chapter I, EGA I, the schematic image Y' of X by a morphism $f : X \longrightarrow Y$ exists in the following two cases: (1) $f_*(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_Y$-module, which is for example the case when f is quasi-compact and quasi-separated; (2) X is reduced. Further, the schematic image satisfies (by definition) an appropriate universal property; its formation is transitive relative to composition of morphisms and behaves well under flat base-change. Topologically, the schematic image is the closure of the set-theoretic image.

2.9. *Specialisations of rank 4 Azumaya algebras*

Until further notice we assume that W is a vector bundle of fixed positive rank on the scheme X with associated coherent locally free sheaf $\mathcal{W}$. Given any X-scheme T, by a T-algebra structure on $W_T := W \times_X T$ (also referred to as a T-algebra bundle), we mean a morphism $W_T \times_T W_T \longrightarrow W_T$ of vector bundles on T arising from a morphism of the associated locally free sheaves. Given such a T-algebra structure and $T' \longrightarrow T$ an X-morphism, it is clear that one gets by pullback (i.e., by base-change) a canonical T'-algebra structure on $W_{T'}$. Thus one has a contravariant "functor of algebra structures on W" from {X-Schemes} to {Sets} denoted as Alg_W with $\mathsf{Alg}_W(T) = \{T\text{-algebra structures on } W_T\} = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{W}_T \otimes \mathcal{W}_T, \mathcal{W}_T)$. It follows from Proposition 9.6.1, Chapter I of EGA I [3] that Alg_W is represented by $\mathrm{Alg}_W := \mathrm{Spec}(\mathsf{Sym}_X[(\mathcal{W}_X{}^{\vee} \otimes_X \mathcal{W}_X{}^{\vee} \otimes_X \mathcal{W}_X)^{\vee}])$. Hence Alg_W is affine (therefore separated), of finite presentation over X and in fact smooth of relative dimension $\mathrm{rank}_X(W)^3$. If $X' \longrightarrow X$ is an extension of base, then the construction Alg_W base-changes well, i.e., one may canonically identify $\mathrm{Alg}_W \times_X X'$ with $\mathrm{Alg}_{W'}$ where $W' = W \times_X X'$ (cf. Proposition 9.4.11, Chapter I, EGA I [3]).

The general linear groupscheme associated with W, namely GL_W naturally acts on Alg_W on the left, so that for each X-scheme T, $\mathrm{Alg}_W(T) \bmod \mathrm{GL}_W(T)$ is the set of isomorphism classes of T-algebra structures on $W \times_X T$.

We remark that an algebra structure may fail to be associative and may fail to have a (two-sided) identity element for multiplication. However, a multiplicative identity for an associative algebra structure must be a nowhere vanishing section (Lemma 2.3, and (2) ⇒ (4) of Lemma 2.4, Part A, [9]).

Let $w \in \Gamma(X, \mathcal{W})$ be a nowhere vanishing section. For any X-scheme T, let $\mathsf{Assoc}_{W,w}(T)$ denote the subset of $\mathrm{Alg}_W(T)$ consisting of associative algebra structures with multiplicative identity the nowhere vanishing section w_T over T induced from w. Thus we obtain a subfunctor $\mathsf{Assoc}_{W,w}$ of Alg_W.

Let $\mathrm{Stab}_w(T) \subset \mathrm{GL}_W(T)$ denote the stabiliser subgroup of w_T, so that one gets a subfunctor in subgroups $\mathrm{Stab}_w \subset \mathrm{GL}_W$. It is in fact represented by a closed subgroupscheme (also denoted by) Stab_w and further behaves well under base-change relative to X, i.e., $\mathrm{Stab}_w \times_X T$ can be canonically identified with Stab_{w_T} for any X-scheme T. These follow from paragraph 9.6.6 of Chapter I, EGA I [3].

It is clear that the natural action of GL_W on Alg_W induces one of Stab_w on $\mathsf{Assoc}_{W,w}$. It is easy to check (p. 489, Part A, [9]) that the functor $\mathsf{Assoc}_{W,w}$ is a sheaf in the big Zariski site over X and further that this functor is represented by a natural closed subscheme $\mathrm{Assoc}_{W,w} \hookrightarrow \mathrm{Alg}_W$ which is Stab_w-invariant. Further the construction $\mathrm{Assoc}_{W,w}$ behaves well with respect to base-change (relative to X). Consider the subfunctor $\mathsf{Azu}_{W,w} \hookrightarrow \mathrm{Assoc}_{W,w}$ corresponding to Azumaya algebras.

Theorem 2.10 (*Theorem 3.4, Part A, [9]*). (1) $\mathsf{Azu}_{W,w}$ *is represented by a* Stab_w*-stable open subscheme* $\mathrm{Azu}_{W,w} \hookrightarrow \mathrm{Assoc}_{W,w}$ *and the canonical open immersion is an affine morphism.* (2) $\mathrm{Azu}_{W,w}$ *is affine (hence separated) and of* *finite presentation over X, and* $\mathrm{Azu}_{W,w}$ *behaves well with respect to base-change (relative to X).*(3) *Further,* $\mathrm{Azu}_{W,w}$ *is smooth of relative dimension* $(m^2-1)^2$ *and geometrically irreducible over X, where* $m^2 := \mathrm{rank}_X(\mathcal{W})$.

Theorem 2.11 (*Theorem 3.8, Part A, [9]*). (1) $\mathrm{Azu}_{W,w} \hookrightarrow \mathrm{Assoc}_{W,w}$ *has a schematic image denoted as* $\mathrm{SpAzu}_{W,w} \hookrightarrow \mathrm{Assoc}_{W,w}$ *which is affine (and hence separated) and of finite type over X and is naturally a* Stab_w*-stable closed subscheme of* $\mathrm{Assoc}_{W,w}$*, the action extending the natural one on the open subscheme* $\mathrm{Azu}_{W,w}$.(2) *When the rank of W over X is*4, $\mathrm{SpAzu}_{W,w}$ *is locally (over X) isomorphic to relative nine-dimensional affine space; in fact over every open affine subscheme U of X where W becomes trivial and w becomes part of a global basis we have* $\mathrm{SpAzu}_{W,w}|_U \cong \mathbb{A}^9_U$. *For the explicit isomorphism see* 3.11. *Thus* $\mathrm{SpAzu}_{W,w}$ *is smooth of relative dimension 9 and geometrically irreducible over X. In particular, it is of finite presentation over X.*(3) *When* $\mathrm{rank}_X(W) = 4$*, the construction* $\mathrm{SpAzu}_{W,w} \longrightarrow X$ *behaves well with respect to base-change (relative to X).*

3. Statements of the main results

3.1. A limiting version of a theorem in cohomology

For any scheme X, denote by $\mathcal{Q}_3(X)$ (respectively, by $\mathcal{Q}_3^{sr}(X)$) the set of isomorphism classes of line-bundle-valued ternary quadratic bundles (respectively line-bundle-valued semiregular ternary quadratic bundles) on X. Here isomorphism stands for isometry as defined in Section 2.2. Consider the group T-Disc(X) of twisted discriminant bundles on X (2.5). By 2.3, this group acts on $\mathcal{Q}_3(X)$ and on the subset $\mathcal{Q}_3^{sr}(X)$.

For a line-bundle-valued quadratic form (V, q, I), recall from Section 2.4 the Bichsel–Knus even-Clifford algebra $C_0(V, q, I)$, which is the degree zero subalgebra of the generalised Clifford algebra $\widetilde{C}(V, q, I)$ and reduces to the usual even Clifford algebra for a quadratic form with values in the structure sheaf.

Let $\mathcal{SPAZU}_4(X)$ (respectively, $\mathcal{AZU}_4(X)$) denote the set of isomorphism classes of associative unital algebra structures on vector bundles of rank 4 over X that are Zariski-locally isomorphic to even-Clifford algebras of rank 3 quadratic bundles (respectively, that are Azumaya). Recall that 2.9 gives a bijection $\mathsf{Witt\text{-}Invariant}^{sr}_{(X;\mathcal{O}_X)}$: $\mathcal{Q}_3^{sr}(X;\mathcal{O}_X)/\mathrm{Disc(X)} \overset{\cong}{\to} \mathcal{AZU}_4(X)$. It follows that $\mathcal{AZU}_4(X) \subset \mathcal{SPAZU}_4(X)$. By 2.8, the association $(V, q, I) \rightsquigarrow C_0(V, q, I)$ induces a map $\mathsf{Witt\text{-}Invariant}_X : \mathcal{Q}_3(X)/\text{T-Disc}(X) \to \mathcal{SPAZU}_4(X)$, where the left side represents the set of orbits. Since the even-Clifford algebra of a semiregular quadratic module is Azumaya, it follows that the above map restricts to a map: $\mathsf{Witt\text{-}Invariant}^{sr}_X : \mathcal{Q}_3^{sr}(X)/\text{T-Disc}(X) \to \mathcal{AZU}_4(X)$.

Theorem 3.1. *For any scheme X, both the map* $\mathsf{Witt\text{-}Invariant}_X$ *and its restriction* $\mathsf{Witt\text{-}Invariant}^{sr}_X$ *are bijections.*

Thus the above 3.1 may be viewed as a “limiting version” of 2.9 which as noted earlier may be interpreted as a statement in cohomology.

We shall see later ((b1), 3.7) that it is necessary to consider line-bundle-valued quadratic forms to obtain the surjectivity of 3.1 in those cases for which a given A representing an element in $\mathcal{SPAZU}_4(X)$ is such that $\det(A) \notin 2.\mathrm{Pic}(X)$.

Theorem 3.2 (*Part A, [9]*). *Algebra bundles belonging to* $\mathcal{SPAZU}_4(X)$ *are the same as X-valued points of schemes of specialisations (in the sense of* 2.11*) of Azumaya algebra structures on rank 4 vector bundles on X.*

Thus one may also restate the surjectivity as *schematic specialisations of rank 4 Azumaya bundles arise as even-Clifford algebras of ternary quadratic bundles* and the injectivity as follows: *if the even-Clifford algebras of two ternary quadratic bundles are isomorphic, then the quadratic bundles are isometric up to tensoring by a twisted discriminant bundle.*

3.2. Study of groups of similitudes

A bilinear form b, with values in a line bundle I, defined on a vector bundle V over the scheme X induces an I-valued quadratic form q_b given on sections by $x \mapsto b(x, x)$. Let $L[I] := \mathcal{O}_X \oplus (\bigoplus_{n>0}(T^n(I) \oplus T^n(I^{-1})))$ be the Laurent–Rees algebra of I, where sections of V (resp. of I) are declared to be of degree one (resp. of degree two). Then, as we saw in (2d), 2.2, b naturally defines a $\mathbb{Z}$-graded linear isomorphism $\psi_b : \widetilde{C}(V, q_b, I) \cong \Lambda(V) \otimes L[I]$.

In fact we have $\psi_0 : \widetilde{C}(V, 0, I) = \Lambda(V) \otimes L[I]$. Since in general a quadratic bundle (V, q, I) on a non-affine scheme X may not be induced from a global I-valued bilinear form, one is unable to identify the $\mathbb{Z}$-graded vector bundle underlying its generalised Clifford algebra bundle with $\Lambda(V) \otimes L[I]$. The following result overcomes this problem.

Proposition 3.3. *With every isomorphism of algebra bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$, one may naturally associate an isomorphism of bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$ thus inducing a map $\zeta_{\Lambda^2} : \phi \mapsto \phi_{\Lambda^2}$.*

Definition 3.4. When $V = V'$ and $I = I'$, we may thus denote the subset of those ϕ for which $\det(\phi_{\Lambda^2}) \in \mathrm{Aut}[\det(\Lambda^2(V) \otimes I^{-1})] \equiv \Gamma(X, \mathcal{O}_X^*)$ is a square by $\mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)]$ and those for which $\det(\phi_{\Lambda^2}) = 1$ by the smaller subset $\text{S-Iso}[C_0(V, q, I), C_0(V, q', I)]$. Taking $q = q'$ in these sets and replacing “Iso” by “Aut” in their notation respectively defines the groups $\mathrm{Aut}(C_0(V, q, I)) \supset \mathrm{Aut}'(C_0(V, q, I)) \supset \text{S-Aut}(C_0(V, q, I))$.

Theorem 3.5. *For I-valued quadratic forms q and q' on a rank 3 vector bundle V over a scheme X, we have the following commuting diagram of natural maps of sets with the downward arrows being the canonical inclusions, the horizontal arrows being surjective and the top horizontal arrow being bijective:*

$$\begin{array}{ccc}
\text{S-Iso}[(V, q, I), (V, q', I)] & \xrightarrow{\cong} & \text{S-Iso}[C_0(V, q, I), C_0(V, q', I)] \\
\text{inj}\downarrow & & \downarrow\text{inj} \\
\mathrm{Iso}[(V, q, I), (V, q', I)] & \xrightarrow{\text{onto}} & \mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)] \\
\text{inj}\downarrow & & \downarrow\text{inj} \\
\mathrm{Sim}[(V, q, I), (V, q', I)] & \xrightarrow{\text{onto}} & \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)]
\end{array}$$

With respect to the surjections of the horizontal arrows in the diagram above, we further have the following (where l is the function that associates a similarity to its multiplier, $\det(g, l) := \det(g)$ *for an I-similarity g with multiplier l and ζ_{Λ^2} is the map of 3.3 above):*(a) *there is a family of sections $s_{2k+1} : \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)] \longrightarrow \mathrm{Sim}[(V, q, I), (V, q', I)]$ indexed by the integers such that $l \circ s_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ and $(\det^2 \circ s_{2k+1}) \times (l^{-3} \circ s_{2k+1}) = \det \circ \zeta_{\Lambda^2}$;*(b) *there is also a section $s' : \mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)] \longrightarrow \mathrm{Iso}[(V, q, I), (V, q', I)]$ such that $\det^2 \circ s' = \det \circ \zeta_{\Lambda^2}$;*(c) *there is a family of sections $s^+_{2k+1} : \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)] \longrightarrow \mathrm{Sim}[(V, q, I), (V, q', I)]$ indexed by the integers which is multiplicative when followed by the natural inclusions into $\mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$, i.e., if $\phi_i \in \mathrm{Iso}[C_0(V, q_i, I), C_0(V, q_{i+1}, I)]$ then $s^+_{2k+1}(\phi_2 \circ \phi_1) = s^+_{2k+1}(\phi_2) \circ s^+_{2k+1}(\phi_1) \in \mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$. Further, $l \circ s^+_{2k+1} = \det^{2k+1} \circ \zeta_{\Lambda^2}$ and $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det \circ \zeta_{\Lambda^2}$.*(d) *The maps s_{2k+1} and s' above may not be multiplicative but are multiplicative up to $\mu_2(\Gamma(X, \mathcal{O}_X))$, i.e., these maps followed by the canonical quotient map, on taking the quotient of $\mathrm{GL}(V) \times \Gamma(X, \mathcal{O}_X^*)$ by $\mu_2(\Gamma(X, \mathcal{O}_X)).\mathrm{Id}_V \times \{1\}$, become multiplicative.*

Theorem 3.6. *For a rank 3 quadratic bundle (V, q, I) on a scheme X, one has the following natural commutative diagram of groups with exact rows, where the downward arrows are the canonical inclusions and where l is the function that associates with any I-(self-)similarity its multiplier:*

$$\begin{array}{ccccccccc}
 & & & & \mathrm{SO}(V, q, I) & \xrightarrow{\cong} & \text{S-Aut}(C_0(V, q, I)) & & \\
 & & & & \text{inj}\downarrow & & \downarrow\text{inj} & & \\
1 & \longrightarrow & \mu_2(\Gamma(X, \mathcal{O}_X)) & \longrightarrow & \mathrm{O}(V, q, I) & \longrightarrow & \mathrm{Aut}'(C_0(V, q, I)) & \longrightarrow & 1 \\
 & & \text{inj}\downarrow & & \text{inj}\downarrow & & \downarrow\text{inj} & & \\
1 & \longrightarrow & \Gamma(X, \mathcal{O}_X^*) & \longrightarrow & \mathrm{GO}(V, q, I) & \longrightarrow & \mathrm{Aut}(C_0(V, q, I)) & \longrightarrow & 1 \\
 & & & & \det^2 \times l^{-3}\downarrow & & \downarrow\det & & \\
 & & & & \Gamma(X, \mathcal{O}_X^*) & = & \Gamma(X, \mathcal{O}_X^*) & &
\end{array}$$

Further, we have:(a1) $\exists$ *splitting homomorphisms* $s^+_{2k+1} : \mathrm{Aut}(C_0(V,q,I)) \longrightarrow \mathrm{GO}(V,q,I)$ *such that* $l \circ s^+_{2k+1} = \det^{2k+1}$ *and* $(\det^2 \circ s^+_{2k+1}) \times (l^{-3} \circ s^+_{2k+1}) = \det$. *In particular,* $\mathrm{GO}(V,q,I)$ *is a semidirect product.*(a2) *The restriction of* s^+_{2k+1} *to* $\mathrm{Aut}'(C_0(V,q,I))$ *does not necessarily take values in* $\mathrm{O}(V,q,I)$*, but the further restriction to* $\text{S-Aut}(C_0(V,q,I))$ *does take values in* $\mathrm{SO}(V,q,I)$.(a3) *The maps* s_{2k+1} *and* s' *of* 3.5 *above (under the current hypotheses) may not be homomorphisms but are homomorphisms up to* $\mu_2(\Gamma(X,\mathcal{O}_X))$.(b) *Suppose* X *is integral and* $q \otimes \kappa(x)$ *is semiregular at some point* x *of* X *with residue field* $\kappa(x)$*. Then any automorphism of* $C_0(V,q,I)$ *has determinant 1. Hence* $\mathrm{Aut}(C_0(V,q,I)) = \mathrm{Aut}'(C_0(V,q,I)) = \text{S-Aut}(C_0(V,q,I))$ *and* $\mathrm{O}(V,q,I)$ *is the semidirect product of* $\mu_2(\Gamma(X,\mathcal{O}_X))$ *and* $\mathrm{SO}(V,q,I)$.

The proofs of the above results, and of the injectivity part of 3.1, will be given in Sections 4 and 5.

3.3. Study of bilinear forms and interpretation as specialisations

As for the proof of the surjectivity part of 3.1, we have the following which will be proved in Section 6:

Theorem 3.7. *Let* X *be a scheme and* A *a specialisation of rank 4 Azumaya algebra bundles on* X*. Let* $\mathcal{O}_X.1_A \hookrightarrow A$ *be the line subbundle generated by the nowhere-vanishing global section of* A *corresponding to the unit for algebra multiplication.*

(a) *There exist a rank 3 vector bundle* V *on* X*, a quadratic form* q *on* V *with values in the line bundle* $I := \det^{-1}(A)$*, and an isomorphism of algebra bundles* $A \cong C_0(V,q,I)$*. This gives the surjectivity in the statement of* 3.1*. Further, the following linear isomorphisms may be deduced:*(1) $\det(A) \otimes \Lambda^2(V) \cong A/\mathcal{O}_X.1_A$*; from which follow:*(2) $\det(\Lambda^2(V)) \cong (\det(A))^{\otimes -2}$*;*(3) $V \cong (A/\mathcal{O}_X.1_A)^\vee \otimes \det(V) \otimes \det(A)$*;*(4) $\det(A^\vee) \cong (\det(A))^{\otimes -3} \otimes (\det(V))^{\otimes -2}$ *which implies that* $\det(A) \otimes \det(A^\vee) \in 2.\mathrm{Pic}(X)$.

(b) *There exists a quadratic bundle* (V',q',I') *such that* $C_0(V',q',I') \cong A$ *and*(1) *with* $I' = \mathcal{O}_X$ *iff* $\det(A) \in 2.\mathrm{Pic}(X)$*;*(2) *with* q' *induced from a global* I'*-valued bilinear form iff* $\mathcal{O}_X.1_A$ *is an* $\mathcal{O}_X$*-direct summand of* A*;*(3) *with both* $I' = \mathcal{O}_X$ *and with* q' *induced from a global bilinear form (with values in* I'*) iff* $\mathcal{O}_X.1_A$ *is an* $\mathcal{O}_X$*-direct summand of* A *and* $\det(A) \in 2.\mathrm{Pic}(X)$.

If $2 \in \Gamma(X,\mathcal{O}_X^*)$, then any quadratic form is induced from a symmetric bilinear form. Therefore from assertion (b2) of the above, we have the following:

Corollary 3.8. *Suppose* $2 \in \Gamma(X,\mathcal{O}_X^*)$*. Then for any specialisation* A *of rank 4 Azumaya algebras on* X*,* $\mathcal{O}_X.1_A$ *is an* $\mathcal{O}_X$*-direct summand of* A.

There are two ingredients in the proof of part (a) of 3.7. The first is 3.5. The second is the following theorem which describes specialisations as bilinear forms under certain conditions. As a preparation towards its statement, we briefly remind the reader of a few results from Part A, [9] (cf. Section 2.9).

For a rank n^2 vector bundle W on a scheme X and $w \in \Gamma(X,W)$ a nowhere-vanishing global section, recall that if $\mathrm{Azu}_{W,w}$ is the open X-subscheme of Azumaya algebra structures on W with identity w then its schematic image (or the scheme of specialisations or the limiting scheme) in the bigger X-scheme $\mathrm{Assoc}_{W,w}$ of associative w-unital algebra structures on W is the X-scheme $\mathrm{SpAzu}_{W,w}$. By definition, the set of distinct specialised w-unital algebra structures on W corresponds precisely to the set of global sections of this last scheme over X.

If $\mathrm{Stab}_w \subset \mathrm{GL}_W$ is the stabiliser subgroupscheme of w, recall from Theorems 3.4 and 3.8, Part A, [9], that there exists a canonical action of Stab_w on $\mathrm{SpAzu}_{W,w}$ such that the natural inclusions (♣) $\mathrm{Azu}_{W,w} \hookrightarrow \mathrm{SpAzu}_{W,w} \hookrightarrow \mathrm{Assoc}_{W,w}$ are all Stab_w-equivariant. Now let V be a rank 3 vector bundle on the scheme X and $\mathrm{Bil}_{(V,I)}$ be the associated rank 9 vector bundle of bilinear forms on V with values in the line bundle I. We say that a bilinear form b over an open subset $U \hookrightarrow X$ is semiregular if there is a trivialisation $\{U_i\}$ of $I \mid U$, such that over each open subscheme U_i, the quadratic form q_i with values in the trivial line bundle induced from $q_b \mid U_i$ is semiregular (it may turn out that a semiregular bilinear form may be degenerate). This definition is independent of the choice of a trivialisation, since q_i is semiregular iff λq_i is semiregular for every $\lambda \in \Gamma(U_i,\mathcal{O}_X^*)$ (for further details see Section 2.3). In this way we obtain the open subscheme $\mathrm{Bil}^{sr}_{(V,I)} \hookrightarrow \mathrm{Bil}_{(V,I)}$ of semiregular bilinear forms on V with values in I. We next take for W the following special choice: $W := \Lambda^{\mathrm{even}}(V,I) := \bigoplus_{n\geq 0} \Lambda^{2n}(V) \otimes I^{-\otimes n}$ and we let $w \in \Gamma(X,W)$ be the nowhere-vanishing global section corresponding to the unit for the natural multiplication in the twisted even-exterior algebra bundle W. There is an obvious natural action of GL_V on $\mathrm{Bil}_{(V,I)}$. There is also a natural morphism of groupschemes $\mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$ given on valued points by $g \mapsto \bigoplus_{n\geq 0} \Lambda^{2n}(g) \otimes \mathrm{Id}$ and therefore the natural inclusions marked by (♣) above are GL_V-equivariant. Finally, note that there is an obvious involution Σ on $\mathrm{Assoc}_{W,w}$ given by $A \mapsto \mathrm{opposite}(A)$ which leaves the open subscheme $\mathrm{Azu}_{W,w}$ invariant.

Theorem 3.9. (1) *Let V be a rank 3 vector bundle on the scheme X, $W := \Lambda^{\mathrm{even}}(V, I)$ and $w \in \Gamma(X, W)$ correspond to 1 in the twisted even-exterior algebra bundle. There is a natural GL_V-equivariant morphism of X-schemes $\Upsilon' = \Upsilon'_X : \mathrm{Bil}_{(V,I)} \longrightarrow \mathrm{Assoc}_{W,w}$ whose schematic image is precisely the scheme of specialisations $\mathrm{SpAzu}_{W,w}$. Further if Υ' factors canonically through $\Upsilon = \Upsilon_X : \mathrm{Bil}_{(V,I)} \longrightarrow \mathrm{SpAzu}_{W,w}$, then Υ is a GL_V-equivariant isomorphism and it maps the GL_V-stable open subscheme $\mathrm{Bil}^{sr}_{(V,I)}$ isomorphically onto the GL_V-stable open subscheme $\mathrm{Azu}_{W,w}$.(2) The involution Σ of $\mathrm{Assoc}_{W,w}$ defines a unique involution (also denoted by Σ) on the scheme of specialisations $\mathrm{SpAzu}_{W,w}$ leaving the open subscheme $\mathrm{Azu}_{W,w}$ invariant, and therefore via the isomorphism Υ, it defines an involution on $\mathrm{Bil}_{(V,I)}$. This involution is none other than the one on valued points given by $B \mapsto \mathrm{transpose}(-B)$.(3) For an X-scheme T, let V_T (resp. W_T, resp. I_T) denote the pullback of V (resp. W, resp. I) to T, and let w_T be the global section of W_T induced by w. Then the base-changes of Υ'_X and Υ_X to T, namely $\Upsilon'_X \times_X T : \mathrm{Bil}_{(V,I)} \times_X T \longrightarrow \mathrm{Assoc}_{W,w} \times_X T$ and $\Upsilon_X \times_X T : \mathrm{Bil}_{(V,I)} \times_X T \cong \mathrm{SpAzu}_{W,w} \times_X T$, may be canonically identified with the corresponding ones over T namely with $\Upsilon'_T : \mathrm{Bil}_{(V_T,I_T)} \longrightarrow \mathrm{Assoc}_{W_T,w_T}$ and $\Upsilon_T : \mathrm{Bil}_{(V_T,I_T)} \cong \mathrm{SpAzu}_{W_T,w_T}$ respectively.*

For the case when $X = \mathrm{Spec}(k)$, the above theorem was pointed out by S. Ramanan (Section 2.3, [8]). In that case, the target needs to be considered just as a variety, i.e., with the canonical reduced structure. However, over a non-reduced base, the correct closed subscheme structure is the one given by taking the schematic image of the open subscheme of Azumaya structures as in (2.11). The explicit computation of the morphism Υ locally over X is an important step in proving the above theorem. To describe this, suppose that I is trivial and V is free of rank 3 over X, so that we may fix a basis $\{e_1, e_2, e_3\}$ for V, which naturally gives rise to a basis of Bil_V.

For any X-scheme T, a T-valued point B of Bil_V is just a global bilinear form with values in $\mathcal{O}_T$ on the pullback $V \otimes_X T$ of V to T. Such a B is given uniquely by a (3×3)-matrix (b_{ij}) with the b_{ij} being global sections of the trivial line bundle $\mathbb{A}^1_T$ (or equivalently, elements of $\Gamma(T, \mathcal{O}_T)$). The chosen basis for V also gives rise to the basis $\{\epsilon_0 := w = 1; \epsilon_1 := e_1 \wedge e_2, \epsilon_2 := e_2 \wedge e_3, \epsilon_3 := e_3 \wedge e_1\}$ of $W = \Lambda^{\mathrm{even}}(V)$. A T-valued point A of $\mathrm{Assoc}_{W,w}$ is just a $w_T := (w \otimes_X T)$-unital associative algebra structure on the bundle $W_T := W \otimes_X T$. Let $\cdot_A$ denote the multiplication in the algebra bundle A, and for ease of notation, let s° denote the section $s \otimes_X T$ induced from a section s (for example, $(w \otimes_X T) = w^\circ, \epsilon_i \otimes_X T = \epsilon_i^\circ$).

Theorem 3.10. *In addition to the hypothesis of 3.9, assume that V is free of rank 3 and that $I = \mathcal{O}_X$. Then fixing a basis for V and adopting the notation above, the map $\Upsilon(T)$ takes $B = (b_{ij})$ to $(A, 1_A, \cdot) = (W_T, w_T = w^\circ, \cdot_A)$ with multiplication given as follows, where $M_{ij}(B)$ is the determinant of the minor of the element b_{ij} in B :* • $\epsilon_1^\circ \cdot_A \epsilon_1^\circ = -M_{33}(B)w^\circ + (b_{21} - b_{12})\epsilon_1^\circ$ • $\epsilon_2^\circ \cdot_A \epsilon_2^\circ = -M_{11}(B)w^\circ + (b_{32} - b_{23})\epsilon_2^\circ$ • $\epsilon_3^\circ \cdot_A \epsilon_3^\circ = -M_{22}(B)w^\circ + (b_{13} - b_{31})\epsilon_3^\circ$ • $\epsilon_1^\circ \cdot_A \epsilon_2^\circ = -M_{31}(B)w^\circ - b_{23}\epsilon_1^\circ - b_{12}\epsilon_2^\circ - b_{22}\epsilon_3^\circ$ • $\epsilon_2^\circ \cdot_A \epsilon_3^\circ = +M_{12}(B)w^\circ - b_{33}\epsilon_1^\circ - b_{31}\epsilon_2^\circ - b_{23}\epsilon_3^\circ$ • $\epsilon_3^\circ \cdot_A \epsilon_1^\circ = +M_{23}(B)w^\circ - b_{31}\epsilon_1^\circ - b_{11}\epsilon_2^\circ - b_{12}\epsilon_3^\circ$ • $\epsilon_1^\circ \cdot_A \epsilon_3^\circ = +M_{32}(B)w^\circ + b_{13}\epsilon_1^\circ + b_{11}\epsilon_2^\circ + b_{21}\epsilon_3^\circ$ • $\epsilon_2^\circ \cdot_A \epsilon_1^\circ = -M_{13}(B)w^\circ + b_{32}\epsilon_1^\circ + b_{21}\epsilon_2^\circ + b_{22}\epsilon_3^\circ$ • $\epsilon_3^\circ \cdot_A \epsilon_2^\circ = -M_{21}(B)w^\circ + b_{33}\epsilon_1^\circ + b_{13}\epsilon_2^\circ + b_{32}\epsilon_3^\circ$

The key to the proofs of 3.1, 3.5 and 3.6 lies in an analysis of a different identification of the scheme of specialisations, namely one related to the scheme of $\mathcal{O}_X$-valued quadratic forms on a trivial rank 3 bundle in the special situation when W is free and w part of a global basis. Without loss of generality we may in this situation therefore take V to be a free rank 3 vector bundle on X and $(W, w) = (\Lambda^{\mathrm{even}}(V), 1)$, so that we are in the situation of 3.10 above. This relationship with quadratic forms was shown in Theorem 5.3, Part A, [9], which we briefly recall next. Let Quad_V denote the bundle of quadratic forms on V (with values in $\mathcal{O}_X$) and Quad^{sr}_V the open subscheme of semiregular quadratic forms. Let A_0 denote the algebra bundle structure (with unit $w = 1$) on $W = \Lambda^{\mathrm{even}}(V)$ given by $\Lambda^{\mathrm{even}}(V)$ itself. Fix a basis for V and adopt the notation preceding 3.10 above. Then Stab_w is the semidirect product of a commutative three-dimensional subgroupscheme $\mathsf{L}_w \cong (\mathbb{A}^3_X, +)$ with the stabiliser subgroupscheme Stab_{A_0} of A_0 in Stab_w (Lemma 5.1, Part A, [9]).

Theorem 3.11 (*Definition 5.2 & Theorem 5.3, Part A, [9]*). *There is a natural isomorphism* $\Theta : \mathrm{Quad}_V \times_X \mathsf{L}_w \cong \mathrm{SpAzu}_{W,w}$ *which maps the open subscheme* $\mathrm{Quad}_V^{sr} \times_X \mathsf{L}_w$ *isomorphically onto the open subscheme* $\mathrm{Azu}_{W,w}$.

The isomorphism Θ was first defined by Seshadri in [7] for the case $X = \mathrm{Spec}(k)$, k an algebraically closed field of characteristic $\neq 2$. In his case, the target needs to be considered just as a variety, i.e., with the canonical reduced structure. However, over a non-reduced base, the correct closed subscheme structure is the one given by taking the schematic image of the open subscheme of Azumaya structures as in (2.11). Section 5 is essentially devoted to studying Θ. There we compute Θ explicitly and in 5.1 we write out the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit that is part of a global basis. It turns out that Θ is not equivariant with respect to GL_V, but nevertheless satisfies a 'twisted' form of equivariance (5.4). A T-valued point q of $\mathrm{Quad}_V \cong \mathbb{A}_X^6$ may be identified with a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})$ corresponding to the quadratic form $(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j$. A T-valued point $\underline{t}$ of $\mathsf{L}_w \cong (\mathbb{A}_X^3, +)$ may be identified with a 3-tuple (t_1, t_2, t_3) which corresponds to the valued point of Stab_w given by the (4×4)-matrix with $(1, t_1, t_2, t_3)$ as first row, and the next 3 rows given by the (3×4)-submatrix $(0, I_3)$ where I_3 is the (3×3)-identity matrix. With this notation, the identifications in 3.9 and 3.10 may be compared with that of the above 3.11 as follows.

Theorem 3.12. *The isomorphism* $\Upsilon^{-1} \circ \Theta : \mathrm{Quad}_V \times_X \mathsf{L}_w \cong \mathrm{Bil}_V$ *takes the valued point* $(q, \underline{t}) = ((\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}), (t_1, t_2, t_3))$ *to the valued point* $B = (b_{ij})$ *such that* $b_{ii} = \lambda_i$, $b_{12} = t_1$, $b_{23} = t_2$, $b_{31} = t_3$, $b_{13} = \lambda_{13} - t_3$, $b_{21} = \lambda_{12} - t_1$, $b_{32} = \lambda_{23} - t_2$. *Moreover, under this identification, the involution* $B \mapsto (-B)^t$ *on* Bil_V *(induced from the isomorphism* Υ *of 3.9) translates into the involution on* $\mathrm{Quad}_V \times_X \mathsf{L}_w$ *given by* $(q, (t_1, t_2, t_3)) \mapsto (-q, (t_1 - \lambda_{12}, t_2 - \lambda_{23}, t_3 - \lambda_{13}))$.

Proposition 3.13. *Let S be a commutative semilocal ring that is 2-perfect, i.e., such that the square map $S \longrightarrow S : s \mapsto s^2$ is surjective, and V a free rank 3 S-module. Then the set of semiregular quadratic S-forms on V forms a single* $\mathrm{GL}(V)$*-orbit; in other words, up to isometry, $\exists$ only one semiregular quadratic S-module structure on V.*

Corollary 3.14. *Let S be a commutative local ring that is 2-perfect. Then any two rank 4 Azumaya S-algebras are isomorphic. If S is only semilocal, the conclusion still holds provided the identity elements for multiplication for each of the two Azumaya S-algebras can be completed to an S-basis.*

The proof of 3.13 is given in Section 6. In view of 2.9, taking $X = \mathrm{Spec}(S)$ with S as in 3.13 proves the first assertion of the above corollary. The second may be deduced by an application of 3.11 along with 3.13.

4. Injectivity: Reduction to lifting to similarities in the free case

Proof of 3.3. Start with an isomorphism of algebra bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$. Let $\{U_i\}_{i \in \mathfrak{I}}$ be an affine open covering of X (which may also be chosen so as to trivialise some or any of the bundles involved if needed). Choose bilinear forms $b_i \in \Gamma(U_i, \mathrm{Bil}_{(V,I)})$ and $b'_i \in \Gamma(U_i, \mathrm{Bil}_{(V',I')})$ such that $q \mid U_i = q_{b_i}$ and $q' \mid U_i = q_{b'_i}$ for each $i \in \mathfrak{I}$. By (2d), 2.2, we have isomorphisms of vector bundles ψ_{b_i} and $\psi_{b'_i}$, which preserve 1 by (2a) of the same Theorem, and we define the isomorphism of vector bundles $(\phi_{\Lambda^{ev}})_i$ so as to make the following diagram commute:

$$\begin{array}{ccc} C_0(V,q,I)|U_i & \xrightarrow[\cong]{\phi|U_i} & C_0(V',q',I')|U_i \\ \psi_{b_i}\downarrow\cong & & \cong\downarrow\psi_{b'_i} \\ (\mathcal{O}_X \oplus \Lambda^2(V)\otimes I^{-1})|U_i & \xrightarrow[(\phi_{\Lambda^{ev}})_i]{\cong} & (\mathcal{O}_X \oplus \Lambda^2(V')\otimes (I')^{-1})|U_i \end{array}$$

The linear isomorphism $(\phi_{\Lambda^{ev}})_i$ preserves 1 and therefore it induces a linear isomorphism $(\phi_{\Lambda^2})_i : (\Lambda^2(V) \otimes I^{-1}) \mid U_i \xrightarrow{\cong} (\Lambda^2(V') \otimes (I')^{-1}) \mid U_i$. Observe that $(\phi_{\Lambda^2})_i$ is independent of the choice of the bilinear forms b_i and b'_i. For, replacing these respectively by $\widehat{b_i}$ and $\widehat{b'_i}$, it follows from (2f), 2.2, that $\psi_{b_i} \circ (\psi_{\widehat{b_i}})^{-1}$ (resp. $\psi_{b'_i} \circ (\psi_{\widehat{b'_i}})^{-1}$) followed by the canonical projection onto $(\Lambda^2(V) \otimes I^{-1}) \mid U_i$ (resp. onto $(\Lambda^2(V') \otimes (I')^{-1}) \mid U_i$) is the same as the projection itself. By this observation, it is also clear that the isomorphisms $\{(\phi_{\Lambda^2})_i\}_{i \in \mathfrak{I}}$ agree on (any open affine subscheme of, and hence on all of) any intersection $U_i \cap U_j$. Therefore they glue to give a global isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$ as required. □ (3.3)

Reduction of proof of injectivity of 3.1 to 3.5. We start with an isomorphism of algebra bundles $\phi : C_0(V, q, I) \cong C_0(V', q', I')$, construct the isomorphism of vector bundles $\phi_{\Lambda^2} : \Lambda^2(V) \otimes I^{-1} \cong \Lambda^2(V') \otimes (I')^{-1}$ and keep the notation introduced in the proof of Proposition 3.3. Firstly we deduce a linear isomorphism $\det((\phi_{\Lambda^2})^\vee)^{-1} : \det((\Lambda^2(V) \otimes I^{-1})^\vee) \cong \det((\Lambda^2(V') \otimes (I')^{-1})^\vee)$. Since V and V' are of rank 3, there are canonical isomorphisms $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ and $\eta' : \Lambda^2(V') \equiv (V')^\vee \otimes \det(V')$. It follows therefore that if we set $L := \det(V') \otimes (\det(V))^{-1}andJ := I' \otimes I^{-1}$ then we get a twisted discriminant line bundle $(L \otimes J^{-1}, h, J)$ and a vector bundle isomorphism $\alpha : V' \cong V \otimes (L \otimes J^{-1})$. Now for each $i \in \mathcal{I}$, the bilinear form $b_i \in \Gamma(U_i, \mathrm{Bil}_{(V,I)})$ induces, via $\alpha \mid U_i$ and $(LJ^{-1}, h, J) \mid U_i$ and (1), 2.6, a bilinear form $b''_i \in \Gamma(U_i, \mathrm{Bil}_{(V',IJ)})$. By (3) of the same Proposition, over each U_i we get an isometry of bilinear form bundles $\alpha \mid U_i : (V' \mid U_i, b''_i, IJ \mid U_i) \cong (V \mid U_i, b_i, I \mid U_i) \otimes (LJ^{-1}, h, J) \mid U_i$ and also an isometry of quadratic bundles $\alpha \mid U_i : (V' \mid U_i, q_{b''_i}, IJ \mid U_i) \cong (V \mid U_i, q_{b_i} = q \mid U_i, I \mid U_i) \otimes (LJ^{-1}, h, J) \mid U_i$. On the other hand, by an assertion in (3), 2.6, we could also define the global quadratic bundle (V', q'', IJ) using (V, q, I), α and (LJ^{-1}, h, J), so that we have an isometry of quadratic bundles $\alpha : (V', q'', IJ) \cong (V, q, I) \otimes (LJ^{-1}, h, J)$. It follows therefore that the $q_{b''_i}$ glue to give q''. Notice that in general the b''_i (resp. the b_i) need not glue to give a global bilinear form b'' (resp. b) such that $q_{b''} = q''$ (resp. $q_b = q$). By (1), 2.7, there exists a unique isomorphism of algebra bundles $C_0(\alpha, 1, IJ) : C_0(V', q'', IJ) \cong C_0((V, q, I) \otimes (LJ^{-1}, h, J))$ and by 2.8 we have a unique isomorphism of algebra bundles $\gamma_{(LJ^{-1},h,J)} : C_0((V, q, I) \otimes (LJ^{-1}, h, J)) \cong C_0(V, q, I)$. Therefore the composition of the following sequence of isomorphisms of algebra bundles on X

$$C_0(V', q'', I') \overset{\text{using} I' \cong IJ}{\cong} C_0(V', q'', IJ) \overset{C_0(\alpha,1,IJ)(\cong)}{\longrightarrow} C_0((V, q, I) \otimes (LJ^{-1}, h, J))$$
$$\overset{\gamma_{(LJ^{-1},h,J)}(\cong)}{\longrightarrow} C_0(V, q, I) \overset{\phi(\cong)}{\longrightarrow} C_0(V', q', I')$$

is an element of $\mathrm{Iso}[C_0(V', q'', I'), C_0(V', q', I')]$, which, granting 3.5, is induced by a similarity $(g, l) \in \mathrm{Sim}[(V', q'', I'), (V', q', I')]$. Hence we would have $g : (V', q'', I') \cong (V', q', I') \otimes (\mathcal{O}_X, (s \otimes s' \mapsto s.s'.l^{-1}), \mathcal{O}_X)$ where $l \in \Gamma(X, \mathcal{O}_X^*)$. This combined with the fact that (V, q, I) and $(V', q'', I' = IJ)$ are isomorphic up to the twisted discriminant bundle (LJ^{-1}, h, J) by the construction above, would imply that (V, q, I) and (V', q', I') also differ by a twisted discriminant bundle. Therefore the proof of the injectivity asserted in 3.1 reduces to the proof of 3.5.

Reduction of 3.5 to the case when I is free. For a similarity g with multiplier l, we have $C_0(g, l, I)$ given by (1), 2.7, so that we may define the map $\mathrm{Sim}[(V, q, I), (V, q', I)] \longrightarrow \mathrm{Iso}[C_0(V, q, I), C_0(V, q', I)] : g \mapsto C_0(g, l, I)$. The equality $\det(C_0(g, l, I)) = \det((C_0(g, l, I))_{\Lambda^2}) = l^{-3}\det^2(g)$ will be shown to hold (locally, and hence globally) in (1), 5.11. Thus $\mathrm{Iso}[(V, q, I), (V, q', I)]$ and $\text{S-Iso}[(V, q, I), (V, q', I)]$ are respectively mapped into $\mathrm{Iso}'[C_0(V, q, I), C_0(V, q', I)]$ and $\text{S-Iso}[C_0(V, q, I), C_0(V, q', I)]$ as claimed.

We start with an isomorphism of algebra bundles $\phi : C_0(V, q, I) \cong C_0(V, q', I)$, which by 3.3 leads to the automorphism ϕ_{Λ^2} of the vector bundle $\Lambda^2(V) \otimes I^{-1}$. Firstly, define the global bundle automorphism $g' \in \mathrm{GL}(V \otimes (\det(V))^{-1} \otimes I)$ so that the following diagram commutes:

$$\begin{array}{ccc}
(\Lambda^2(V))^\vee \otimes I & \xrightarrow[\cong]{((\phi_{\Lambda^2})^\vee)^{-1}} & (\Lambda^2(V))^\vee \otimes I \\
{\scriptstyle (\eta^\vee)^{-1} \otimes I} \downarrow \equiv & & \equiv \downarrow {\scriptstyle (\eta^\vee)^{-1} \otimes I} \\
V \otimes (\det(V))^{-1} \otimes I & \xrightarrow[g']{\cong} & V \otimes (\det(V))^{-1} \otimes I
\end{array}$$

where $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ is the canonical isomorphism (since V is of rank 3). Now let $g \in \mathrm{GL}(V) \overset{\cong}{\longleftarrow} \mathrm{GL}(V \otimes (\det(V))^{-1} \otimes I)$ be the image of g', i.e., the image of $g' \otimes \det(V) \otimes I^{-1}$ under the canonical identification $\mathrm{GL}(V \otimes (\det(V))^{-1} \otimes I \otimes \det(V) \otimes I^{-1}) \equiv \mathrm{GL}(V)$. Next, let $l \in \Gamma(X, \mathcal{O}_X^*)$ be a global section such that

$\gamma(l) := (l^3).\det(\phi_{\Lambda^2})$ has a square root in $\Gamma(X, \mathcal{O}_X^*)$. For example, we have the following independent special cases when this is true:

Case 1. If $\phi \in$ S-Iso$[C_0(V, q, I), C_0(V, q', I)]$, i.e., if $\det(\phi_{\Lambda^2}) = 1$, then set $l = 1$ and $\sqrt{\gamma(l)} = 1$.

Case 2. If $\phi \in$ Iso$'[C_0(V, q, I), C_0(V, q', I)]$, i.e., $\det(\phi_{\Lambda^2})$ is a square, then set $l = 1$ and let $\sqrt{\gamma(l)}$ denote any fixed square root of $\det(\phi_{\Lambda^2})$.

Case 3. Given an integer k, take $l = (\det(\phi_{\Lambda^2}))^{2k+1}$ and let $\sqrt{\gamma(l)}$ denote any fixed square root of $(\det(\phi_{\Lambda^2}))^{6k+4}$.

For each integer k, we now associate with ϕ the element $g_l^{\phi} := (l^{-1}\sqrt{\gamma(l)})g$ with g as defined above. We shall show the following locally for the Zariski topology on X (more precisely, for each open subscheme of X over which V and I are free): (1) that g_l^{ϕ} is an I-similarity from (V, q, I) to (V, q', I) with multiplier l (5.9); (2) that g_l^{ϕ} induces ϕ, i.e., with the notation of (1), 2.7, that $C_0(g_l^{\phi}, l, I) = \phi$ (5.10); (3) that $\det(g_l^{\phi}) = \sqrt{\gamma(l)}$ so that $\det^2(g_l^{\phi}) = \det(\phi_{\Lambda^2})$ in cases 1 and 2 (5.8) and (4) that the map S-Iso$[(V, q, I), (V, q', I)] \longrightarrow$ S-Iso$[C_0(V, q, I), C_0(V, q', I)]$ is injective (5.12).

It would follow then that these statements are also true globally. The maps $s_{2k+1} : \phi \mapsto g_l^{\phi}$ with l as in Case 3 and $s' : \phi \mapsto g_l^{\phi}$ with l as in Case 2 will then give the sections to the maps (which would imply their surjectivities) as mentioned in 3.5. But these maps are not necessarily multiplicative since a computation reveals that if $\phi_i \in$ Iso$[C_0(V, q_i, I), C_0(V, q_{i+1}, I)]$ is associated with $g_{l_i}^{\phi_i} \in$ Sim$[(V, q_i, I), (V, q_{i+1}, I)]$, and $\phi_2 \circ \phi_1$ to $g_{l_{21}}^{\phi_2\circ\phi_1}$, then $g_{l_{21}}^{\phi_2\circ\phi_1} = \delta g_{l_2}^{\phi_2} \circ g_{l_1}^{\phi_1}$ for $\delta \in \mu_2(\Gamma(X, \mathcal{O}_X))$ because of the ambiguity in the initial global choices of square roots for $\gamma(l_i)$ and $\gamma(l_{21})$. However this can be remedied as follows. For any given $\phi \in$ Iso$[C_0(V, q, I), C_0(V, q', I)]$, irrespective of whether or not $\det(\phi_{\Lambda^2})$ is a square, take $l = (\det(\phi_{\Lambda^2}))^{2k+1}, \gamma(l) = l^3\det(\phi_{\Lambda^2}), \sqrt{\gamma(l)} := (\det(\phi_{\Lambda^2}))^{3k+2}$ and $s_{2k+1}^{+}(\phi) := g_l^{\phi} = (l^{-1}\sqrt{\gamma(l)})g$. Then it is clear that each s_{2k+1}^{+} is multiplicative with the properties as claimed in the statement. We thus reduce the proof of 3.5 to the case when the rank 3 vector bundle V and the line bundle I are free. This will be treated in the next section.

5. The free case: Investigation of the isomorphism theta

Throughout this section, we work with $I = \mathcal{O}_X$ and shorten our earlier notation $(V, q, I), C_0(V, q, I), C_0(g, l, I)$ etc respectively to $(V, q), C_0(V, q), C_0(g, l)$ etc. We conclude the proofs of the injectivity of 3.1 and 3.5 which were begun in Section 4 and also prove 3.6.

As means to these ends, we carry out two explicit computations. Firstly we compute the isomorphism Θ of 3.11. This provides us with the multiplication table of every specialised algebra structure on any fixed free rank 4 vector bundle with fixed unit which is part of a global basis (5.1 below). This result will also be used in Section 6 in the proof of 3.12. It turns out that Θ is not equivariant with respect to GL_V, but nevertheless satisfies a 'twisted' form of equivariance (5.4). Secondly, we explicitly compute the algebra bundle isomorphism $C_0(g, l) : C_0(V, q) \cong C_0(V, q')$ of (1), 2.7 induced by a similarity $g : (V, q) \cong_l (V, q')$ with multiplier $l \in \Gamma(X, \mathcal{O}_X^*)$ in the case when V is free of rank 3 (5.5).

5.1. The action of GL on forms

Let V be a vector bundle over a scheme X with associated locally free sheaf $\mathcal{V}$. The X-smooth X-groupscheme GL_V acts naturally on the left on the sheaves $\mathrm{Alt}^2_{\mathcal{V}}$, $\mathrm{Bil}_{\mathcal{V}}$ and $\mathrm{Quad}_{\mathcal{V}}$ of alternating, bilinear and quadratic forms on $\mathcal{V}$ (with values in $\mathcal{O}_X$). Namely, for $U \hookrightarrow X$ an open subscheme, and $b \in \Gamma(U, \mathrm{Bil}_{\mathcal{V}})$ (resp. $a \in \Gamma(U, \mathrm{Alt}^2_{\mathcal{V}})$, resp. $q \in \Gamma(U, \mathrm{Quad}_{\mathcal{V}})$) and $g \in \Gamma(U, \mathrm{GL}_V) = \mathrm{GL}(V \mid U)$, the corresponding form of the same type $g.b$ (resp. $g.a$, resp. $g.q$) is defined on sections (over open subsets of U) by $(g.b)(v, v') := b(g^{-1}(v), g^{-1}(v'))$ (resp. $(g.a)(v, v') := a(g^{-1}(v), g^{-1}(v'))$, resp. $(g.q)(v) := q(g^{-1}(v))$). It is immediate that the following short exact sequence of sheaves is equivariant with respect to this action: (♠) $0 \longrightarrow \mathrm{Alt}^2_{\mathcal{V}} \longrightarrow \mathrm{Bil}_{\mathcal{V}} \longrightarrow \mathrm{Quad}_{\mathcal{V}} \longrightarrow 0$. Equivalently, the X-groupscheme GL_V acts on the corresponding geometric vector bundles such that both of the X-morphisms of X-vector bundles in the following sequence are GL_V-equivariant: $\mathrm{Alt}^2_V \hookrightarrow \mathrm{Bil}_V \twoheadrightarrow \mathrm{Quad}_V$. Notice that it is one and the same thing to require that $\mathrm{GL}(V \mid U) \ni g : (V \mid U, q) \cong_l (V \mid U, q')$ be a similitude (=similarity) with multiplier $l \in \Gamma(U, \mathcal{O}_X)$, and to require that $g.q = l^{-1}q'$.

5.2. Computation of the isomorphism theta

We briefly recall the definition of Θ from Part A of [9]. We keep the notation introduced just before 3.11; for ease of notation, the pullback of a section s (of a vector bundle or its associated sheaf) is denoted by s°. Since V is free of rank 3 on X, we choose an identification $\mathcal{V} \equiv \mathcal{O}_X.e_1 \oplus \mathcal{O}_X.e_2 \oplus \mathcal{O}_X.e_3$. This gives the identification of the dual bundle as $\mathcal{V}^\vee \equiv \mathcal{O}_X.f_1 \oplus \mathcal{O}_X.f_2 \oplus \mathcal{O}_X.f_3$ (defined uniquely by $f_i(e_j) = \delta_{ij}$, the Kronecker delta). Therefore the dual of the sheaf of quadratic forms on V, which is $(\mathrm{Quad}_{\mathcal{V}})^\vee := (\mathrm{Bil}_{\mathcal{V}}/\mathrm{Alt}^2_{\mathcal{V}})^\vee = ((T^2\mathcal{V})^\vee/(\Lambda^2\mathcal{V})^\vee)^\vee$, has global $\mathcal{O}_X$-basis given by $\{e_i \otimes e_i; (e_i \otimes e_j + e_j \otimes e_i)\}$. This leads to an identification of the associated sheaf of symmetric algebras $\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Quad}^\vee_{\mathcal{V}}) \equiv \mathcal{O}_X[Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23}]$, where $e_i \otimes e_i \equiv Y_i$ and $e_i \otimes e_j + e_j \otimes e_i \equiv Y_{ij}$, and so $\mathrm{Quad}_V := \mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{Quad}^\vee_{\mathcal{V}})) \equiv \mathbf{Spec}(\mathcal{O}_X[Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23}]) = \mathbb{A}^6_X$. Consider the universal quadratic bundle $(\mathbf{V}, \mathbf{q})$ where $\mathbf{V}$ is the pullback of V by $\mathrm{Quad}_V \longrightarrow X$. The universal quadratic form $\mathbf{q}$ is given by $(x_1, x_2, x_3) \mapsto \Sigma_i Y_i.(x_i)^2 + \Sigma_{i<j} Y_{ij}.x_i.x_j$ and moreover the global bilinear form on $\mathbf{V}$ given by $\mathbf{b(q)}$: $((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) \mapsto \Sigma_i Y_i.x_i.x'_i + Y_{12}.x_2.x'_1 + Y_{23}.x_3.x'_2 + Y_{13}.x_1.x'_3$ induces $\mathbf{q}$ (the bilinear form 'associated in the usual sense' with $\mathbf{q}$, namely $b_{\mathbf{q}}$, is not $\mathbf{b(q)}$ but in fact its symmetrisation). Therefore, by (2d), 2.2, we get an isomorphism of vector bundles $\psi_{\mathbf{b(q)}} : C_0(\mathbf{V}, \mathbf{q} = q_{\mathbf{b(q)}}) \cong \Lambda^{\mathrm{even}}(\mathbf{V}) =: \mathbf{W}$ which, according to (2a) and (2f) of the same Theorem, carries the ordered Poincaré–Birkhoff–Witt basis $\{1; e^\circ_1.e^\circ_2, e^\circ_2.e^\circ_3, e^\circ_3.e^\circ_1\}$ onto the corresponding ordered basis of the even exterior algebra (=even Clifford algebra of the zero quadratic form on $\mathbf{V}$) given by $\{w^\circ = 1^\circ = 1; e^\circ_1 \wedge e^\circ_2, e^\circ_2 \wedge e^\circ_3, e^\circ_3 \wedge e^\circ_1\}$. The choices $e^\circ_3.e^\circ_1$ and $e^\circ_3 \wedge e^\circ_1$ instead of the usual $e^\circ_1.e^\circ_3$ and $e^\circ_1 \wedge e^\circ_3$ are deliberate—for example, $\psi_{\mathbf{b(q)}}$ would carry $\{1; e^\circ_1.e^\circ_2, e^\circ_2.e^\circ_3, e^\circ_1.e^\circ_3\}$ onto $\{w^\circ = 1^\circ = 1; e^\circ_1 \wedge e^\circ_2, e^\circ_2 \wedge e^\circ_3, e^\circ_1 \wedge e^\circ_3 + Y_{13}.w^\circ\}$ which depends on Y_{13}. Thus the even Clifford algebra bundle $C_0(\mathbf{V}, \mathbf{q} = q_{\mathbf{b(q)}})$ induces via $\psi_{\mathbf{b(q)}}$ a w°-unital algebra structure on the pullback bundle $\mathbf{W}$ of $W := \Lambda^{\mathrm{even}}(V)$ (where w corresponds to 1 in $\Lambda^{\mathrm{even}}(V)$). But by definition, this algebra structure corresponds precisely to an X-morphism $\theta : \mathrm{Quad}_V \longrightarrow \text{Id-}w\text{-Sp-Azu}_W$. The isomorphism Θ is now given by the composition of the following X-morphisms (cf. Definition 5.2, Part A, [9]): $\mathrm{Quad}_V \times_X \mathsf{L}_w \xrightarrow{\theta\times\mathrm{ID}} \text{Id-}w\text{-Sp-Azu}_W \times_X \mathsf{L}_w \xrightarrow{\mathrm{SWAP}(\equiv)} \mathsf{L}_w \times_X \text{Id-}w\text{-Sp-Azu}_W \xrightarrow{\mathrm{ACTION}} \text{Id-}w\text{-Sp-Azu}_W$. The association of $\mathbf{q}$ with $\mathbf{b(q)}$ also defines a splitting of the exact sequence (♠) as above, so that more generally, given a valued point $q \in (\mathrm{Quad}_V)(T)$, we may associate uniquely a valued point $b(q) \in (\mathrm{Bil}_V)(T)$ which induces it. That this association is not GL_V-equivariant is reflected in the lack of equivariance of the isomorphism Θ (5.4).

Theorem 5.1. *Let T be an X-scheme. Let q be a T-valued point of $\mathrm{Quad}_V \equiv \mathbb{A}^6_X$ which is identified uniquely with a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23})$ corresponding to the quadratic form $(x_1, x_2, x_3) \mapsto \Sigma_i \lambda_i x_i^2 + \Sigma_{i<j} \lambda_{ij} x_i x_j$. Let $\underline{t}$ be a T-valued point of $\mathsf{L}_w \equiv (\mathbb{A}^3_X, +)$ which is identified uniquely with a 3-tuple (t_1, t_2, t_3) that corresponds to the T-valued point of Stab_w given by the (4×4)-matrix with first row $(1, t_1, t_2, t_3)$ and with the remaining (3×4)-submatrix given by $(0, I_3)$ where I_3 is the (3×3)-identity matrix. Then in terms of the global basis $\{w^\circ = 1^\circ = 1; \epsilon^\circ_1 := e^\circ_1 \wedge e^\circ_2, \epsilon^\circ_2 := e^\circ_2 \wedge e^\circ_3, \epsilon^\circ_3 := e^\circ_3 \wedge e^\circ_1\}$ induced from that of $W = \Lambda^{\mathrm{even}}(V)$, the multiplication table for the specialised algebra structure $\Theta(q, \underline{t}) = \underline{t}.\theta(q)$ on the pullback bundle W_T with unit $w^\circ = w_T$ is given as follows:* • $\epsilon^\circ_1.\epsilon^\circ_1 = (t_1\lambda_{12} - \lambda_1\lambda_2 - t_1^2).w^\circ + (\lambda_{12} - 2t_1).\epsilon^\circ_1$; • $\epsilon^\circ_2.\epsilon^\circ_2 = (t_2\lambda_{23} - \lambda_2\lambda_3 - t_2^2).w^\circ + (\lambda_{23} - 2t_2).\epsilon^\circ_2$; • $\epsilon^\circ_3.\epsilon^\circ_3 = (t_3\lambda_{13} - \lambda_1\lambda_3 - t_3^2).w^\circ + (\lambda_{13} - 2t_3).\epsilon^\circ_3$; • $\epsilon^\circ_1.\epsilon^\circ_2 = (\lambda_2\lambda_{13} - \lambda_2 t_3 - t_1 t_2).w^\circ - t_2\epsilon^\circ_1 - t_1\epsilon^\circ_2 - \lambda_2\epsilon^\circ_3$; • $\epsilon^\circ_2.\epsilon^\circ_3 = (\lambda_3\lambda_{12} - \lambda_3 t_1 - t_2 t_3).w^\circ - \lambda_3\epsilon^\circ_1 - t_3\epsilon^\circ_2 - t_2\epsilon^\circ_3$; • $\epsilon^\circ_3.\epsilon^\circ_1 = (\lambda_1\lambda_{23} - \lambda_1 t_2 - t_1 t_3).w^\circ - t_3\epsilon^\circ_1 - \lambda_1\epsilon^\circ_2 - t_1\epsilon^\circ_3$; • $\epsilon^\circ_2.\epsilon^\circ_1 = (\lambda_2 t_3 - (\lambda_{12} - t_1)(\lambda_{23} - t_2)).w^\circ + (\lambda_{23} - t_2)\epsilon^\circ_1 + (\lambda_{12} - t_1)\epsilon^\circ_2 + \lambda_2\epsilon^\circ_3$; • $\epsilon^\circ_3.\epsilon^\circ_2 = (\lambda_3 t_1 - (\lambda_{13} - t_3)(\lambda_{23} - t_2)).w^\circ + \lambda_3\epsilon^\circ_1 + (\lambda_{13} - t_3)\epsilon^\circ_2 + (\lambda_{23} - t_2)\epsilon^\circ_3$; • $\epsilon^\circ_1.\epsilon^\circ_3 = (\lambda_1 t_2 - (\lambda_{12} - t_1)(\lambda_{13} - t_3)).w^\circ + (\lambda_{13} - t_3)\epsilon^\circ_1 + \lambda_1\epsilon^\circ_2 + (\lambda_{12} - t_1)\epsilon^\circ_3$.

Proof of 5.1. For clarity, let $*_q$ denote the multiplication in the algebra $C_0(V_T, q)$, and for uniformity, let $\epsilon_0 := w$. Since $q = q_{b(q)}$, we have by (2d), 2.2, the isomorphism $\psi_{b(q)} : C_0(V_T, q) \cong \Lambda^{\mathrm{even}}(V_T) = W_T$. Let $*_{b(q)}$ denote the product in the algebra structure $\theta(q)$ thus induced on W_T. Since the ϵ°_i are a basis for W_T, it is enough to compute the products $\epsilon^\circ_i *_{b(q)} \epsilon^\circ_j$ for $1 \le i, j \le 3$. For example, consider the product $\epsilon^\circ_2 *_{b(q)} \epsilon^\circ_1$. Using the properties of the multiplication in $C(V_T, q)$, and the properties of the isomorphism $\psi_{b(q)}$ from (2), 2.2, we get the following:

$$\begin{aligned}
\epsilon^\circ_2 *_{b(q)} \epsilon^\circ_1 &= \psi_{b(q)}\left(\{\psi^{-1}_{b(q)}(e^\circ_2 \wedge e^\circ_3)\} *_q \{\psi^{-1}_{b(q)}(e^\circ_1 \wedge e^\circ_2)\}\right) \\
&= \psi_{b(q)}((e^\circ_2 *_q e^\circ_3) *_q (e^\circ_1 *_q e^\circ_2)) \\
&= \psi_{b(q)}((\lambda_{23}(1^\circ) - e^\circ_3 *_q e^\circ_2) *_q (\lambda_{12}(1^\circ) - e^\circ_2 *_q e^\circ_1)) \\
&= (-\lambda_{12}\lambda_{23})w^\circ + \lambda_{23}\epsilon^\circ_1 + \lambda_{12}\epsilon^\circ_2 + \lambda_2\epsilon^\circ_3.
\end{aligned}$$

In a similar fashion, the other products may be computed; this amounts to computing θ on T-valued points. The following result is needed to compute Θ from θ.

Lemma 5.2. *Let $*_{(b(q),\underline{t})}$ denote the multiplication in the algebra $\Theta(q,\underline{t}) = \underline{t}.\theta(q)$ and as before, $*_{b(q)}$ denote the multiplication in $\theta(q)$. Then we have*(1) $\underline{t}(\epsilon_i^\circ) = t_i w^\circ + \epsilon_i^\circ$ *for* $1 \le i \le 3$;(2) $(\underline{t})^{-1}(\epsilon_i^\circ) = -t_i w^\circ + \epsilon_i^\circ$ *for* $1 \le i \le 3$;(3) $\epsilon_i^\circ *_{(b(q),\underline{t})} \epsilon_j^\circ = \underline{t}(\epsilon_i^\circ *_{b(q)} \epsilon_j^\circ) - t_j\epsilon_i^\circ - t_i\epsilon_j^\circ - t_it_jw^\circ$.

While the first two of the above formulae follow easily by direct computation, the third follows by using the first two along with the following: $\epsilon_i^\circ *_{(b(q),\underline{t})} \epsilon_j^\circ = \underline{t}((\underline{t}^{-1}(\epsilon_i^\circ)) *_{b(q)} (\underline{t}^{-1}(\epsilon_j^\circ)))$. We may now compute the multiplication in the algebra $\Theta(q,\underline{t}) = \underline{t}.\theta(q)$ by making use of the formulae listed in the above lemma and the expressions for the products of the form $\epsilon_i^\circ *_{b(q)} \epsilon_j^\circ$ whose computation had already been illustrated before the lemma. □ (5.1)

5.3. Computation of the isomorphism arising from a similarity

We continue with the notation introduced above. In the following we study the lack of equivariance of the isomorphism Θ relative to GL_V and show that it satisfies a curious 'twisted' version of equivariance. The aim is to compute the isomorphism induced by a similarity. Firstly we consider the morphism of X-groupschemes $\Lambda^{\mathrm{even}} : \mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$ given on valued points by $g \mapsto \Lambda^{\mathrm{even}}(g)$. Recall that $\Lambda^{\mathrm{even}}(V) =: A_0 \in \text{Id-}w\text{-Sp-Azu}_W(X)$ is the even graded part of the Clifford algebra of the zero quadratic form on V. A simple computation reveals the following result.

Lemma 5.3. *For each X-scheme T, define the map* $\mathrm{GL}(V_T) \longrightarrow \mathrm{Stab}((A_0)_T)$ *given by sending g to the matrix with first row* $(1, 0)$ *and with the remaining rows given by the* (3×4)*-submatrix* $(0, B(g))$*, where $B(g)$ is the* (3×3)*-matrix given by* $\det(g)\left(E_{12}E_{23}(g^{-1})^t E_{23}E_{12}\right)$ *and the matrices E_{ij} are given row-wise by* $E_{12} = (0, 1, 0; 1, 0, 0; 0, 0, 1)$ *and* $E_{23} = (1, 0, 0; 0, 0, 1; 0, 1, 0)$. *Then the above maps define a morphism of X-groupschemes which in fact is none other than $\Lambda^{\mathrm{even}} : \mathrm{GL}_V \longrightarrow \mathrm{Stab}_w$; in other words: $B(g) = \Lambda^2(g)$.*

Recall from Lemma 5.1, Part A, [9], that Stab_w is the semidirect product of Stab_{A_0} and L_w, so that Stab_{A_0} naturally acts on L_w by "outer conjugation". Let GL_V act on L_w via the homomorphism Λ^{even}, i.e., for $g \in \mathrm{GL}(V_T)$ and $\underline{t} \in \mathsf{L}_w(T)$, $g.\underline{t} := \Lambda^{\mathrm{even}}(g).\underline{t} := \Lambda^{\mathrm{even}}(g)\underline{t}\Lambda^{\mathrm{even}}(g^{-1})$. Any element $h \in \mathrm{Stab}(w_T)$ can be uniquely written as $h = h_sh_l = h_l'h_s$ where $h_s \in \mathrm{Stab}((A_0)_T)$ and $h_l, h_l' \in \mathsf{L}_w(T)$. Then the relation between h_l and h_l' can be written as $h_l' = h_s.h_l$ or $h_l = h_s^{-1}.h_l'$ where "." stands for the action of Stab_{A_0} on L_w. Thus one has a GL_V-action on $\mathrm{Quad}_V \times_X \mathsf{L}_w$ induced by the diagonal embedding $\mathrm{GL}_V \overset{\Delta}{\hookrightarrow} \mathrm{GL}_V \times_X \mathrm{GL}_V$. Since Id-$w$-*Sp-Azu*$_W$ comes with a canonical action of Stab_w on it, we let GL_V act on Id-w-Sp-Azu$_W$ via Λ^{even}. The following result describes the lack of GL_V-equivariance of the isomorphism Θ.

Theorem 5.4. *Let T be an X-scheme. For T-valued points $g, q, \underline{t}$ respectively of GL_V, Quad_V, and L_w, there exists a unique T-valued point of L_w given by an isomorphism $h_l'(g,q)$ of $\mathcal{O}_T$-algebra bundles $h_l'(g,q) : g.\Theta(q,\underline{t}) \overset{\cong}{\longrightarrow} \Theta(g.q, g.\underline{t})$. Further, $h_l'(g,q)$ satisfies the formula $h_l'(gg',q) = h_l'(g, g'.q)(g.h_l'(g',q))$.*

Therefore Θ satisfies a 'twisted' version of GL_V-equivariance. The next theorem, which was originally motivated by the proof of this 'twisted equivariance', will be of central importance to us for the rest of this section.

Theorem 5.5. *Given a similarity $g : (V_T, q) \cong_l (V_T, q')$ with multiplier $l \in \Gamma(T, \mathcal{O}_T^*)$, let $h(g,l,q,q')$ be the automorphism of (W_T, w_T) given by the composition of the following isomorphisms:*

$$W_T \overset{(\psi_{b(q)})^{-1}(\cong)}{\longrightarrow} C_0(V_T,q) \overset{C_0(g,l)(\cong)}{\longrightarrow} C_0(V_T,q') \overset{\psi_{b(q')}(\cong)}{\longrightarrow} W_T$$

where the algebra bundle isomorphism $C_0(g,l)$ comes from (1)*, 2.7 and the linear isomorphisms $\psi_{b(q)}$ and $\psi_{b(q')}$ come from*(2d)*, 2.2. In terms of actions, this means that $h(g,l,q,q').\theta(q) = \theta(q')$. Write $h(g,l,q,q') \in \mathrm{Stab}(w_T)$ uniquely as a product $h(g,l,q,q') = h_s(g,l,q,q')h_l(g,l,q,q')$ with the first factor in $\mathrm{Stab}_{A_0}(T)$ and the second in $\mathsf{L}_w(T)$ as explained earlier. Then $h_s(g,l,q,q')$ depends only on g and l and not on q or q'. In fact, one has*

$$h_s(g,l,q_1,q_2) = h_s(g,l) := \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l^{-1}\Lambda^2(g) \end{pmatrix} \quad \forall q_1, q_2 \in \mathrm{Quad}(V_T).$$

Proof. We directly compute the $\mathcal{O}_T$-linear automorphism $h(g, l, q, q')$ of W_T as follows. Of course, this automorphism fixes $w^\circ = w_T$. So we only need to compute the images of the three remaining basis elements $\epsilon_1^\circ = e_1^\circ \wedge e_2^\circ$, $\epsilon_2^\circ = e_2^\circ \wedge e_3^\circ$ and $\epsilon_3^\circ = e_3^\circ \wedge e_1^\circ$ in terms of the basis elements w° and ϵ_i°. Let q and $l(g.q) = q'$ respectively correspond to the 6-tuples $(\mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{13}, \mu_{23})$ and $(\nu_1, \nu_2, \nu_3, \nu_{12}, \nu_{13}, \nu_{23})$ in $\Gamma(T, {\mathcal{O}_T}^{\oplus 6})$. (We caution the reader that $l(g.q) \neq (lg).q = l^{-2}(g.q)$!). Let $g \in \mathrm{GL}(V_T) \equiv \mathrm{GL}_3(\Gamma(T, \mathcal{O}_T))$ be given by the matrix (g_{ij}). Observe that the ν are polynomials in the μ and g_{ij}. In the following computation, for the sake of clarity, we denote the product in $C(V_T, q)$ by $*_q$. For example, we have

$$\begin{aligned}
h(g, l, q, q')\epsilon_1 &= \psi_{b(q')} \circ C_0(g, l)\left((\psi_{b(q)})^{-1}(e_1^\circ \wedge e_2^\circ)\right) \\
&= \psi_{b(q')}\left(C_0(g, l)\left(e_1^\circ *_q e_2^\circ\right)\right) \qquad \text{(by (2f), 2.2)} \\
&= \psi_{b(q')}\left(l^{-1}(g(e_1^\circ) *_{q'} g(e_2^\circ))\right) \qquad \text{(by (1), 2.7)} \\
&= l^{-1}\psi_{b(q')}\left((g_{11}e_1^\circ + g_{21}e_2^\circ + g_{31}e_3^\circ) *_{q'} (g_{12}e_1^\circ + g_{22}e_2^\circ + g_{32}e_3^\circ)\right) \\
&= l^{-1}\left(P_1(g, l, q, q')w^\circ + C_{33}(g)\epsilon_1^\circ + C_{13}(g)\epsilon_2^\circ + C_{23}(g)\epsilon_3^\circ\right) \\
&\qquad \text{(by (2f), 2.2)}
\end{aligned}$$

where $P_1(g, l, q, q')$ is a polynomial in the ν and g_{ij} and where $C_{ij}(g)$ represents the cofactor determinant of the element g_{ij} of the matrix $g = (g_{ij})$. Similarly one computes the values of $h(g, l, q, q')\epsilon_2$ and $h(g, l, q, q')\epsilon_3$. Then the matrix of $h(g, l, q, q')$ is given by

$$h(g, l, q, q') = \begin{bmatrix} 1 & l^{-1}P_1(g, l, q, q') & l^{-1}P_2(g, l, q, q') & l^{-1}P_3(g, l, q, q') \\ 0 & l^{-1}C_{33}(g) & l^{-1}C_{31}(g) & l^{-1}C_{32}(g) \\ 0 & l^{-1}C_{13}(g) & l^{-1}C_{11}(g) & l^{-1}C_{12}(g) \\ 0 & l^{-1}C_{23}(g) & l^{-1}C_{21}(g) & l^{-1}C_{22}(g) \end{bmatrix}$$

which implies that

$$h_s(g, l, q, q') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & l^{-1}C_{33}(g) & l^{-1}C_{31}(g) & l^{-1}C_{32}(g) \\ 0 & l^{-1}C_{13}(g) & l^{-1}C_{11}(g) & l^{-1}C_{12}(g) \\ 0 & l^{-1}C_{23}(g) & l^{-1}C_{21}(g) & l^{-1}C_{22}(g) \end{bmatrix} \text{ depends only on } g \text{ and } l.$$

Next define the matrix

$$\widehat{g} = \begin{bmatrix} g_{33} & g_{13} & g_{23} \\ g_{31} & g_{11} & g_{21} \\ g_{32} & g_{12} & g_{22} \end{bmatrix} \text{ so that } h_s(g, l, q, q') = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & l^{-1}C(\widehat{g})^t \end{bmatrix}$$

where $C(\widehat{g})$ is the cofactor matrix of $\widehat{g}$. Now if E_{12} and E_{23} are the matrices defined in 5.3 above, premultiplying by E_{ij} has the effect of interchanging the ith and jth rows, while postmultiplying has a similar effect on the columns. Thus we get $\widehat{g} = E_{12}E_{23}(g^t)E_{23}E_{12}$ from which it follows that $C(\widehat{g})^t = \text{Adjoint}\,(\widehat{g}) = \det\,(\widehat{g}).(\widehat{g})^{-1} = \det\,(g).(E_{12}E_{23}(g^{-1})^t E_{23}E_{12})$, showing that $C(\widehat{g})^t = \Lambda^2(g)$ by 5.3. □ (5.5)

The proof of 5.4 is given in p. 53 of [10] and involves the use of 2.7, 3.11 and 5.5. We stated 5.4 only to emphasise that its proof led us to the crucial 5.5.

5.4. Conclusion of proof of injectivity

We remind the reader that towards the end of Section 4, we had reduced the proof of the injectivity of 3.1 to that of 3.5, and had indicated that it would be enough to prove the latter in the case when V and I are both free—which has been the case in this section so far. Starting with an isomorphism of algebra bundles $\phi : C_0(V, q) \cong C_0(V, q')$ we arrive at the element $g_l^\phi \in \mathrm{GL}(V)$ as defined earlier; to briefly recall this, firstly $g \in \mathrm{GL}(V)$ was defined by the following commuting diagram:

$$
\begin{array}{ccccccc}
(C_0(V,q))^\vee & \xrightarrow[\cong]{((\psi_{b(q)})^\vee)^{-1}} & (\Lambda^{\text{even}}(V))^\vee & \xrightarrow{\text{surjection}} & (\Lambda^2(V))^\vee & \xrightarrow{=} & \\
\phi^\vee \uparrow \cong & & (\phi_{\Lambda^{ev}})^\vee \uparrow \cong & & (\phi_{\Lambda^2})^\vee \uparrow \cong & & \\
(C_0(V,q'))^\vee & \xrightarrow[\cong]{((\psi_{b(q')})^\vee)^{-1}} & (\Lambda^{\text{even}}(V))^\vee & \xleftarrow{\text{inclusion}} & (\Lambda^2(V))^\vee & \xrightarrow{=} &
\end{array}
$$

$$
\begin{array}{ccccc}
(\Lambda^2(V))^\vee & \xrightarrow[\equiv]{(\eta^\vee)^{-1}} & V \otimes (\det(V))^{-1} & \xrightarrow[\equiv]{\otimes \det(V)} & V \\
(\phi_{\Lambda^2})^\vee \uparrow \cong & & (g')^{-1} \uparrow \cong & & g^{-1} \uparrow \cong \\
(\Lambda^2(V))^\vee & \xrightarrow[\equiv]{(\eta^\vee)^{-1}} & V \otimes (\det(V))^{-1} & \xrightarrow[\equiv]{\otimes \det(V)} & V
\end{array}
$$

Secondly, we had defined $g_l^\phi := (l^{-1}\sqrt{\gamma(l)})g$. Our current special choices of bilinear forms $b(q)$ and $b(q')$ that induce q and q' respectively do not affect the generality, as was observed in the proof of 3.3. We shall now show that g_l^ϕ is a similitude from (V, q) to (V, q') with multiplier l and that this similitude induces ϕ, i.e., with the notation of (1), 2.7, that $C_0(g_l^\phi, l) = \phi$. We proceed with the proof which will follow from several lemmas.

Lemma 5.6. *Consider the element $h_s h_l = h'_l h_s = h := \phi_{\Lambda^{ev}} \in (\mathrm{Stab}_w)(X)$ written uniquely as an ordered product in two different ways with $h_l, h'_l \in (\mathsf{L}_w)(X)$ and $h_s \in (\mathrm{Stab}_{A_0})(X)$ as explained previously; let B be the matrix corresponding to ϕ_{Λ^2}, and let the matrices E_{ij} be as defined in 5.3. Then h_s is the matrix with first row* $(1, 0)$ *and the remaining* (3×4)*-submatrix given by* $(0, B)$ *and* $g_l^\phi = (l^{-1}\sqrt{\gamma(l)})E_{23}E_{12}((B)^t)^{-1}E_{12}E_{23}$.

The proof of the above lemma follows from the fact that the matrix of the canonical isomorphism $\eta : \Lambda^2(V) \equiv V^\vee \otimes \det(V)$ is given by $E_{23}E_{12}$, which can be verified by a simple computation.

Lemma 5.7. *We have the formulae* $h_s(\text{Identity}, l^{-1}, q', l^{-1}q') = (1, 0; 0, l \times I_3)$ *and* $h_s(\text{Identity}, l^{-1}, q', l^{-1}q').\theta(q') = \theta(l^{-1}q')$.

The identity map on V is obviously a similarity with multiplier l^{-1} from (V, q') to $(V, l^{-1}q')$. Hence the above lemma follows by taking $T = X$, $g =$ Identity, and the l^{-1} and the q' at hand for the l and the q of 5.5 (caution: the q' there would have to be replaced by $l^{-1}q'$!). This can also be seen directly from the multiplication tables for $\theta(l^{-1}q') = \Theta(l^{-1}q', I_4)$ and $\theta(q') = \Theta(q', I_4)$ written out in 5.1, where we must take $T = X$ and $\underline{t} = I_4$, i.e., $t_i = 0 \forall i$. We observe from the multiplication table that each of the coefficients of ϵ_i for $1 \le i \le 3$ is a single λ, whereas each coefficient of $w = 1 = \epsilon_0$ is a product of two λs, and this observation implies the lemma above. As the reader might have noticed, there are two crucial facts about the identifications in this section; namely, firstly, for any X-scheme T, each of the maps $\psi_{b(q)}$ (for different q) identify $(C_0(V_T, q), 1)$ with the same (W_T, w_T) and secondly, relative to the bases chosen, all these identifying maps have trivial determinant. The latter is also true of the identification η, since it is given by the matrix $E_{23}E_{12}$ (as was noted after 5.6). It therefore follows that $\det(\phi) = \det(\phi_{\Lambda^{ev}}) = \det(\phi_{\Lambda^2}) = \det(g') = \det(g) = \det(B^{-1})$. But we had chosen $l \in \Gamma(X, \mathcal{O}_X^*)$ such that $\gamma(l) := (l^3).\det(\phi_{\Lambda^2}) = l^3\det(B)$. Using these facts along with 5.6 above, a straightforward computation gives the following.

Lemma 5.8. *We have the equality* $\det(g_l^\phi) = \sqrt{\gamma(l)}$ *from which it follows that $B(g_l^\phi) = l \times B$ where $B(g_l^\phi)$ and B were defined in 5.3 and 5.6 respectively. In particular,* $\det^2(g_l^\phi) = \det(\phi_{\Lambda^2})$ *when* $\det(\phi_{\Lambda^2})$ *is itself a square and for the cases* 1 *and* 2 *earlier where we had chosen* $l := 1$.

Lemma 5.9. g_l^ϕ *is a similitude from* (V, q) *to* (V, q') *with multiplier* l.

The hypothesis $\phi : C_0(V, q) \cong C_0(V, q')$ is an algebra isomorphism translates in terms of actions into $h.\theta(q) = \theta(q')$ where $h = \phi_{\Lambda^{ev}} \in (\mathrm{Stab}_w)(X)$. Let $h(g_l^\phi, q) := h(g_l^\phi, 1, q, g_l^\phi.q)$ where $h(g, l, q, q')$ was defined in

5.5 above. Then we have $\Theta(g_l^\phi.q, I_4) := \theta(g_l^\phi.q) = h(g_l^\phi, q).\theta(q) = h(g_l^\phi, q).(h^{-1}.\theta(q'))$, so

$$\begin{aligned}\Theta(g_l^\phi.q, I_4) &= \left(h_l'(g_l^\phi, q)h_s(g_l^\phi, q){h_s}^{-1}(h_l')^{-1}\right).\theta(q')\\ &= \left(h_l'(g_l^\phi, q)\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & B(g_l^\phi)\end{pmatrix}\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & B^{-1}\end{pmatrix}(h_l')^{-1}\right).\theta(q') \\ &\qquad \text{(by 5.5, 5.3 \& 5.6)}\\ &= \left(h_l'(g_l^\phi, q)\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & l\times I_3\end{pmatrix}(h_l')^{-1}\right).\theta(q') \qquad \text{(by 5.8)}\\ &= \left(h_l'(g_l^\phi, q)h_s(\text{Identity}, l^{-1}, q', l^{-1}q')(h_l')^{-1}\right).\theta(q') \qquad \text{(by 5.7)}\\ &= (h_l'(g_l^\phi, q)h_l'').\left(h_s(\text{Identity}, l^{-1}, q', l^{-1}q').\theta(q')\right)\\ &\qquad \text{(since Stab}_w\text{ is a semidirect product)}\\ &= (h_l'(g_l^\phi, q)h_l'').\theta(l^{-1}q') \qquad \text{(by 5.7)}\end{aligned}$$

Thus $\Theta(g_l^\phi.q, I_4) = \Theta(l^{-1}q', (h_l'(g_l^\phi, q)h_l''))$. But since Θ is an isomorphism (3.11), this implies the claim of the above lemma, namely, that $g_l^\phi.q = l^{-1}q'$ and further that $h_l'(g_l^\phi, q) = (h_l'')^{-1}$.

Lemma 5.10. *The similarity $g_l^\phi : (V, q) \cong_l (V, q')$ induces ϕ, i.e., with the notation of* (1), 2.7, $C_0(g_l^\phi, l) = \phi$.

We have $C_0(g_l^\phi, l) = \phi$ iff $h(g_l^\phi, l, q, q') = \psi_{b(q')}C_0(g_l^\phi, l)\psi_{b(q)}^{-1} = \psi_{b(q')}\phi\psi_{b(q)}^{-1} =: \phi_{\Lambda^{ev}} =: h$. Now using successively 5.5, 5.3, 5.6 and 5.8, we get the following sequence of equalities: $h_s(g_l^\phi, l, q, q') = (1, 0; 0, l^{-1} \times \Lambda^2(g_l^\phi)) = (1, 0; 0, l^{-1} \times B(g_l^\phi)) = (1, 0; 0, B) = h_s$. So the present hypotheses in terms of actions give $h(g_l^\phi, l, q, q').\theta(q) = \theta(q') = h.\theta(q)$ implying $h_s(g_l^\phi, l, q, q').(h_l(g_l^\phi, l, q, q').\theta(q)) = h_s.(h_l.\theta(q))$. This implies $\Theta(q, h_l(g_l^\phi, l, q, q')) = \Theta(q, h_l)$. But Θ being an isomorphism (3.11), the last equality implies that $h_l(g_l^\phi, l, q, q') = h_l$ which gives $h(g_l^\phi, l, q, q') = h$.

Lemma 5.11. (1) *For a similarity $g \in$ Sim$[(V, q), (V, q')]$ with multiplier l and the induced isomorphism $C_0(g, l) \in$ Iso$[C_0(V, q), C_0(V, q')]$ given by* (1), 2.7, *we have the equality* $\det((C_0(g, l))_{\Lambda^2}) = l^{-3}\det^2(g)$. *Therefore the map from* Sim$[(V, q), (V, q')]$ *to* Iso$[C_0(V, q), C_0(V, q')]$ *given by $g \mapsto C_0(g, l)$ maps the subsets* Iso$[(V, q), (V, q')]$ *and* S-Iso$[(V, q), (V, q')]$ *respectively into the subsets* Iso$'[C_0(V, q), C_0(V, q')]$ *and* S-Iso$[C_0(V, q), C_0(V, q')]$.

(2) *In the case $q' = q$, if $C_0(g, l)$ is the identity on $C_0(V, q)$, then $g = l^{-1}\det(g) \times \mathrm{Id}_V$, and further if $g \in \mathrm{O}(V, q)$ then $g = \det(g) \times \mathrm{Id}_V$ with* $\det^2(g) = 1$.

By definition, $(C_0(g, l))_{\Lambda^{ev}} = \psi_{b(q')} \circ C_0(g, l) \circ \psi_{b(q)}^{-1}$, and the latter isomorphism is $h(g, l, q, q')$ from 5.5 which further gives a formula for $h_s(g, l, q, q')$. Now using the facts that $\psi_{b(q)}$ and $\psi_{b(q')}$ have trivial determinant (as noted before 5.8) we get assertion (1): $\det((C_0(g, l))_{\Lambda^2}) = \det((C_0(g, l))_{\Lambda^{ev}}) = \det(h(g, l, q, q')) = \det(h_s(g, l, q, q')) = l^{-3}\det^2(g)$. If $q = q'$ and $C_0(g, l)$ is the identity, then the same argument in fact shows that $l^{-1}\Lambda^2(g) = I_3$ and by using the formula in 5.3 for $B(g) = \Lambda^2(g)$, we get $g = l^{-1}\det(g)I_3$; taking determinants in the last equality gives $\det^2(g) = l^3$, so that when $g \in \mathrm{O}(V, q)$, i.e., $l = 1$, $\det^2(g) \in \mu_2(\Gamma(X, \mathcal{O}_X))$ and assertion (2) follows.

Lemma 5.12. *The map from* S-Iso$[(V, q), (V, q')]$ *to* S-Iso$[C_0(V, q), C_0(V, q')]$ *given by $g \mapsto C_0(g, 1, q, q')$ is a bijection.*

Given $\phi \in$ S-Iso$[C_0(V, q), C_0(V, q')]$, by Definition 3.4 we have $\det(\phi_{\Lambda^2}) = 1$, so by 5.8 $\det(g_l^\phi) = \sqrt{\gamma(l)} := 1$ for our choice under Case 1 earlier. Therefore the corresponding element $g_l^\phi \in$ S-Iso$[(V, q), (V, q')]$ and is, according to Lemma 5.10, such that $C_0(g_l^\phi, l = 1, q, q') = \phi$ which gives the surjectivity. As for the injectivity, if $g_1, g_2 : (V, q) \cong_1 (V, q')$ are isometries with determinant 1 such that $C_0(g_1, 1, q, q') = C_0(g_2, 1, q, q')$, then we have $h(g_1, 1, q, q') = h(g_2, 1, q, q')$ so that $h_s(g_1, 1, q, q') = h_s(g_2, 1, q, q')$ whence by 5.5 and 5.3 $B(g_1) = B(g_2) \Rightarrow g_1 = g_2$. □ (3.5 and injectivity of 3.1)

Proof of 3.6. Taking $q' = q$ in 3.5 gives the commutative diagram of groups and homomorphisms as asserted in the statement of the theorem. We continue with the notation above. For $g \in \mathrm{GO}(V, q, I)$ with multiplier l, that the equality $\det(C_0(g, l, I)) = \det((C_0(g, l, I))_{\Lambda^2}) = l^{-3}\det^2(g)$ holds (locally, and hence globally) was shown in (1), 5.11. Assertion (2) of the same Lemma shows the following (locally, and hence globally): if $C_0(g, l, I)$ is the identity on $C_0(V, q, I)$, then $g = l^{-1}\det(g).\mathrm{Id}_V$, and further if $g \in \mathrm{O}(V, q, I)$ then $g = \det(g).\mathrm{Id}_V$ with $\det^2(g) = 1$. The map $\Gamma(X, \mathcal{O}_X^*) \longrightarrow \mathrm{GO}(V, q, I)$ is the natural one given by sending λ to the similarity $\lambda.\mathrm{Id}_V$ with multiplier λ^2. It follows from the formula in (1), 2.7 that $C_0(\lambda.\mathrm{Id}_V, \lambda^2, I) = $ Identity. This gives exactness at $\mathrm{GO}(V, q, I)$ and at $\mathrm{O}(V, q, I)$. We proceed to prove assertion (b). Let $\phi \in \mathrm{Aut}(C_0(V, q, I))$, and consider the self-similarity $g_l^{\phi} = s_{2k+1}^{+}(\phi)$ with multiplier $l = \det(\phi)^{2k+1}$. For the moment, assume that V and I are trivial over X. Fix a global basis $\{e_1, e_2, e_3\}$ for V and set $e_i' = g_l^{\phi}(e_i)$. It follows from Kneser's definition of the half-discriminant d_0—see formula (3.1.4), Chapter IV, [6]—that $d_0(q, \{e_i\}) = d_0(q, \{e_i'\})\det^2(g_l^{\phi})$. Since we have $g_l^{\phi}.q = l^{-1}q$, a simple computation shows that $d_0(q, \{e_i'\}) = l^3 d_0(q, \{e_i\})$. The hypothesis that $q \otimes \kappa(x)$ is semiregular means that the image of the element $d_0(q, \{e_i\}) \in \Gamma(X, \mathcal{O}_X)$ in $\kappa(x)$ is nonzero. Since X is integral, we therefore deduce that $\det^2(g_l^{\phi}) = l^{-3}$. On the other hand, we know that $\det^2(g_l^{\phi})l^{-3} = \det(\phi)$. It follows that $\det^{12k+7}(\phi) = 1 \forall k \in \mathbb{Z}$, which forces $\det(\phi) = 1$. In general, even if V and I are not necessarily trivial, since this equality holds over a covering of X which trivialises both V and I, it also holds over all of X. □ (3.6)

6. Surjectivity of 3.1: Bilinear forms as specialisations

In this section we reduce the proof of 3.9 to 3.10. We prove the latter and using it along with 5.1, deduce 3.12. The surjectivity of 3.1 is also established. We also prove 3.13.

Reduction of proof of 3.9 to the case $I = \mathcal{O}_X$. We adopt the notation introduced just before 3.9. Let T be an X-scheme. Given a bilinear form $b \in \mathrm{Bil}_{(V,I)}(T) = \Gamma(T, \mathrm{Bil}_{(V_T,I_T)})$, consider the linear isomorphism $\psi_b : C_0(V_T, q_b, I_T) \cong \mathcal{O}_T \oplus \Lambda^2(V_T) \otimes (I_T)^{-1} = W_T$ of (2d), 2.2. Let A_b denote the algebra bundle structure on W_T with unit $w_T = 1$ induced via ψ_b from the even Clifford algebra $C_0(V_T, q_b, I_T)$. By definition, $A_b \in \mathrm{Assoc}_{W,w}(T)$ and we get a map of T-valued points $\Upsilon'(T) : \mathrm{Bil}_{(V,I)}(T) \longrightarrow \mathrm{Assoc}_{W,w}(T) : b \mapsto A_b$. This is functorial in T because of (3), 2.2, and hence defines an X-morphism $\Upsilon' : \mathrm{Bil}_{(V,I)} \longrightarrow \mathrm{Assoc}_{W,w}$. The morphism Υ' is GL_V-equivariant due to (2i), 2.2. Notice that the schemes $\mathrm{Bil}_{(V,I)}$, $\mathrm{Bil}_{(V,I)}^{sr}$ and $\mathrm{Assoc}_{W,w}$ are well behaved relative to X with respect to base-change. In fact, so are $\mathrm{Azu}_{W,w}$ and $\mathrm{SpAzu}_{W,w}$, as may be recalled from 2.10 and 2.11. In view of these observations, by taking a trivialisation for I over X, we may reduce to the case when I is trivial. We treat this case next.

Reduction of proof of 3.9 for $I = \mathcal{O}_X$ to 3.10. We first recall the following (see (1), Proposition 3.2.4, Chapter IV [6]): *The even Clifford algebra of a semiregular quadratic form is an Azumaya algebra.* Using this and the definition of Υ', we see that the morphism Υ' when restricted to Bil_V^{sr} factors through $\mathrm{Azu}_{W,w}$ by a morphism Υ^{sr} such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Bil}_V & \xrightarrow{\Upsilon'} & \mathrm{Assoc}_{W,w} \\ \uparrow & & \uparrow \\ \mathrm{Bil}_V^{sr} & \xrightarrow[\Upsilon^{sr}]{} & \mathrm{Azu}_{W,w} \end{array}$$

where the vertical arrows are the canonical open immersions. The above diagram base-changes well in view of (2), 2.10, 2.1 and (3), 2.2. Notice that since the structure morphism $\mathrm{Bil}_V \longrightarrow X$ is an affine morphism, and since the same is true of $\mathrm{Assoc}_{W,w} \longrightarrow X$, it is also true of Υ'. In particular, Υ' is quasi-compact and separated, and therefore has a schematic image (see Section 2.8). The same is true of each of the two vertical arrows and of Υ^{sr} in view of 2.1 and (1) of 2.10. Further, as noted in 2.1, $\mathrm{Bil}_V^{sr} \hookrightarrow \mathrm{Bil}_V$ is schematically dominant and therefore the limiting scheme of the former in the latter is the latter itself. So using the commutative diagram above, the transitivity of the schematic image, and the definition of $\mathrm{SpAzu}_{W,w}$ (assertion (1), 2.11), we see that in order to prove (1), 3.9, it is enough to show that **(*):** Υ^{sr} is schematically dominant and surjective, and Υ' is a closed immersion. We now claim that these are equivalent to **(**):** Υ^{sr} is proper and Υ' is a closed immersion. For, suppose **(**)** holds. To show **(*)**, we only need to show that Υ^{sr} is surjective and schematically dominant. From **(**)** it follows that $\Upsilon^{sr}_K := \Upsilon^{sr} \otimes_X K$ is functorially injective and proper for each algebraically closed field K with an X-morphism $\mathrm{Spec}(K) \longrightarrow X$. That both the K-schemes $\mathrm{Bil}^{sr}_V \otimes_X K$ and $\mathrm{Azu}_{W,w} \otimes_X K$ are integral and smooth of the same dimension follows from the smoothness of relative dimension 9 and geometric irreducibility $/X$ of Bil^{sr}_V (which is obvious), and of $\mathrm{Azu}_{W,w}$ from (3), 2.10. Since Υ^{sr}_K is differentially injective at each closed point, it has to be a smooth morphism by Theorem 17.11.1 of EGA IV [2] and thus has to be an open map. But by **(**)** it is also proper and hence a closed map. Thus Υ^{sr}_K is bijective étale, and hence an isomorphism. This also gives that Υ^{sr} is surjective. Now from Corollary 11.3.11 of EGA IV and from the flatness of Bil^{sr}_V over X, it follows that Υ^{sr} is itself flat, and hence schematically dominant since it is faithfully flat (being already surjective). Therefore **(**)** $\Longrightarrow$ **(*)**.

The property of a morphism being proper is local on the target (see, for example, (f), Corollary 4.8, Chapter IV, [4]) and the same is true of the property of being a closed immersion. Therefore, in verifying **(**)**, we may assume that V is free over X so that $W = \Lambda^{\mathrm{even}}(V)$ is also free over X and w is part of a global basis. We are now in the situation of 3.10. Granting it, we see immediately from the multiplication table that **(**)** holds. For the table shows that the composition of the X-morphisms $\mathrm{Bil}_V \xrightarrow{\Upsilon'} \mathrm{Assoc}_{W,w} \overset{\text{CLOSED}}{\hookrightarrow} \mathrm{Alg}_W$ is a closed immersion, which implies that Υ' is also a closed immersion. Further, the multiplication table also shows that both Υ' and Υ^{sr} satisfy the valuative criterion for properness, and are therefore proper. Thus the conditions **(**)** are verified. So we have reduced the proof of (1), 3.9 to 3.10.

As for statement (2) of 3.9, firstly, the involution Σ of $\mathrm{Assoc}_{W,w}$ defines a unique involution (also denoted by Σ) on the scheme of specialisations $\mathrm{SpAzu}_{W,w}$ (leaving the open subscheme $\mathrm{Azu}_{W,w}$ invariant) because of the defining property of the schematic image involved; for we may verify that an automorphism of a scheme T which leaves an open subscheme U stable will also leave stable the limiting scheme of U in T (of course we assume here that the canonical open immersion $U \hookrightarrow T$ is a quasi-compact open immersion, which ensures the existence of the limiting scheme). Secondly, a glance at the multiplication table of 3.10 keeping in view the definition of opposite algebra shows that the induced $\Sigma \in \mathrm{Aut}_X(\mathrm{Bil}_V)$ does indeed take the T-valued point $B = (b_{ij})$ to $\mathrm{transpose}(-B) = (-b_{ji})$. Finally, assertion (3) of 3.9 is a consequence of (1) taking into account (3), 2.11.

Proof of 3.10. Given $B = (b_{ij}) \in \mathrm{Bil}_V(T)$, by definition, $(\Upsilon'(T))(B) = A_B$ is the algebra structure induced from the linear isomorphism $\psi_B : C_0(V_T, q_B) \cong \Lambda^{\mathrm{even}}(V_T)$ of (2d), 2.2. The stated multiplication table for $A = A_B$ is a consequence of straightforward calculation, keeping in mind (2f), 2.2 and the standard properties of the multiplication in the even Clifford algebra $C_0(V_T, q_B)$. □ (3.10 and 3.9)

Proof of 3.12. The proof follows by comparing the multiplication table relative to Θ as computed in 5.1 with the multiplication table relative to Υ of 3.10 computed above. □ (3.12)

Proofs of assertions in (a), 3.7 and the surjectivity part of 3.1. Let W be the rank 4 vector bundle underlying the specialised algebra A and $w \in \Gamma(X, W)$ be the global section corresponding to 1_A. We choose an affine open covering $\{U_i\}_{i \in \mathcal{I}}$ of X such that $W \mid U_i$ is trivial and $w \mid U_i$ is part of a global basis $\forall i$. Therefore on the one hand, for each $i \in \mathcal{I}$, we can find a linear isomorphism $\zeta_i : \Lambda^{\mathrm{even}}(\mathcal{O}_X^{\oplus 3} \mid U_i) \cong W \mid U_i$ taking $1_{\Lambda^{\mathrm{even}}}$ onto $w \mid U_i$. The $(w \mid U_i)$-unital algebra structure $A \mid U_i$ induces via ζ_i an algebra structure A_i on $\Lambda^{\mathrm{even}}(\mathcal{O}_X^{\oplus 3} \mid U_i)$ (so that ζ_i becomes an algebra isomorphism). Recall that A_i is also a specialised algebra structure by (3), 2.11. Hence by 3.9 applied to $X = U_i$, $V = \mathcal{O}_{U_i}^{\oplus 3}$ and $I = \mathcal{O}_{U_i}$, we can also find an $\mathcal{O}_{U_i}$-valued quadratic form q_i on $\mathcal{O}_X^{\oplus 3} \mid U_i$ induced from a bilinear form b_i so that the algebra structure A_i is precisely the one induced by the linear isomorphism $\psi_{b_i} : C_0(\mathcal{O}_X^{\oplus 3} \mid U_i, q_i) \cong \Lambda^{\mathrm{even}}(\mathcal{O}_X^{\oplus 3} \mid U_i)$ given by (2d) of 2.2. For each pair of indices $(i, j) \in \mathcal{I} \times \mathcal{I}$, let ζ_{ij} and ϕ_{ij} be defined so that the following diagram commutes:

$$\begin{array}{ccccc}
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_i|U_{ij}) & \xrightarrow[\cong]{\psi_{b_i}|U_{ij}} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) & \xrightarrow[\cong]{\zeta_i|U_{ij}} & A|U_{ij} \\
\phi_{ij}\downarrow\cong & & \zeta_{ij}\downarrow\cong & & =\downarrow \\
C_0(\mathcal{O}_X^{\oplus 3}|U_{ij}, q_j|U_{ij}) & \xrightarrow[\psi_{b_j}|U_{ij}]{\cong} & \Lambda^{ev}(\mathcal{O}_X^{\oplus 3}|U_{ij}) & \xrightarrow[\zeta_j|U_{ij}]{\cong} & A|U_{ij}
\end{array}$$

The above diagram means that the algebras A_i glue along $U_{ij} := U_i \cap U_j$ via ζ_{ij} to give (an algebra bundle isomorphic to) A, and in the same vein, the even Clifford algebras $C_0(\mathcal{O}_X^{\oplus 3} \mid U_i, q_i)$ glue along the U_{ij} via ϕ_{ij} to give A as well.

Now consider the similarity $g_{l_{ij}}^{\phi_{ij}} = s_{-1}^{+}(\phi_{ij}) : (\mathcal{O}_X^{\oplus 3} \mid U_{ij}, q_i \mid U_{ij}) \cong_{l_{ij}} (\mathcal{O}_X^{\oplus 3} \mid U_{ij}, q_j \mid U_{ij})$ with multiplier $l_{ij} := \det(\phi_{ij})^{-1}$ given by (c), 3.5. Since s_{-1}^{+} is multiplicative, and since the ϕ_{ij} satisfy the cocycle condition, it follows that the $s_{-1}^{+}(\phi_{ij})$ also satisfy the cocycle condition and therefore glue the $\mathcal{O}_X^{\oplus 3} \mid U_i$ along the U_{ij} to give a rank 3 vector bundle V on X. While the q_i do not glue to give an $\mathcal{O}_X$-valued quadratic form on V, the facts that the multipliers $\{l_{ij}\}$ form a cocycle for $I := \det^{-1}(A)$ and that $s_{(-1)}^{+}$ is a section together imply, taking into account the uniqueness in (1), 2.7, that actually the q_i glue to give an I-valued quadratic form q on V and that $C_0(V, q, I) \cong A$. We shall now revert to the notation of Section 5. By 5.5, we have $h_s(g_{l_{ij}}^{\phi_{ij}}, l_{ij}, q_i \mid U_{ij}, q_j \mid U_{ij}) = (1, 0; 0, l_{ij}^{-1} \Lambda^2(g_{l_{ij}}^{\phi_{ij}}))$ which means that $(\phi_{ij})_{\Lambda^2} = \det(\phi_{ij}) \times \Lambda^2(g_{l_{ij}}^{\phi_{ij}})$. This immediately implies part (1) of assertion (a) of 3.7, from which parts (2)—(4) can be deduced using the standard properties of the determinant and the perfect pairings of suitable exterior powers of a bundle.

Proofs of assertions in (b), 3.7. We first prove (b1). Let A be a given specialisation, and let $A \cong C_0(V, q, I)$ as in part (a) of 3.7 with $I = \det^{-1}(A)$. By the injectivity part of 3.1, we have $C_0(V, q, I) \cong A \cong C_0(V', q', \mathcal{O}_X)$ iff there exists a twisted discriminant bundle (L, h, J) and an isomorphism $(V, q, I) \cong (V', q', \mathcal{O}_X) \otimes (L, h, J)$. The latter implies that $I \cong J \cong L^2$ and hence $\det(A) \in 2.\mathrm{Pic}(X)$. On the other hand, if this last condition holds, we could take for L a square root of $J := I^{-1}$, along with an isomorphism $h : L^2 \cong J$ and we would have by Proposition 2.8 an algebra isomorphism $\gamma_{(L,h,J)} : C_0(V \otimes L, q \otimes h, \mathcal{O}_X) \equiv C_0((V, q, I) \otimes (L, h, J)) \cong C_0(V, q, I) \cong A$. For the proof of (b2), suppose that the line subbundle $\mathcal{O}_X.1_A \hookrightarrow A$ is a direct summand of A. We may choose a splitting $A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A)$. Using assertion (1) of (a), 3.7, we see that there exists a rank 3 vector bundle V on X such that $A \cong \mathcal{O}_X.1_A \oplus (A/\mathcal{O}_X.1_A) \cong \mathcal{O}_X.1_A \oplus (\Lambda^2(V) \otimes I^{-1}) \cong \mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1} =: W$ where $I := \det^{-1}(A)$ and the last isomorphism is chosen so as to map $\mathcal{O}_X.1_A$ isomorphically onto $\mathcal{O}_X.1$. Therefore if $(W, w) := (\mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1}, 1)$, then by the above identification A induces an element of $\mathrm{SpAzu}_{W,w}(X)$, and since $\Upsilon : \mathrm{Bil}_{(V,I)} \cong \mathrm{SpAzu}_{W,w}$ is an X-isomorphism by (1), 3.9, it follows that there exists an I-valued global quadratic form $q = q_b$ induced from an I-valued global bilinear form b on V such that the algebra structure $\Upsilon(b) \cong A$. (We recall that $\Upsilon(b)$ is the algebra structure induced from the linear isomorphism $\psi_b : C_0(V, q = q_b, I) \cong \mathcal{O}_X.1 \oplus \Lambda^2(V) \otimes I^{-1} = W$ of (2d), 2.2, which preserves 1 by (2a) of the same Theorem.) The proof of (b3) follows from a combination of those of (b1) and (b2). □ (3.7 and surjectivity of 3.1)

Proof of 3.13. Fix an S-basis $\{e_1, e_2, e_3\}$ for V, and with respect to this basis, let q^1 denote the quadratic form given by $(x_1e_1 + x_2e_2 + x_3e_3) \longmapsto x_1x_2 + x_3^2$. It is easy to see that this quadratic form is semiregular. We show that any semiregular quadratic form q can be moved to q^1, i.e., that $\exists g \in \mathrm{GL}(V)$ such that $g \cdot q = q^1$. By Proposition 3.17, Chapter IV, [6], there exists a basis $\{e_1', e_2', e_3'\}$ for V such that q restricted to the submodule generated by e_1' and e_2' is *regular* and further such that $q(e_3') \in S^*$, $b_q(e_1', e_2') = 1$ and $b_q(e_1', e_3') = 0 = b_q(e_2', e_3')$. Let $g' \in \mathrm{GL}(V)$ be the automorphism that maps e_i' onto the e_i for each i and consider the quadratic form $q' := g' \cdot q$. Then by definition of the $\mathrm{GL}(V)$-action on the set $\mathrm{Quad}(V)$ of quadratic S-forms on V we have $q'(e_3) \in S^*$, $b_{q'}(e_1, e_2) = 1$ and $b_{q'}(e_1, e_3) = 0 = b_{q'}(e_2, e_3)$. So if we assume that $q'(e_i) = \lambda_i (\Leftrightarrow q(e_i') = \lambda_i)$, then we would have $q'(x_1e_1 + x_2e_2 + x_3e_3) = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2 + x_1x_2\ \forall x_i \in S$. Thus it is enough to show that q' can be moved to q^1. We look for an invertible matrix $g'' = (u_{ij}) \in \mathrm{GL}(V)$ such that $g'' \cdot q' = q^1$. Writing this condition equivalently as $q' = (g'')^{-1} \cdot q^1$ and comparing the polynomials in the x_i gives six equations in terms of the u_{ij} and the λ_i which are to be satisfied. We choose the u_{ij} as follows. First set $u_{11} = u_{22} = 0$ and let $u_{12} \in S^*$ be a free parameter. Since every element of S has square roots in S, it makes sense to choose $u_{31} = \pm\sqrt{\lambda_1}$ and $u_{32} = \pm\sqrt{\lambda_2}$. We let $\alpha = 1 + 2u_{31}u_{32}$, $\beta = 1 - 2u_{31}u_{32}$ and $u_{21} = \beta/u_{12}$. Since $q' = g \cdot q$ and since q is semiregular, q' is also semiregular. Its half-discriminant relative to the present basis of V is (remembering that $\lambda_3 \in S^*$) $d_{q'}(e_1, e_2, e_3) = \lambda_3.(4\lambda_1\lambda_2 - 1) \in S^*$. This implies that $\alpha\beta = 1 - 4\lambda_1\lambda_2 \in S^* \Longrightarrow \alpha, \beta \in S^*$. Therefore it makes sense to define $u_{33} = \pm\sqrt{(\beta\lambda_3)/\alpha}$, $u_{13} = -2u_{31}u_{33}u_{12}/\beta$ and $u_{23} = -2u_{32}u_{33}/u_{12}$. Note that $u_{33} \in S^*$. A computation shows that the determinant of the matrix $g'' = (u_{ij})$ defined above is $-u_{33}\alpha \in S^*$ and hence g'' is invertible. It is also easily checked that $g'' \cdot q' = q^1$. □ (3.13)

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