

# INTERSECTION POINCARÉ POLYNOMIAL FOR NAGARAJ-SESHADRI MODULI SPACE

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ABSTRACT. We compute Betti numbers of both the components of the moduli space of rank 2 semi-stable torsion-free sheaves with fixed determinant over a reducible nodal curve with two smooth components intersecting at a node. We also compute the intersection Betti numbers of the moduli space.

## 1. INTRODUCTION

Let  $Y$  be a smooth irreducible projective curve over  $\mathbb{C}$  degenerating to a reducible nodal curve  $X$  with two smooth components meeting transversally at one point and  $L$  be a line bundle of odd degree on  $Y$  specializing to a line bundle  $\xi$  on  $X$ . Then the moduli space  $M_L$  of stable vector bundles of rank two and determinant  $L$  specializes to the moduli space  $M_\xi$  of stable torsion-free sheaves on  $X$  of rank two and determinant  $\xi$  [8, Remark 7.1, 7.2]. It is known that  $M_L$  is a smooth irreducible projective variety. The 'limiting' moduli space  $M_\xi$ , though, is neither smooth nor irreducible but has two smooth components  $M_{12}$  and  $M_{21}$ , intersecting transversally along a smooth divisor  $N$  [11, Proposition 6.5].

The topological invariants of  $M_L$  have been quite well studied and its Poincaré Polynomial has been computed by many authors using different techniques [1], [5], [6]. It is natural to investigate similar topological invariants of  $M_\xi$ . Recently, it has been shown that the irreducible components  $M_{12}$  and  $M_{21}$  of  $M_\xi$  are rational varieties (cf. [4]). In particular, they are unirational and hence by a result of Serre [9] they are simply connected. The first three Betti numbers of each component have been computed in [11, Theorem 3.9] by using topological techniques. In this paper, we find all the Betti numbers of both the components (see §4). We follow Harder–Narasimhan method [6] which is based on Weil conjectures that tell us that for any smooth complex projective variety  $V$  defined over a number ring, the number of  $\mathbb{F}_{q^n}$  rational points of the corresponding variety  $V$  over  $\mathbb{F}_{q^n}$  determines the Betti numbers of  $V$ . In order to compute rational points of  $M_{12}$  and  $M_{21}$ , we use their description in terms of triples [8] and the stratification obtained in [3]. The idea of proof is based on [3], where the authors compute the Betti numbers for Seshadri desingularization of the moduli space of rank 2 semi-stable vector bundles with trivial determinant over a smooth complex projective curve.

Since the components  $M_{12}$  and  $M_{21}$  of  $M_\xi$  intersect transversally along a smooth divisor  $N$ , the normalization  $\widetilde{M}_\xi$  is isomorphic to  $M_{12} \sqcup M_{21}$ . Using this fact and a result of Goresky-Macpherson [12, §4.2] we compute the intersection Poincaré polynomial of  $M_\xi$  (see Theorem 5.1).

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## 2. PRELIMINARIES

**2.1. A Brief Description of the Nagaraj-Seshadri Moduli Space.** In this section, we shall briefly recall some of the results proved in [8] which will be useful in later sections. Let  $X$  be a reducible projective nodal curve defined over an algebraically closed field with two smooth irreducible components  $X_1$  and  $X_2$  meeting at the nodal point  $p$ . Any torsion-free sheaf  $E$  on  $X$  can be identified with a triple  $(E_1, E_2, \vec{T})$  or  $(E'_1, E'_2, \overleftarrow{S})$  where  $E_i$ 's are locally free sheaves over  $X_i$ 's and  $\vec{T}$  and  $\overleftarrow{S}$  are linear maps from  $E_1(p)$  to  $E_2(p)$  and  $E'_2(p)$  to  $E'_1(p)$  respectively.

**Definition 2.1.** (i) A morphism between two triples  $(E_1, E_2, \vec{T})$  and  $(F_1, F_2, \vec{U})$  is a pair  $(h_1, h_2)$ , where  $h_i : E_i \rightarrow F_i$  is a map of  $\mathcal{O}_{X_i}$ -modules for each  $i$ , such that the following diagram commutes -

$$\begin{array}{ccc} E_1(p) & \xrightarrow{T} & E_2(p) \\ \downarrow h_1 & & \downarrow h_2 \\ F_1(p) & \xrightarrow{U} & F_2(p). \end{array}$$

(ii) If  $h_i$ 's are isomorphic as  $\mathcal{O}_{X_i}$ -modules, we say  $(E_1, E_2, \vec{T})$  and  $(F_1, F_2, \vec{U})$  are isomorphic as triples.

We denote the isomorphism class of triple  $(E_1, E_2, \vec{T})$  by  $[E_1, E_2, \vec{T}]$ . Similar definitions hold for triples in other direction. In fact in [8, Lemma 2.3], an equivalence is shown between the category of torsion-free sheaves and the category of triples.

Let  $E$  be a torsion-free sheaf on  $X$  identified by the triple  $(E_1, E_2, \vec{T})$  as well as  $(E'_1, E'_2, \overleftarrow{S})$ . Then we have the following equality of Euler characteristics between them (see [8], Remark 2.11)-

$$\chi(E) = \chi(E_1, E_2, \vec{T}) = \chi(E'_1, E'_2, \overleftarrow{S}), \quad (2.1)$$

where

$$\chi(E_1, E_2, \vec{T}) := \chi(E_1) + \chi(E_2) - rk(E_2), \quad (2.2)$$

and

$$\chi(E'_1, E'_2, \overleftarrow{S}) := \chi(E'_1) + \chi(E'_2) - rk(E'_1). \quad (2.3)$$

Let  $\mathbf{a} = (a_1, a_2)$  be a polarization on  $X$  with  $a_i > 0$  rational numbers and  $a_1 + a_2 = 1$ . For every non-zero triple  $(E_1, E_2, \vec{T})$ , we define

$$\mu((E_1, E_2, \vec{T})) = \frac{\chi(E_1, E_2, \vec{T})}{a_1 rk(E_1) + a_2 rk(E_2)}.$$

**Definition 2.2.** Let  $(E_1, E_2, \vec{T})$  be a triple. We say that the triple  $(E_1, E_2, \vec{T})$  is stable (resp. semi-stable) if for every proper sub-triple  $(G_1, G_2, \vec{U})$ ,  $\mu((G_1, G_2, \vec{U})) < \mu((E_1, E_2, \vec{T}))$  (resp.  $\leq$ ).

(Similar definitions can be given for the triple  $(E'_1, E'_2, \overleftarrow{S})$ ).

When  $\chi$  is odd and  $a_1\chi$  is not an integer the moduli space  $M(2, \mathbf{a}, \chi)$  of stable torsion-free sheaves on  $X$  with Euler characteristic  $\chi$  is a reduced, connected projective variety with the two smooth irreducible components  $M_{12}$  and  $M_{21}$  intersecting transversally on a smooth projective variety  $N$  [8, Theorem 4.1]. The irreducible components  $M_{12}$  and  $M_{21}$  have the following description in terms of triples:

The first component  $M_{12}$  is a smooth projective variety which is a fine moduli space consisting of isomorphism classes of stable triples  $[E_1, E_2, \overrightarrow{T}]$  such that  $E_i$ 's are rank 2 vector bundles over  $X_i$ 's, and  $\overrightarrow{T} : E_1(p) \rightarrow E_2(p)$  is a nonzero linear map such that

$$a_1\chi < \chi(E_1) < a_1\chi + 1, \quad a_2\chi + 1 < \chi(E_2) < a_2\chi + 2, \quad (2.4)$$

The second component  $M_{21}$  also has a similar description in terms of triples. It is a smooth projective variety which is a fine moduli space consisting of isomorphism classes of stable triples  $[E'_1, E'_2, \overleftarrow{S}]$  such that  $E'_i$ 's are rank 2 vector bundles over  $X_i$ 's, and  $\overleftarrow{S} : E'_2(p) \rightarrow E'_1(p)$  is a nonzero linear map such that

$$a_1\chi + 1 < \chi(E'_1) < a_1\chi + 2, \quad a_2\chi < \chi(E'_2) < a_2\chi + 1. \quad (2.5)$$

Form now on, we fix an odd integer  $\chi$  and a polarization  $\mathbf{a} = (a_1, a_2)$  such that  $a_1\chi$  is not an integer.

**Lemma 2.3.** *Let  $E_1$  be a rank 2 semi-stable vector bundle on  $X_1$  and  $E_2$  be a rank 2 semi-stable vector bundle on  $X_2$  such that at least one of them is stable and  $\chi(E_1), \chi(E_2)$  satisfy inequalities (2.4). Then for any nonzero linear map  $\overrightarrow{T} : E_1(p) \rightarrow E_2(p)$ , the triple  $(E_1, E_2, \overrightarrow{T})$  is stable.*

*Proof.* One can easily prove this lemma by a case-by-case analysis.  $\square$

**Remark 2.4.** *The converse of the above Lemma is also true, i.e., if  $(E_1, E_2, \overrightarrow{T})$  is a rank two, semi-stable triple, such that  $\chi(E_1)$  and  $\chi(E_2)$  satisfy inequalities (2.4), then  $E_1$  and  $E_2$  are semi-stable vector bundles on  $X_1$  and  $X_2$  respectively [8, Theorem 5.1]. Similar results hold for triples in the other direction also.*

The intersection subvariety  $N$  is given by

$$\{[E_1, E_2, \overrightarrow{T}] \in M_{12} \mid \text{rk}(\overrightarrow{T}) = 1\},$$

which can be identified with

$$\{[E'_1, E'_2, \overleftarrow{S}] \in M_{21} \mid \text{rk}(\overleftarrow{S}) = 1\}.$$

Further  $N$  can be described as a product  $N_1 \times N_2$ , where  $N_1$  and  $N_2$  are rank 2 parabolic moduli spaces over  $X_1$  and  $X_2$  with Euler characteristics  $\chi_1$  and  $\chi_2$  respectively satisfying the inequalities (2.4), having full flag and parabolic weights  $(a_1/2, a_2/2)$ . In characteristic 0, this was proved in [8, Theorem 6.1]. Here we will give a characteristic-free proof.

**Proposition 2.5.**  $N \simeq N_1 \times N_2$

*Proof.* Let  $\mathbf{F}_i$  be the universal quotient sheaf on  $X_i \times R_i$  parametrizing vector bundles with Euler characteristic  $\chi_i$ , where  $R_i$  is the quot scheme parametrizing locally free quotients satisfying certain cohomological conditions (see [8, Theorem 5.3] for details). Let  $F_i =$

$p_i^*(\mathbf{F}_i |_{\{p\} \times R_i})$  on  $R_1 \times R_2$ , where  $p_i$ 's are projections from  $R_1 \times R_2$  to  $R_i$ 's. Consider the projective bundle  $\mathbb{P}(F_1^*)$ . The fibre over any closed point  $(q_1, q_2)$  parametrizes all surjective 1-dimensional quotients  $\{F_1(q_1) \rightarrow W(q_1)\}$ . Similarly, for the projective bundle  $\mathbb{P}(F_2)$ , the fiber over any closed point  $(q_1, q_2)$  parametrizes all 1 dimensional subspaces  $\{V(q_2) \subset F_2(q_2)\}$ . Let  $\mathbb{P}(F_1^*)^{ss}$  (resp.  $\mathbb{P}(F_2)^{ss}$ ) denote the set of closed points of  $\mathbb{P}(F_1^*)$  (resp.  $\mathbb{P}(F_2)$ ) such that corresponding parabolic bundle on  $X_1$  (resp. on  $X_2$ ) is parabolic semi-stable (= parabolic stable)<sup>1</sup> with respect to parabolic weights  $(\frac{a_1}{2}, \frac{a_2}{2})$ . Let  $((F_1(q_1) \rightarrow W(q_1), V(q_2) \subset F_2(q_2))$  be a point in  $\mathbb{P}(F_1^*)^{ss} \times \mathbb{P}(F_2)^{ss}$  and  $\lambda : W(q_1) \simeq V(q_2)$  be an isomorphism. By composing  $\lambda$  with the quotient  $F_1(q_1) \rightarrow W(q_1)$ , we get a rank 1 linear map  $T_\lambda$  from  $F_1(q_1) \rightarrow F_2(q_2)$ . This gives rise to a triple  $(\mathbf{F}_{1,q_1}, \mathbf{F}_{2,q_2}, \vec{T}_\lambda)$ . From the choice of parabolic weights one can easily see that  $\mathbf{F}_{1,q_1}$  and  $\mathbf{F}_{2,q_2}$  are semi-stable on  $X_1$  and  $X_2$  respectively and at least one of them is stable. So by Lemma 2.3, the triple  $(\mathbf{F}_{1,q_1}, \mathbf{F}_{2,q_2}, \vec{T}_\lambda)$  is semi-stable. Since there is nothing canonical in  $\lambda$ , we naturally get an element in  $\mathbb{P}(\mathcal{H}om(F_1, F_2))^{ss}$ . Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(F_1^*)^{ss} \times_{R_1 \times R_2} \mathbb{P}(F_2)^{ss} & \xrightarrow{C} & \mathbb{P}(\mathcal{H}om(F_1, F_2))^{ss} \\ & \searrow & \downarrow \\ & & R_1^{ss} \times R_2^{ss}. \end{array}$$

There is a natural action of the group  $G = PGL(k_1) \times PGL(k_2)$  (for a suitable choice of  $k_1$  and  $k_2$ ) on both the domain and codomain of  $C$  and since  $C$  is  $G$ -equivariant, the morphism  $C$  descends to respective quotients. So we have the following commutative diagram

$$\begin{array}{ccc} N_1 \times N_2 & \xrightarrow{\bar{C}} & M_{12} \\ & \searrow & \downarrow \\ & & M_1 \times M_2, \end{array}$$

where  $M_1$  (resp.  $M_2$ ) is the moduli space of rank 2 stable (resp. semi-stable) vector bundles with Euler characteristic  $\chi_1$  (resp.  $\chi_2$ ) over  $X_1$  (resp.  $X_2$ ). It is clear that the image of  $\bar{C}$  is inside  $N$ . Further there is a natural map  $D : N \rightarrow N_1 \times N_2$  which sends any semi-stable triple  $(E_1, E_2, \vec{T})$  to  $((E_1, E_1(p) \rightarrow E_1(p)/\text{Ker}(\vec{T})), (E_2, 0 \subset \text{Image}(\vec{T}) \subset E_2(p)))$  (see also [8, Theorem 6.1]). One can easily check that  $D \circ \bar{C} = \bar{C} \circ D = \text{id}$ .  $\square$

### The notion of fixed determinant moduli space :

Let  $E$  be an element of  $M(2, \mathbf{a}, \chi)$  and  $(E_1, E_2, \vec{T})$ ,  $(E'_1, E'_2, \vec{S})$  be the triples corresponding to  $E$ . Let  $\chi_i := \chi(E_i)$  and  $\chi'_i := \chi(E'_i)$ , for  $i = 1, 2$ . If  $\chi_i$ 's satisfy the inequalities in (2.4), then  $E$  belongs to  $M_{12}$  and if  $\chi'_i$ 's satisfy the inequalities in (2.5), then  $E$  belongs to  $M_{21}$ . Let  $J^{d_i}(X_i)$  be the Jacobian of line bundles of degree  $d_i$  on  $X_i$ , where  $d_i = \chi_i - 2(1 - g_i)$ , for  $i = 1, 2$ . By [8, Proposition 7.1], there is a well-defined surjective morphism

$$\det : M(2, \mathbf{a}, \chi \neq 0) \longrightarrow J^{d_1}(X_1) \times J^{d_2}(X_2),$$

<sup>1</sup>Since we are in rank 2 and there is only one parabolic point the parabolic stability coincides with parabolic semi-stability

given by

$$\begin{aligned} E &\mapsto (\Lambda^2(E_1), \Lambda^2(E_2)) \quad \text{if } E \in M_{12}, \text{ and} \\ E &\mapsto \Psi((\Lambda^2(E'_1)), \Lambda^2(E'_2)) \quad \text{if } E \in M_{21}, \end{aligned}$$

where

$$\Psi : J^{d_1+1}(X_1) \times J^{d_2-1}(X_2) \rightarrow J^{d_1}(X_1) \times J^{d_2}(X_2)$$

is an isomorphism defined by

$$(L_1, L_2) \mapsto (L_1 \otimes \mathcal{O}_{X_1}(-p), L_2 \otimes \mathcal{O}_{X_2}(p)).$$

Now let us fix  $L_1 \in J^{d_1}(X_1)$  and  $L_2 \in J^{d_2}(X_2)$ , we write  $\xi = (L_1, L_2)$ . Then the fixed determinant moduli space  $M(2, \mathbf{a}, \chi, \xi)$  is by definition  $\det^{-1}(\xi)$ . By [8, Proposition 7.2], it is reduced. Let  $\det_{12}$  (resp.  $\det_{21}$ ) be the morphism  $\det|_{M_{12}}$  (resp.  $\det|_{M_{21}}$ ). For notational convenience we write  $M_{12}(\xi)$  (resp.  $M_{21}(\xi)$ ) for  $\det_{12}^{-1}(\xi)$  (resp.  $\det_{21}^{-1}(\xi)$ ). Then we have

$$M_{12}(\xi) = \{[E_1, E_2, \vec{T}] \in M_{12} \mid \Lambda^2(E_1) \simeq L_1, \Lambda^2(E_2) \simeq L_2\},$$

and

$$M_{21}(\xi) = \{[E'_1, E'_2, \overleftarrow{S}] \in M_{21} \mid \Lambda^2(E'_1) \simeq L_1 \otimes \mathcal{O}_{X_1}(p), \Lambda^2(E'_2) \simeq L_2 \otimes \mathcal{O}_{X_2}(-p)\}.$$

By [11, Proposition 6.5], the fixed determinant moduli space is a connected, projective scheme with exactly two smooth irreducible components  $M_{12}(\xi)$  and  $M_{21}(\xi)$ , meeting transversally along the smooth divisor  $N(\xi) = M_{12}(\xi) \cap N$  (which is identified with  $M_{21}(\xi) \cap N$ ). Since  $\chi$  is assumed to be an odd integer, and  $\chi = \chi_1 + \chi_2 - 2$ , we can conclude that either  $\chi_1$  is odd or  $\chi_2$  is odd and not both (Same argument applies to  $\chi'_1$  and  $\chi'_2$  also).

**2.2. The Intersection Homology.** Suppose  $Y$  is a compact oriented manifold such that  $\dim_{\mathbb{R}} Y = n$ . Then Poincaré duality states that there exists a perfect pairing

$$H_i(Y) \otimes H_{n-i}(Y) \rightarrow \mathbb{C}, \quad (2.6)$$

for each  $0 \leq i \leq n$ , and in particular

$$\dim_{\mathbb{C}} H_i(Y) = \dim_{\mathbb{C}} H_{n-i}(Y).$$

It is known that Poincaré duality does not hold for the cohomology of singular spaces (cf. [14, pp. 3-4] for examples). To remedy this, a generalization to singular spaces of the Poincaré-Lefschetz theory of intersections of homology cycles on manifolds, was introduced by Goresky and Macpherson in [12] and [13]. Their main objective was to study the intersection of cycles on an  $n$  dimensional oriented pseudomanifold.

**Definition 2.6** ([13] §1.1). *A topological pseudomanifold of dimension  $n$  is an  $n$ -dimensional stratified paracompact Hausdorff topological space  $Y$  which admits a stratification*

$$Y = Y_n \supset Y_{n-1} \supset Y_{n-2} \supset \cdots \supset Y_1 \supset Y_0 \quad (2.7)$$

such that  $Y_{n-1} = Y_{n-2}$  and  $Y - Y_{n-1}$  is dense in  $Y$ .

For a topological pseudomanifold of dimension  $n$ , they introduced a new class of groups  $IH_*^{\overline{p}}(Y)$  called the intersection homology groups. These groups depend on the choice of a perversity  $\overline{p}$  which is a function from the set  $\{2, \dots, n\}$  to  $\mathbb{N}$  such that  $\overline{p}(2) = 0$  and  $\overline{p}(i+1) = \overline{p}(i)$  or  $\overline{p}(i) + 1$  (cf. [14, Definition 4.2.3]).

Further, if  $\bar{p}, \bar{q}$ , and  $\bar{r}$  are perversities such that  $\bar{p} + \bar{q} \leq \bar{r}$ , then there is a unique intersection pairing

$$\cap : IH_i^{\bar{p}}(Y) \times IH_j^{\bar{q}}(Y) \rightarrow IH_{i+j-n}^{\bar{r}}(Y) \quad (2.8)$$

such that  $[C \cap D] = [C] \cap [D]$ , for every dimensionally transverse pair of cycles  $C \in IH_i^{\bar{p}}(Y)$  and  $D \in IH_j^{\bar{q}}(Y)$  (cf. [12, §2.1 Definition and §2.3 Theorem 1]).

Moreover, if  $i$  and  $j$  are complementary dimensions ( $i + j = n$ ),  $\bar{t}$  is the top perversity (which is defined to be the function  $i \mapsto (i - 2)$  for  $2 \leq i \leq n$ ) and if  $\bar{p}$  and  $\bar{q}$  are complementary perversities ( $\bar{p} + \bar{q} = \bar{t}$ ) then the pairing

$$IH_i^{\bar{p}}(Y) \times IH_j^{\bar{q}}(Y) \xrightarrow{\cap} IH_0^{\bar{t}}(Y) \xrightarrow{\epsilon} \mathbb{Z}$$

is a perfect pairing when tensored with rationals  $\mathbb{Q}$ , where

$$\epsilon : IH_0^{\bar{t}}(Y) \rightarrow \mathbb{Z}$$

is the "augmentation" which counts points with multiplicities (cf. [12, §3.3 Theorem]).

It is also known that these intersection homology groups  $IH_*^{\bar{p}}(Y)$  of a topological pseudomanifold  $Y$  are topological invariants. This fact is proved using the sheaf theoretic construction of the intersection complex  $\mathbf{IC}$  (cf. [13, §3 and §4]).

Another important property of intersection homology groups is their behavior under normalization. The following theorems will be very useful in §5 where we compute the intersection homology groups of the fixed determinant moduli space  $M(2, \mathbf{a}, \chi, \xi)$ .

**Theorem 2.7** ([12] §4.2). *If  $Y$  is a pseudo-manifold with normalization  $\pi : \tilde{Y} \rightarrow Y$ , then the map  $\pi$  induces isomorphisms  $IH_i^{\bar{p}}(\tilde{Y}) \simeq IH_i^{\bar{p}}(Y)$  for any perversity  $\bar{p}$ .*

**Theorem 2.8** ([12] §4.3). *Let  $Y$  be a topologically pseudomanifold of dimension  $n$ . If  $Y$  is topologically normal then there are canonical isomorphisms  $IH_i^{\bar{t}}(Y) \simeq H_i(Y)$  and  $IH_i^{\bar{0}}(Y) \simeq H^{n-i}(Y)$ , where  $\bar{t}$  is the top perversity,  $\bar{0}$  is the zero perversity,  $H_i(Y)$  and  $H^{n-i}(Y)$  stand for the usual singular homology and cohomology groups of  $Y$ .*

An important example of a pseudomanifold (which we shall be concerned with in this article) is that of a complex quasi-projective variety of pure dimension  $n$ . It is a theorem of Whitney ([15, Theorem 19.2]) that any quasi-projective variety  $Y$  of pure dimension  $n$  has a Whitney stratification (cf. [14, Definition 4.10.2]). It is this stratification of  $Y$  which makes it into a topological pseudomanifold ([16, IV §2]).

**Definition 2.9.** *Let  $Y$  be a complex quasi-projective variety of pure dimension  $n$ . We define the intersection Poincaré polynomial of  $Y$  with respect to the perversity  $\bar{p}$  to be the polynomial  $\sum_{j=0}^{2n} \beta_j^{\bar{p}} t^j$ , where  $\beta_j^{\bar{p}} = \dim_{\mathbb{R}}(IH_j^{\bar{p}}(Y))$ .*

**Remark 2.10.** *If  $Y$  is a complex quasi-projective variety then by [12, §4.1 pp. 151], it is known that the algebraic normalization  $\widetilde{Y}_{alg}$  of  $Y$  is homeomorphic to the topological normalization  $\tilde{Y}$  of  $Y$ . So by the Theorem 2.7, in order to compute the intersection homology groups for such a  $Y$ , it is enough to compute it for  $\widetilde{Y}_{alg}$ .*

**2.3. Notation.** From now on, we will drop some extra baggages and proceed with the following notations -

- $X$  denotes the projective reducible nodal curve with two smooth components  $X_1$  and  $X_2$  meeting at the nodal point  $p$ .
- $\xi = (L_1, \mathcal{O}_{X_2})$ , where  $L_1$  is an invertible sheaf on  $X_1$  of degree 1.
- By  $M$  we mean the moduli space of rank 2, Euler-characteristic  $3 - 2g$ , stable torsion free sheaves over  $X$  with the fixed determinant  $\xi$ .
- The moduli space  $M_{12} = \{[E_1, E_2, \overrightarrow{T}] : [E_1, E_2, \overrightarrow{T}] \text{ is an isomorphism class of stable triples with the respective Euler characteristics satisfying inequalities (2.4), } \text{rk}(E_i) = 2, \Lambda^2(E_1) \simeq L_1, \Lambda^2(E_2) \simeq \mathcal{O}_{X_2}, \text{deg}(L_1) = 1\}$ .
- The moduli space  $M_{21} = \{[E'_1, E'_2, \overleftarrow{S}] : [E'_1, E'_2, \overleftarrow{S}] \text{ is an isomorphism class of stable triples with the respective Euler characteristics satisfying inequalities (2.5), } \text{rk}(E'_i) = 2, \Lambda^2(E'_1) \simeq L_1 \otimes \mathcal{O}_{X_1}(p), \Lambda^2(E'_2) \simeq \mathcal{O}_{X_2}(-p)\}$ .
- Let  $N = M_{12} \cap M_{21}$  and  $M_{12}^0 = M_{12} \setminus N$ .
- The moduli space  $M_1 = \{[E_1] : [E_1] \text{ is the isomorphism class of stable bundles on } X_1, \text{rk}(E_1) = 2, \Lambda^2(E_1) \simeq L_1\}$ .
- The moduli space  $M_2 = \{[E_2] : [E_2] \text{ is the gr-equivalence class of semi-stable vector bundles on } X_2, \text{rk}(E_2) = 2, \Lambda^2(E_2) \simeq \mathcal{O}_{X_2}\}$ .
- $N_1 = \{E_{1*} := (E_1, 0 \subset F^2 E_1(p) \subset E_1(p)) : E_{1*} \text{ is parabolic semi-stable bundle with respect to the parabolic weight } (\frac{a_1}{2}, \frac{a_2}{2}) \text{ on } X_1, \text{ where } 0 < a_1 < a_2 < 1, \text{rk}(E_1) = 2, \Lambda^2(E_1) \simeq L_1\}$ .
- $N_2 = \{E_{2*} := (E_2, 0 \subset F^2 E_2(p) \subset E_2(p)) : E_{2*} \text{ is parabolic semi-stable bundle with respect to the parabolic weight } (\frac{a_1}{2}, \frac{a_2}{2}) \text{ on } X_2, \text{ where } 0 < a_1 < a_2 < 1, \text{rk}(E_2) = 2, \Lambda^2(E_2) \simeq \mathcal{O}_{X_2}\}$ .

### 3. COUNTING $\mathbb{F}_q$ RATIONAL POINTS

Throughout this section, we assume our base field to be an algebraically closed field  $\overline{\mathbb{F}_l}$ , where  $l$  is a prime number. Let  $M$  be the moduli space of rank 2 stable torsion free sheaves on  $X$  with Euler characteristic  $\chi$  and fixed determinant  $\xi$  as in Notation 2.3 such that  $a_1 \chi$  is not an integer. By §2,  $M$  is a reduced, connected projective variety with two smooth irreducible components  $M_{12}$  and  $M_{21}$  intersecting transversally on a smooth projective variety  $N$ . Hence without loss of generality, we can assume both  $X$  and  $M$  are defined over a finite field extension  $\mathbb{F}_q$  of  $\mathbb{F}_l$ .

In this section, we are interested in computing  $\mathbb{F}_q$ -rational points of  $M_{12}$  and  $M_{21}$  respectively. We will explicitly compute it for the component  $M_{12}$ , the computation for other component follows similarly. Now we consider the following morphism-

$$\Phi : M_{12} \rightarrow M_1 \times M_2 \quad (3.1)$$

which sends the equivalence class of triple  $[E_1, E_2, \overrightarrow{T}]$  in  $M_{12}$  to equivalence class of pairs  $([E_1], [E_2])$  in  $M_1 \times M_2$ . By Lemma 2.3 and Remark 2.4,  $\Phi$  is well defined and surjective. A priori,  $\Phi$  is defined over  $\overline{\mathbb{F}_l}$  but without loss of generality, we can assume that it is defined over  $\mathbb{F}_q$ . We have the following stratification on  $M_1 \times M_2$ -

$$M_1 \times M_2 = (M_1 \times M_2^s) \sqcup (M_1 \times (K - K_0)) \sqcup (M_1 \times K_0), \quad (3.2)$$

where  $M_2^s$  is the stable locus of vector bundles over  $X_2$  with fixed determinant  $\mathcal{O}_{X_2}$ ,  $K$  is the Kummer variety associated to the Jacobian  $J_2$  of  $X_2$  which can be identified with the isomorphism classes of vector bundles of the form  $(L \oplus L^{-1})$  where  $L$  is a line bundle of degree 0 over  $X_2$ ,

$$K - K_0 = \{[L \oplus L^{-1}] : L \in J_2, L^2 \text{ not isomorphic to } \mathcal{O}_{X_2}\},$$

and

$$K_0 = \{[L \oplus L] : L \in J_2, L^2 \simeq \mathcal{O}_{X_2}\}.$$

Since  $K$  is a projective variety, without loss of generality we can assume  $K$  is defined over  $\mathbb{F}_q$ . It is also known that  $K_0$  is a finite set having  $2^{2g_2}$  points. Now since

$$M_{12} = M_{12}^0 \sqcup N,$$

where  $M_{12}^0$  and  $N$  are as in §2.3, we have

$$N_q(M_{12}) = N_q(M_{12}^0) + N_q(N),$$

where, by  $N_q(V)$  we mean the number of  $\mathbb{F}_q$ -rational points of a quasi-projective variety  $V$  defined over  $\overline{\mathbb{F}}_q$ .

Denoting the restriction of the map  $\Phi$  to the open set  $M_{12}^0$  by  $\phi$  and using the stratification (3.2), we obtain

$$\begin{aligned} N_q(M_{12}) &= N_q(\phi^{-1}(M_1 \times M_2^s)) + N_q(\phi^{-1}(M_1 \times (K - K_0))) \\ &\quad + N_q(\phi^{-1}(M_1 \times K_0)) + N_q(N). \end{aligned} \quad (3.3)$$

**3.1. Computation of  $N_q(\phi^{-1}(M_1 \times M_2^s))$ .** Let

$$A = \{[L \oplus L^{-1}] \in K - K_0 \mid L \text{ and } L^{-1} \text{ are both defined over } \mathbb{F}_q\} \quad (3.4)$$

and

$$B = (K - K_0) - A. \quad (3.5)$$

Then it is clear that

$$N_q(A) = (1/2)(N_q(J_2) - 2^{2g_2}) \quad (3.6)$$

and

$$N_q(B) = N_q(K - K_0) - N_q(A). \quad (3.7)$$

Let

$$\beta_1 = \frac{N_q(A)}{(q-1)^2} + \frac{N_q(B)}{q^2-1} + \frac{N_q(J_2) - 2^{2g_2} N_q(\mathbb{P}^{g_2-2})}{q-1} \quad (3.8)$$

and

$$\beta_2 = \frac{2^{2g_2}}{N_q(\text{GL}(2, \mathbb{F}_q))} + \frac{2^{2g_2} N_q(\mathbb{P}^{g_2-1})}{q(q-1)}. \quad (3.9)$$

By [[3], Proposition 3.7],

$$N_q(M_2^s) = N_q(M_L(2, 1)) + \frac{q^{g_2-1} N_q(J_2)}{q^2-1} - (\beta_1 + \beta_2)(q-1), \quad (3.10)$$



where  $M_L(2, 1)$  is the moduli space of stable bundles of rank 2 and degree 1 with determinant  $L$ .

**Lemma 3.1.**

$$\begin{aligned} N_q(\phi^{-1}(M_1 \times M_2^s)) &= N_q(M_1)N_q(\mathrm{PGL}(2, \mathbb{F}_q))[N_q(M_L(2, 1)) \\ &\quad + \frac{q^{g_2-1}N_q(J_2)}{q^2-1} - (\beta_1 + \beta_2)(q-1)]. \end{aligned} \quad (3.11)$$

*Proof.* Let  $([E_1], [E_2]) \in M_1 \times M_2^s$ . As  $E_2$  is stable,  $\mathrm{gr}(E_2) = E_2$ . Since  $E_1$  is also stable,  $\mathrm{Aut}(E_1)$  and  $\mathrm{Aut}(E_2)$  will just be non-zero scalars.

We have

$$\phi^{-1}([E_1], [E_2]) = \{[E_1, E_2, \vec{T}] \mid \vec{T} : E_1(p) \rightarrow E_2(p) \text{ is a linear isomorphism}\},$$

where  $[E_1, E_2, \vec{T}]$  is an isomorphism class with the triple  $(E_1, E_2, \vec{T})$  as its representative. Now if two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are in the same isomorphism class, then they satisfy the following commutative diagram

$$\begin{array}{ccc} E_1(p) & \xrightarrow{\vec{T}} & E_2(p) \\ \downarrow \lambda & & \downarrow \mu \\ E_1(p) & \xrightarrow{\vec{S}} & E_2(p). \end{array}$$

Hence we have  $\vec{S} = \left(\frac{\mu}{\lambda}\right)\vec{T}$ , where  $\lambda \in \mathrm{Aut}(E_1)$  and  $\mu \in \mathrm{Aut}(E_2)$ .

Conversely if two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are such that  $\vec{S} = \lambda\vec{T}$ , where  $\lambda$  is a non-zero scalar, then again by the following commutative diagram

$$\begin{array}{ccc} E_1(p) & \xrightarrow{\vec{T}} & E_2(p) \\ \downarrow 1 & & \downarrow \lambda \\ E_1(p) & \xrightarrow{\vec{S}} & E_2(p), \end{array}$$

the triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are in the same isomorphism class. Hence two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are in same isomorphism class if and only if  $\vec{S}$  is a scalar multiple of  $\vec{T}$ . Now if we fix a basis for  $E_1(p)$  and  $E_2(p)$ , then  $\vec{T}$  can be identified with an element of  $\mathrm{GL}(2, \mathbb{F}_q)$ . But by the above arguments, the triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \lambda\vec{T})$  are in same isomorphism class for every non-zero scalar  $\lambda$ . This proves that each distinct isomorphism class of triples in  $\phi^{-1}([E_1], [E_2])$  corresponds to a unique element of  $\mathrm{PGL}(2, \mathbb{F}_q)$ . So

$$N_q(\phi^{-1}(M_1 \times M_2^s)) = N_q(M_1)N_q(M_2^s)N_q(\mathrm{PGL}(2, \mathbb{F}_q)). \quad (3.12)$$

Now using equation (3.10) in the equation (3.12), we get the equation (3.11). This proves the Lemma.  $\square$

**3.2. Computation of  $N_q(\phi^{-1}(M_1 \times (K - K_0)))$ .** Let  $([E_1], [L \oplus L^{-1}]) \in M_1 \times (K - K_0)$ . Our aim is to find out the fiber  $\phi^{-1}([E_1], [L \oplus L^{-1}])$ . Suppose  $E_2$  is a vector bundle on  $X_2$  defined over  $\mathbb{F}_q$  such that  $\text{gr}(E_2) \simeq L \oplus L^{-1}$ . Every such  $E_2$  can be given as an extension

$$0 \rightarrow L \rightarrow E_2 \rightarrow L^{-1} \rightarrow 0, \quad (3.13)$$

where the extension (3.13) and the line bundles  $L$  and  $L^{-1}$  are defined over  $\overline{\mathbb{F}}_q$  (which is same as  $\overline{\mathbb{F}}_l$ ).

We know that  $(K - K_0) = A \sqcup B$ , where  $A$  and  $B$  are as defined in the equations (3.4) and (3.5) respectively. Therefore

$$N_q(\phi^{-1}(M_1 \times K - K_0)) = N_q(\phi^{-1}(M_1 \times A)) + N_q(\phi^{-1}(M_1 \times B)). \quad (3.14)$$

We now compute the expressions for  $N_q(\phi^{-1}(M_1 \times A))$  and  $N_q(\phi^{-1}(M_1 \times B))$  separately.

**Lemma 3.2.**

$$N_q(\phi^{-1}(M_1 \times A)) = N_q(M_1)[N_q(A)N_q\left(\frac{GL(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}\right) + 2N_q(A)N_q(\mathbb{P}^{g_2-2})N_q(PGL(2, \mathbb{F}_q))].$$

*Proof.* Suppose  $E_2$  is a vector bundle on  $X_2$  defined over  $\mathbb{F}_q$  such that  $\text{gr}(E_2) \simeq L \oplus L^{-1} \in A$ . Then the extension (3.13) that corresponds to  $E_2$  either splits or does not split. We now analyze the fibers in both the cases separately.

**Case (i):** If the extension (3.13) splits, then  $E_2 \simeq L \oplus L^{-1}$ . In this case  $\text{Aut}(E_2) \simeq \mathbb{G}_m \times \mathbb{G}_m$  [3, Lemma 3.3(b)]. Now if  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  belong to the same isomorphism class, then by the following commutative diagram

$$\begin{array}{ccc} E_1(p) & \xrightarrow{\vec{T}} & E_2(p) \\ \downarrow \lambda & & \downarrow \psi \\ E_1(p) & \xrightarrow{\vec{S}} & E_2(p) \end{array}$$

we have

$$\vec{S} = \begin{pmatrix} \psi \\ \lambda \end{pmatrix} \vec{T},$$

where  $\lambda \in \text{Aut}(E_1)$  and  $\psi \in \text{Aut}(E_2)$ . Conversely if two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are such that  $\vec{S} = \gamma \vec{T}$ , where  $\gamma \in \mathbb{G}_m \times \mathbb{G}_m$ , then one can easily see that they both belong to the same isomorphism class. Hence two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are in the same isomorphism class if and only if  $\vec{S}$  is a  $\mathbb{G}_m \times \mathbb{G}_m$  multiple of  $\vec{T}$ , i.e., if and only if both  $\vec{S}$  and  $\vec{T}$  represent the same coset of  $\frac{GL(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}$ . So for a fixed  $L$ , each distinct isomorphism class of triples  $[E_1, L \oplus L^{-1}, \vec{T}]$  corresponds to a unique element in  $\frac{GL(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}$ . Hence **Case (i)** contributes  $N_q(M_1)N_q(A)N_q\left(\frac{GL(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}\right)$  to  $N_q(\phi^{-1}(M_1 \times A))$ .

**Case (ii) :** If the extension (3.13) does not split, then  $\text{Aut}(E_2) \simeq \mathbb{G}_m$  [3, Lemma 3.3(a)]. For each fixed isomorphism class of bundles  $E_2$ , repeating the arguments done for the fiber over the stable locus  $M_1 \times M_2^s$ , one can conclude that the set of distinct isomorphism classes of triples of the form  $[E_1, E_2, \vec{T}]$  is in bijection with  $PGL(2, \mathbb{F}_q)$ . But then each distinct isomorphism class of bundles  $E_2$  corresponds to an isomorphism class of

non-split extensions of type (3.13) and each such isomorphism class of non-split extensions corresponds to a unique element of  $(\text{Ext}^1(L^{-1}, L) \setminus \{0\}) \bmod \mathbb{G}_m$ . It is well known that  $\text{Ext}^1(L^{-1}, L) \bmod \mathbb{G}_m$  is same as  $\mathbb{P}^{g_2-2}$ . Since the extensions of  $L$  by  $L^{-1}$  and the extensions of  $L^{-1}$  by  $L$  give rise to distinct isomorphism classes of bundles, **Case (ii)** contributes  $N_q(M_1)2N_q(A)N_q(\mathbb{P}^{g_2-2})N_q(\text{PGL}(2, \mathbb{F}_q))$  to  $N_q(\phi^{-1}(M_1 \times A))$ .

Combining both the cases we get

$$N_q(\phi^{-1}(M_1 \times A)) = N_q(M_1)[N_q(A)N_q\left(\frac{\text{GL}(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}\right) + 2N_q(A)N_q(\mathbb{P}^{g_2-2})N_q(\text{PGL}(2, \mathbb{F}_q))].$$

which proves the Lemma.  $\square$

**Lemma 3.3.**

$$N_q(\phi^{-1}(M_1 \times B)) = N_q(M_1)N_q(B)\frac{N_q(\text{GL}(2, \mathbb{F}_q))}{q^2 - 1}.$$

*Proof.* Suppose  $E_2$  is a vector bundle on  $X_2$  defined over  $\mathbb{F}_q$  such that  $\text{gr}(E_2) \simeq L \oplus L^{-1} \in B$ . Then from the arguments in [3, Remark 3.2], the corresponding extension (3.13) splits over  $\mathbb{F}_{q^2}$  (i.e.,  $L$  and  $L^{-1}$  are defined over  $\mathbb{F}_{q^2}$  and  $E_2 \simeq L \oplus L^{-1}$  over  $\mathbb{F}_{q^2}$ ). By the following commutative diagram

$$\begin{array}{ccc} E_1(p) & \xrightarrow{\vec{T}} & E_2(p) \\ \downarrow \lambda & & \downarrow \tau \\ E_1(p) & \xrightarrow{\vec{S}} & E_2(p) \end{array}$$

we have

$$\vec{S} = \begin{pmatrix} \tau \\ \lambda \end{pmatrix} \vec{T},$$

where  $\lambda \in \text{Aut}(E_1)$  and  $\tau \in \text{Aut}(E_2)$ . Since the group  $\text{Aut}(E_2)$  acts on  $\mathcal{L}(E_1(p), E_2(p))$  (where  $\mathcal{L}(E_1(p), E_2(p))$  denotes the set of all linear isomorphisms from  $E_1(p)$  to  $E_2(p)$ ) by composition, we can conclude that two triples  $(E_1, E_2, \vec{T})$  and  $(E_1, E_2, \vec{S})$  are in the same isomorphism class if and only if  $\vec{T}$  and  $\vec{S}$  are in the same orbit under this action. So it is clear that the set of distinct isomorphism classes of triples will be in bijection with the set of all orbits under this action which can be identified with  $\frac{\text{GL}(2, \mathbb{F}_q)}{\text{Aut}(E_2)}$ . So the fibers over  $M_1 \times B$  contribute  $N_q(M_1)N_q(B)N_q\left(\frac{\text{GL}(2, \mathbb{F}_q)}{\text{Aut}(E_2)}\right)$  to  $N_q(\phi^{-1}(M_1 \times B))$ . By [3, Lemma 3.5], we know that  $N_q(\text{Aut}(E_2)) = N_{q^2}(\mathbb{G}_m) = q^2 - 1$ . Therefore we have

$$N_q(\phi^{-1}(M_1 \times B)) = N_q(M_1)N_q(B)\frac{N_q(\text{GL}(2, \mathbb{F}_q))}{q^2 - 1}.$$

$\square$

This completes the Lemma.

**Lemma 3.4.**

$$\begin{aligned} N_q(\phi^{-1}(M_1 \times (K - K_0))) &= N_q(M_1)[N_q(\tilde{K})(q^2 - q) + N_q(J_2)[q + q(q^2 - 1)N_q(\mathbb{P}^{g_2-2})] \\ &\quad - 2^{2g_2}[q + q(q^2 - 1)N_q(\mathbb{P}^{g_2-2}) + (q^2 - q)N_q(\mathbb{P}^{g_2-1})]], \end{aligned} \quad (3.15)$$

where  $\tilde{K}$  is the canonical desingularization of  $K$ .

*Proof.* Using the Lemmas 3.2 and 3.3 in the equation (3.14), we get

$$\begin{aligned}
N_q(\phi^{-1}(M_1 \times (K - K_0))) &= N_q(M_1)[N_q(A)N_q\left(\frac{\mathrm{GL}(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_m}\right) + 2N_q(A)N_q(\mathbb{P}^{g_2-2})N_q(\mathrm{PGL}(2, \mathbb{F}_q)) \\
&\quad + N_q(B)\frac{N_q(\mathrm{GL}(2, \mathbb{F}_q))}{q^2-1}] \\
&= N_q(M_1)[N_q(A)q(q+1) + 2N_q(A)N_q(\mathbb{P}^{g_2-2})q(q^2-1) \\
&\quad + N_q(B)(q^2-q)]. \tag{3.16}
\end{aligned}$$

By [3, Remark 4.4], it is known that

$$N_q(K - K_0) = N_q(\tilde{K}) - 2^{2g_2} N_q(\mathbb{P}^{g_2-1}). \tag{3.17}$$

(One can see [10] for more details on  $K$  and  $\tilde{K}$ ). Now using (3.6), (3.7) and (3.17) in the equation (3.16), we get the equation (3.15). This completes the Lemma.  $\square$

### 3.3. Computation of $N_q(\phi^{-1}(M_1 \times K_0))$ .

**Lemma 3.5.**

$$N_q(\phi^{-1}(M_1 \times K_0)) = N_q(M_1)2^{2g_2}[1 + (q^2-1)N_q(\mathbb{P}^{g_2-1})].$$

*Proof.* Let  $([E_1], [L \oplus L]) \in M_1 \times K_0$ . Suppose  $E_2$  is a vector bundle over  $X_2$  defined over  $\mathbb{F}_q$  such that  $\mathrm{gr}(E_2) \simeq L \oplus L$ . Again every such  $E_2$  can be given as an extension

$$0 \rightarrow L \rightarrow E_2 \rightarrow L \rightarrow 0, \tag{3.18}$$

with  $L^2 \simeq \mathcal{O}_{X_2}$ . Such line bundles are  $2^{2g_2}$  in number and so, without loss of generality we can assume that  $L$  and the extensions of type (3.18) are defined over  $\mathbb{F}_q$ . We again consider two cases, the case when (3.18) splits and when it does not split.

**Case (i):** If the extension (3.18) splits, then  $E_2 \simeq L \oplus L$ . Also in this case  $\mathrm{Aut}(E_2) \simeq \mathrm{GL}(2, \mathbb{F}_q)$  [3, Proposition 3.1(1)]. We claim that for a fixed  $L$ , (fixing a basis for  $E_1(p)$  and  $E_2(p)$ ) there is only one isomorphism class of triples namely  $[E_1, L \oplus L, \vec{I}_2]$ , where  $\vec{I}_2$  stands for  $2 \times 2$  identity matrix, i.e.,  $I_2$  corresponds to the invertible linear transformation  $\vec{I}_2$  which sends the fixed basis of  $E_1(p)$  to the fixed basis of  $E_2(p)$ .

Suppose  $[E_1, L \oplus L, \vec{T}]$  is an isomorphism class of triples. Fix a basis  $\{v_1, v_2\}$  of  $E_1(p)$  and a basis  $\{w_1, w_2\}$  of  $E_2(p)$ . Once the bases are fixed,  $\vec{T}$  can be identified with an element of  $\mathrm{GL}(2, \mathbb{F}_q)$ , call it  $T$ . Consider the following commutative diagram-

$$\begin{array}{ccc}
E_1(p) & \xrightarrow{T} & E_2(p) \\
\downarrow 1 & & \downarrow T^{-1} \\
E_1(p) & \xrightarrow{I_2} & E_2(p).
\end{array}$$

Here 1 denotes the identity element of  $\mathrm{Aut}(E_1)$ , and  $T^{-1} \in \mathrm{Aut}(E_2)$  is the inverse of the matrix  $T$ . From the above diagram, we have  $[E_1, L \oplus L, \vec{T}] = [E_1, L \oplus L, \vec{I}_2]$ . So there is only one isomorphism class of triples once  $L$  is fixed. Since there are  $2^{2g_2}$  line bundles of this type, **Case (i)** contributes  $N_q(M_1)2^{2g_2}$  to  $N_q(\phi^{-1}(M_1 \times K_0))$ .

**Case (ii):** Suppose the extension (3.18) does not split. Then  $\mathrm{Aut}(E_2) \simeq \mathbb{G}_m \times \mathbb{G}_a$  [3, Proposition 3.1(2)]. As earlier, each distinct isomorphism class of bundles  $E_2$  corresponds to an

isomorphism class of non-split extensions of type (3.18) and each non-split extension class corresponds to a unique element of  $(\text{Ext}^1(L, L) \setminus \{0\}) \bmod \mathbb{G}_m$  which is nothing but  $\mathbb{P}^{g_2-1}$ . So, for a fixed  $L$  such that  $L \oplus L \in K_0$ , each distinct isomorphism class of triples  $[E_1, E_2, \vec{T}]$  (where  $E_2$  is given by an isomorphism class of non-split extensions of type (3.18)), corresponds to a unique element of  $\mathbb{P}^{g_2-1} \times \frac{\text{GL}(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_a}$ . Finally since  $K_0$  has  $2^{2g_2}$  elements, **Case (ii)** contributes  $2^{2g_2} N_q(M_1) N_q(\mathbb{P}^{g_2-1}) N_q(\frac{\text{GL}(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_a})$  to  $N_q(\phi^{-1}(M_1 \times K_0))$ . So we have

$$N_q(\phi^{-1}(M_1 \times K_0)) = N_q(M_1) 2^{2g_2} [1 + N_q(\frac{\text{GL}(2, \mathbb{F}_q)}{\mathbb{G}_m \times \mathbb{G}_a}) N_q(\mathbb{P}^{g_2-1})].$$

This implies

$$N_q(\phi^{-1}(M_1 \times K_0)) = N_q(M_1) 2^{2g_2} [1 + (q^2 - 1) N_q(\mathbb{P}^{g_2-1})]. \quad (3.19)$$

□

### 3.4. Computation of $N_q(N)$ .

#### Lemma 3.6.

$$N_q(N) = N_q(M_1) N_q(M_2(-1)) N_q(\mathbb{P}^1) N_q(\mathbb{P}^1), \quad (3.20)$$

where by  $M_2(-1)$  we mean the moduli space of stable bundles of rank two and degree -1 with fixed determinant.

*Proof.* By Lemma 2.3,  $N \simeq N_1 \times N_2$ , where  $N_1$  and  $N_2$  are as described in the notation in §2.3. Therefore

$$N_q(N) = N_q(N_1) N_q(N_2). \quad (3.21)$$

By the choice of weights, it is automatic that

- (i) parabolic stability coincides with the semi-stability of the underlying bundle and
- (ii) underlying bundle is stable implies it is parabolic stable for any quasi-parabolic structure.

Then it is known that the canonical map  $N_1 \rightarrow M_1$  (which sends a parabolic vector bundle  $E_{1*} \in N_1$  to the underlying bundle  $E_1 \in M_1$ ) is a  $\mathbb{P}^1$ -fibration over  $M_1$ , locally trivial in Zariski topology. So we have

$$N_q(N_1) = N_q(\mathbb{P}^1) N_q(M_1). \quad (3.22)$$

Now to find  $N_q(N_2)$ , let  $E_{2*} \in N_2$  be arbitrary. Then we have a canonical surjection  $E_2(p) \rightarrow \frac{E_2(p)}{F^2 E_2(p)}$ , where  $F^2 E_2(p)$  is a one-dimensional subspace of  $E_2(p)$ . Let  $W$  be the kernel of this map. Then clearly  $W$  is locally free of rank two and degree -1. By [2, Proposition 6],  $W$  is stable and the map  $\psi : N_2 \rightarrow M_2(-1)$  which sends  $E_2$  to  $W$  is a  $\mathbb{P}^1$ -fibration, locally trivial in Zariski topology. Therefore

$$N_q(N_2) = N_q(\mathbb{P}^1) N_q(M_2(-1)). \quad (3.23)$$

Using equations (3.22) and (3.23) in the equation (3.21), we get the equation (3.20). □

Using equations (3.11), (3.15), (3.19) and (3.20) in the equation (3.3), we have

$$\begin{aligned}
N_q(M_{12}) &= N_q(M_1)N_q(\mathrm{PGL}(2, \mathbb{F}_q)[N_q(M_L(2, 1)) + \frac{q^{g_2-1}N_q(J_2)}{q^2-1} - (\beta_1 + \beta_2)(q-1)] \\
&\quad + N_q(M_1)[N_q(\tilde{K})(q^2 - q) + N_q(J_2)[q + q(q^2 - 1)N_q(\mathbb{P}^{g_2-2})] \\
&\quad - 2^{2g_2}[q + q(q^2 - 1)N_q(\mathbb{P}^{g_2-2}) + (q^2 - q)N_q(\mathbb{P}^{g_2-1})]] \\
&\quad + N_q(M_1)2^{2g_2}[1 + (q^2 - 1)N_q(\mathbb{P}^{g_2-1})] \\
&\quad + N_q(M_1)N_q(M_2(-1))N_q(\mathbb{P}^1)N_q(\mathbb{P}^1). \tag{3.24}
\end{aligned}$$

#### 4. THE POINCARÉ POLYNOMIAL COMPUTATION

Let  $R$  be an algebraic number ring and  $\mathfrak{q}$  be a maximal ideal of  $R$  contracting to a nonzero maximal ideal  $p\mathbb{Z}$ . The field  $R/\mathfrak{q}$  is a finite extension of  $\mathbb{Z}/p\mathbb{Z}$ . We denote it by  $\mathbb{F}_q$ . Let  $X_R$  be a projective curve over  $R$  with two smooth components  $X_{1,R}$  and  $X_{2,R}$  intersecting transversally at a nodal point  $p_R$ . Let  $X_{\mathbb{F}_q}$  and  $X_{\mathbb{C}}$  be the curves obtained by base changing  $X_R$  to  $\mathbb{F}_q$  and  $\mathbb{C}$  respectively.

Fix an odd integer  $\chi$  and a polarization  $(a_1, a_2)$  such that  $a_1\chi$  is not an integer. Let  $\xi_R = (L_{1,R}, \mathcal{O}_{X_{2,R}})$  be a line bundle on  $X_R$ , where  $L_{1,R}$  is a line bundle of degree 1 on  $X_{1,R}$ . Let  $\xi_{\mathbb{F}_q}$  and  $\xi_{\mathbb{C}}$  be the corresponding line bundles over  $X_{\mathbb{F}_q}$  and  $X_{\mathbb{C}}$  respectively. Let  $M_{12,R}$  (resp.  $M_{12,\mathbb{F}_q}$  and  $M_{12,\mathbb{C}}$ ) be the moduli space of stable torsion free sheaves with Euler characteristic  $\chi$  and fixed determinant  $\xi_R$  (resp.  $\xi_{\mathbb{F}_q}$  and  $\xi_{\mathbb{C}}$ ), where  $\chi = \chi_1 + \chi_2 - 2$  and  $\chi_1, \chi_2$  satisfy inequalities (2.4). Since  $M_{12,R}$  itself is a fine moduli space (cf. §2), by the base change property  $M_{12,R} \times \mathrm{Spec}(\mathbb{F}_q) \simeq M_{12,\mathbb{F}_q}$  and  $M_{12,R} \times \mathrm{Spec}(\mathbb{C}) \simeq M_{12,\mathbb{C}}$ . Similar description holds for  $M_{21,R}$  also. For notational convenience, from now on we drop the subscripts  $\mathbb{F}_q$  and  $\mathbb{C}$  and write  $M_{12}$  (resp.  $M_{21}$ ). Subscripts will be understood from the context.

Now we recall few generalities which are used in computing Poincaré polynomial using Weil conjectures. Let  $Y_R$  be a smooth projective variety of dimension  $n$  defined over an algebraic number ring  $R$  and  $Y_{\mathbb{C}}$  (resp.  $Y$ ) be a smooth projective variety obtained from  $Y_R$  by base changing to  $\mathbb{C}$  (resp.  $\mathbb{F}_q$ ). Following Weil, we define the *zeta* function of  $Y$  as

$$Z(Y, t) = \exp\left(\sum_{r=0}^{\infty} N_{q^r} \frac{t^r}{r}\right).$$

It is clear that  $Z(Y, t)$  is a formal power series with rational coefficients. Now the Weil Conjectures state that  $Z(Y, t)$  can be written in the form

$$Z(Y, t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}.$$

where  $P_0(t) = (1-t)$ ;  $P_{2n}(t) = (1-q^n t)$ ; and for each  $1 \leq i \leq (2n-1)$ ,  $P_i(t)$  is a polynomial with integer coefficients, which can be written as

$$P_i(t) = \prod (1 - \beta_{ij} t),$$

where  $\beta_{ij}$ 's are algebraic integers with  $|\beta_{ij}| = q^{\frac{i}{2}}$ .

Furthermore, the Betti numbers of  $Y_{\mathbb{C}}$  are given by  $B_i(Y_{\mathbb{C}}) = \deg(P_i(t))$ .

For  $r \geq 1$ , and  $Y$  a projective variety over  $\mathbb{F}_q$  of dimension  $n$ , set  $\widetilde{N}_{q^r}(Y) = q^{-r n} N_{q^r}(Y)$ .

We recall the following lemma of Kirwan which is the main tool in the computation of the Poincaré Polynomial of  $M_{12}$ -

**Lemma 4.1.** [7, Page-186] *Suppose that  $Y_1, \dots, Y_k$  are smooth projective varieties over  $\mathbb{F}_q$  obtained as reduction mod  $p$  of varieties defined in characteristic 0. Suppose that  $f$  is a rational function in  $(k+1)$  variables with  $\mathbb{Z}$  coefficients such that*

$$f(q^{-r}, \widetilde{N}_{q^r}(Y_1), \dots, \widetilde{N}_{q^r}(Y_k)) = 0,$$

for all  $r \geq 1$ . Then

$$f(t^2, P_t(Y_1), \dots, P_t(Y_k)) = 0,$$

where  $P_t(Y_i)$  is the Poincaré Polynomial of the corresponding variety over  $\mathbb{C}$ .

Multiplying both sides of the equation (3.24) by  $q^{-\dim(M_{12})}$  and using the fact that  $N_q(\text{PGL}(2, \mathbb{F}_q)) = q(q^2 - 1)$ , and  $N_q(\mathbb{P}^1) = (q + 1)$ , and simplifying, we have

$$\begin{aligned} \widetilde{N}_q(M_{12}) &= \widetilde{N}_q(M_1)(1 - q^{-2})[\widetilde{N}_q(M_L(2, 1)) + \widetilde{N}_q(J_2)\{\frac{q^{-g_2}}{1 - q^{-2}}\} - q^{-3g_2+3}(\beta_1 + \beta_2)(q - 1)] \\ &\quad + \widetilde{N}_q(M_1)2^{2g_2}[q^{-3g_2} + q^{-2g_2+1}(1 - q^{-2})\widetilde{N}_q(\mathbb{P}^{g_2-1})] \\ &\quad + \widetilde{N}_q(M_1)[\widetilde{N}_q(\tilde{K})q^{-2g_2+2}(1 - q^{-1}) + \widetilde{N}_q(J_2)[q^{-2g_2+1} + q^{-g_2+1}(1 - q^{-2})\widetilde{N}_q(\mathbb{P}^{g_2-2})] \\ &\quad - 2^{2g_2}[q^{-3g_2+1} + q^{-2g_2+1}(1 - q^{-2})\widetilde{N}_q(\mathbb{P}^{g_2-2}) + q^{-2g_2+1}(1 - q^{-1})\widetilde{N}_q(\mathbb{P}^{g_2-1})] \\ &\quad + \widetilde{N}_q(M_1)\widetilde{N}_q(M_2(-1))q^{-1}(1 + 2q^{-1} + q^{-2})]. \end{aligned} \quad (4.1)$$

It is to be noted that all the varieties that occur in the equation (4.1) are smooth projective varieties. Further, replacing  $q$  by  $q^r$  in the equation (4.1), one can get the expression for  $\widetilde{N}_{q^r}(M_{12})$ , for  $r \geq 1$ .

We can now apply Kirwan's Lemma and obtain the Poincaré Polynomial of  $M_{12}$ .

**Theorem 4.2.** *The Poincaré Polynomial of  $M_{12}$  is given as follows:*

$$\begin{aligned} P_t(M_{12}) &= P_t(M_1)(1 - t^4)[P_t(M_L(2, 1)) + P_t(J_2)\{\frac{t^{2g_2}}{1 - t^4}\} - (\widetilde{\beta}_1(t) + \widetilde{\beta}_2(t))] \\ &\quad + P_t(M_1)2^{2g_2}[t^{6g_2} + t^{4g_2-2}(1 - t^4)P_t(\mathbb{P}^{g_2-1})] \\ &\quad + P_t(M_1)[P_t(\tilde{K})t^{4g_2-4}(1 - t^2) + P_t(J_2)[t^{4g_2-2} + t^{2g_2-2}(1 - t^4)P_t(\mathbb{P}^{g_2-2})] \\ &\quad - 2^{2g_2}[t^{6g_2-2} + t^{4g_2-2}(1 - t^4)P_t(\mathbb{P}^{g_2-2}) + t^{4g_2-2}(1 - t^2)P_t(\mathbb{P}^{g_2-1})] \\ &\quad + P_t(M_1)P_t(M_2(-1))t^2(1 + 2t^2 + t^4), \end{aligned}$$

where

$$\begin{aligned}
P_t(M_1) &= \frac{(1+t^3)^{2g_1} - t^{2g_1}(1+t)^{2g_1}}{(1-t^2)(1-t^4)}, \\
P_t(M_L(2,1)) &= \frac{(1+t^3)^{2g_2} - t^{2g_2}(1+t)^{2g_2}}{(1-t^2)(1-t^4)} = P_t(M_2(-1)), \\
P_t(J_2) &= (1+t)^{2g_2}, \\
\widetilde{\beta}_1(t) &= P_t(\tilde{K})\left[\frac{t^{4g_2-4}}{(1+t^2)}\right] + P_t(J_2)\left[\frac{t^{4g_2-2}}{(1-t^4)} + P_t(\mathbb{P}^{g_2-2})t^{2g_2-2}\right] \\
&\quad - P_t(\mathbb{P}^{g_2-1})\left[2^{2g_2}\frac{t^{4g_2-2}}{(1+t^2)}\right] - 2^{2g_2}\left[\frac{t^{6g_2-2}}{(1-t^4)} + P_t(\mathbb{P}^{g_2-2})t^{4g_2-2}\right], \\
\widetilde{\beta}_2(t) &= 2^{2g_2}\left[\frac{t^{6g_2}}{(1-t^4)} + t^{4g_2-2}P_t(\mathbb{P}^{g_2-1})\right], \\
P_t(\mathbb{P}^n) &= (1+t^2+\dots+t^{2n}), \\
P_t(\tilde{K}) &= \sum_{i=0}^{2g_2} b_i(\tilde{K})t^i,
\end{aligned}$$

where

$$\begin{aligned}
b_i(\tilde{K}) &= 0, \text{ if } i \text{ is odd,} \\
&= 1, \text{ if } i=0 \text{ or } 2g_2, \text{ and} \\
&= \binom{2g_2}{i} + 2^{2g_2}, \text{ if } i \text{ is even and } 0 < i < 2g_2.
\end{aligned}$$

(see [[10], Theorem 2]).

We give below the Betti numbers of the variety  $M_{12}$  for some low genus curves.

Table 1:

Betti Nos. of $M_{12}$ for low genus						
Betti No.	$g_1 = 3$ $g_2 = 3$	$g_1 = 3$ $g_2 = 4$	$g_1 = 3$ $g_2 = 5$	$g_1 = 4$ $g_2 = 3$	$g_1 = 4$ $g_2 = 4$	$g_1 = 4$ $g_2 = 5$
$B_0$	1	1	1	1	1	1
$B_1$	0	0	0	0	0	0
$B_2$	3	3	3	3	3	3
$B_3$	12	14	16	14	16	18
$B_4$	8	8	8	8	8	8
$B_5$	36	42	48	42	48	54
$B_6$	81	106	135	107	136	169
$B_7$	90	106	122	112	128	144
$B_8$	207	284	371	299	388	487
$B_9$	356	542	768	568	808	1096
$B_{10}$	435	657	879	708	974	1240



Continuation of Table 1						
Betti No.	$g_1 = 3$ $g_2 = 3$	$g_1 = 3$ $g_2 = 4$	$g_1 = 3$ $g_2 = 5$	$g_1 = 4$ $g_2 = 3$	$g_1 = 4$ $g_2 = 4$	$g_1 = 4$ $g_2 = 5$
$B_{11}$	698	1224	1850	1320	2040	2880
$B_{12}$	992	1953	3203	2089	3524	5336
$B_{13}$	1120	2412	3988	2592	4520	6756
$B_{14}$	1345	3520	6606	3781	7448	12302
$B_{15}$	1520	4784	9976	5110	11152	19872
$B_{16}$	1345	5386	12271	5731	13821	25555
$B_{17}$	1120	6236	16900	6626	18920	39008
$B_{18}$	992	6884	22191	7314	24621	55967
$B_{19}$	698	6236	24804	6626	27496	68762
$B_{20}$	435	5386	28060	5731	30996	90304
$B_{21}$	356	4784	30512		33584	114202
$B_{22}$	207		28060		30996	126671
$B_{23}$	90		24804		27496	140424
$B_{24}$	81					150346
$B_{25}$	36					140424
$B_{26}$	8					
$B_{27}$	12					
$B_{28}$	3					
$B_{29}$	0					
$B_{30}$	1					
End of Table						

**4.1. The Poincaré Polynomial of  $M_{21}$ .** We now briefly sketch the changes that need to be incorporated in order to find the Poincaré polynomial of  $M_{21}$  (see §2.3 for the definition of  $M_{21}$ ).

Let  $L'_1$  be the invertible sheaf on  $X_1$  such that  $L_1'^{\otimes 2} = L_1^* \otimes \mathcal{O}_{X_1}(-p)$ . This is possible because  $L_1^* \otimes \mathcal{O}_{X_1}(-p)$  is of even degree. Let  $L'_2 = \mathcal{O}_{X_2}(p)$ . Now tensoring  $M_{21}$  by the invertible sheaf  $L' = (L'_1, L'_2, \overrightarrow{\lambda}')$ , one gets an isomorphism between  $M_{21}$  and  $M'_{21}$ , where

$$M'_{21} = \{[F'_1, F'_2, \overleftarrow{S}'] : [F'_1, F'_2, \overleftarrow{S}'] \text{ is an isomorphism class of stable triples,} \\ \text{rk}(F'_i) = 2, \Lambda^2(F'_1) \simeq \mathcal{O}_{X_1}, \Lambda^2(F'_2) \simeq \mathcal{O}_{X_2}(p)\}.$$

So finding the Poincaré Polynomial of  $M_{21}$  is equivalent to finding the Poincaré Polynomial of  $M'_{21}$ . Now analogous to the natural map  $\Phi$  which we had in (3.1), we have a surjective map

$$\Theta : M'_{21} \rightarrow M'_1 \times M'_2$$

which sends equivalence class of the triple  $[F'_1, F'_2, \overleftarrow{S}']$  to  $([F'_1], [F'_2])$ , where

$$M'_1 = \{[F'_1] : [F'_1] \text{ is the gr-equivalence class of semi-stable vector bundles on } X_1, \\ \text{rk}(F'_1) = 2, \Lambda^2(F'_1) \simeq \mathcal{O}_{X_1}\},$$

and

$$M'_2 = \{[F'_2] : [F'_2] \text{ is the isomorphism class of stable bundles on } X_2, \text{rk}(F'_2) = 2, \\ \Lambda^2(F'_2) \simeq \mathcal{O}_{X_2}(p)\}.$$

Now we apply exactly the same stratification for  $M'_1$  as we did for  $M_2$  earlier and repeat the same arguments as before to obtain the Poincaré polynomial of  $M'_{21}$  and hence for  $M_{21}$ .

**Remark 4.3.** (i) *It is easy to see that the Poincaré polynomial of  $M_{12}$  for the case  $g_1 = i$ ,  $g_2 = j$  is same as that of  $M_{21}$  for the case  $g_1 = j$ ,  $g_2 = i$ , where  $i, j \in \mathbb{Z}, i \geq 3, j \geq 3$ .*

(ii) *The topological Euler characteristic  $\chi$  of the spaces  $M_{12}$  and  $M_{21}$  can be obtained by evaluating the respective Poincaré polynomials at  $t = -1$ . By Theorem 4.2 and the above remark, it follows that  $\chi(M_{12}) = 0 = \chi(M_{21})$ .*

## 5. THE INTERSECTION POINCARÉ POLYNOMIAL COMPUTATION

By [11, Lemma 2.4], it is known that the fixed determinant moduli space  $M$  is a union of two irreducible, smooth varieties  $M_{12}$  and  $M_{21}$  intersecting transversally along the smooth divisor  $N$ . Since both the irreducible components of  $M$  are smooth and are of dimension  $3g - 3$ , by [15, Theorem 19.2],  $M$  has a Whitney stratification which makes it into a topological pseudomanifold ([16, IV §2]).

Let  $\widetilde{M}, \widetilde{M}_{12}$  and  $\widetilde{M}_{21}$  be the algebraic normalizations of  $M, M_{12}$  and  $M_{21}$  respectively. Since  $M_{12}$  and  $M_{21}$  intersect transversally along  $N$ , we have

$$\begin{aligned} \widetilde{M} &= \widetilde{M}_{12} \sqcup \widetilde{M}_{21} \\ &\simeq M_{12} \sqcup M_{21}. \end{aligned} \tag{5.1}$$

Now since  $\widetilde{M}, M_{12}$  and  $M_{21}$  are normal, their intersection homology groups (with top perversity  $\bar{t}$ ) will coincide with the singular homology groups by the Theorem 2.8. So Theorem 2.8 combined with the isomorphism (5.1) gives

$$IH_*^{\bar{t}}(\widetilde{M}) \simeq H_*(\widetilde{M}) \simeq H_*(M_{12}) \oplus H_*(M_{21}). \tag{5.2}$$

Now using the Remark 2.10 and the Theorem 2.7 in the equation (5.2), we get

$$IH_*^{\bar{t}}(M) \simeq H_*(M_{12}) \oplus H_*(M_{21}) \tag{5.3}$$

Let  $IP^{\bar{t}}(M)$  denote the intersection Poincaré polynomial of  $M$  with respect to the top perversity  $\bar{t}$  (see definition 2.9). Then by the equation (5.3), we have

**Theorem 5.1.**

$$IP^{\bar{t}}(M) = P_t(M_{12}) + P_t(M_{21}),$$

where  $P_t(M_{12})$  and  $P_t(M_{21})$  are the Poincaré polynomials of  $M_{12}$  and  $M_{21}$  that have been computed in the previous section.

**Remark 5.2.** *We notice that, since  $\chi(M_{12})$  and  $\chi(M_{21})$  are zero, the intersection Euler characteristic of  $M$  also turns out to be zero by Theorem 5.1.*

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