

# Interconnection and Damping Assignment Passivity-Based Control of Mechanical Systems With Underactuation Degree One

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**Abstract**—Interconnection and damping assignment passivity-based control is a new controller design methodology developed for (asymptotic) stabilization of nonlinear systems that does not rely on, sometimes unnatural and technique-driven, linearization or decoupling procedures but instead endows the closed-loop system with a Hamiltonian structure with a desired energy function—that qualifies as Lyapunov function for the desired equilibrium. The assignable energy functions are characterized by a set of partial differential equations that must be solved to determine the control law. We prove in this paper that for a class of mechanical systems with *underactuation degree one* the partial differential equations can be *explicitly solved*. Furthermore, we introduce a suitable parametrization of assignable energy functions that provides the designer with a handle to address transient performance and robustness issues. Finally, we develop a speed estimator that allows the implementation of *position-feedback* controllers. The new result is applied to obtain an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling, and a controller for the pendulum in a cart that can swing-up the pendulum from any position in the upper half plane and stop the cart at any desired location. In both cases we obtain very simple and intuitive position-feedback solutions.

**Index Terms**—Energy shaping, Hamiltonian systems, nonlinear control, passivity, underactuated mechanical systems.

## I. INTRODUCTION

**I**N [31], we introduced a controller design technique, called interconnection and damping assignment passivity-based control (IDA-PBC), that achieves stabilization for underactuated mechanical systems invoking the physically motivated principles of *energy shaping* and *damping injection*. IDA-PBC endows the closed-loop system with a Hamiltonian structure

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where the kinetic and potential energy functions have some desirable features, a minimal requirement being to have a minimum at the desired operating point to ensure its stability. Similar techniques have been reported for general port-controlled Hamiltonian and Lagrangian systems in [30], [40] and [32], respectively; see also [12]–[14] for the case of Lagrangian mechanical systems and [29] which contains an extensive list of references on this topic. The success of these methods relies on the possibility of solving a set of partial differential equations (PDEs) that identify the energy functions that can be assigned to the closed-loop. The PDE associated to the kinetic energy defines the admissible closed-loop inertia matrices and is nonlinear, while the PDE of assignable potential energy functions is linear. In [12] the authors identify a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved. Also, techniques to solve the PDEs have been reported in [8] and [11] and some geometric aspects of the equations are investigated in [23]. In [18] it is shown that the kinetic energy PDE reduces to an ordinary differential equation (ODE) if the system is of underactuation degree one, that is, if the difference between the number of degrees of freedom and the number of control actions is one—see also [9] for a detailed study of this case for the Controlled Lagrangian Method. In spite of all these developments the need to solve the PDEs remains the main stumbling block for a wider applicability of these methods.

In this paper we are interested in the application of IDA-PBC to *mechanical systems with underactuation degree one*. The main contributions of the paper are as follows.

- 1) Identification of a class of underactuation degree one mechanical systems for which the PDEs of IDA-PBC can be *explicitly solved*. Roughly speaking, we assume that the open-loop systems inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate.
- 2) Derivation of conditions to effectively assign a minimum to the energy function at the desired operating point—providing in this way a complete constructive procedure for stabilization. The conditions are given in terms of single *algebraic inequality* that measures our ability to influence, through the modification of the inertia matrix, the unactuated component of the force induced by potential energy.

- 3) Development, using the recently introduced method of Immersion and Invariance [6], [22], of a speed estimator that allows the implementation of the proposed controllers *measuring position* only. To the best of our knowledge, this is the first position-feedback solutions reported for these systems—at this level of generality.
- 4) Last, but not least, the introduction of a suitable parametrization of assignable energy functions—via two *free functions* and a gain matrix—giving the designer the possibility to address transient performance and robustness issues. In spite of their great practical importance these issues are rarely studied in the literature. Indeed, most of the controllers reported for this class of systems rely on the rather unnatural, technique-driven and fragile operations of linearization and decoupling. Other existing schemes give very little freedom to the designer to tune the controller—basically only the selection of saturation and domination functions or the adjustment of high-gain injection or damping.

The new result is applied to obtain an (almost) globally stabilizing scheme for the vertical takeoff and landing aircraft with strong input coupling, and a controller for the pendulum on a cart that can swing-up the pendulum from any position in the (open) upper half plane and stop the cart at any desired location. In both cases we obtain very simple and intuitive position-feedback solutions that endow the closed-loop system with a Hamiltonian structure with desired potential and kinetic energy functions. For other applications we refer the reader to to [1]–[3].

## II. IDA-PBC METHOD FOR (SIMPLE) MECHANICAL SYSTEMS

In this section, we briefly review the material of [31] that introduces the IDA-PBC approach to regulate the position of underactuated mechanical systems with total energy

$$H(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q) \quad (1)$$

where  $q \in \mathbb{R}^n, p \in \mathbb{R}^n$  are the generalized position and momenta, respectively,  $M = M^\top > 0$  is the inertia matrix, and  $V$  is the potential energy. If we assume that the system has no natural damping, then the equations of motion can be written as<sup>1</sup>

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u \quad (2)$$

where  $u \in \mathbb{R}^m$  and  $G \in \mathbb{R}^{n \times m}$  with  $\text{rank} G = m < n$ .

In IDA-PBC stabilization is achieved assigning to the closed-loop a desired total energy function. The main result of [31] is contained in the proposition below, that we prove for the sake of completeness.

*Proposition 1:* Assume there is  $M_d(q) = M_d^\top(q) \in \mathbb{R}^{n \times n}$  and a function  $V_d(q)$  that satisfy the PDEs

$$G^\perp \{ \nabla_q (p^\top M^{-1} p) - M_d M^{-1} \nabla_q (p^\top M_d^{-1} p) + 2J_2 M_d^{-1} p \} = 0 \quad (3)$$

$$G^\perp \{ \nabla V - M_d M^{-1} \nabla V_d \} = 0 \quad (4)$$

for some  $J_2(q, p) = -J_2^\top(q, p) \in \mathbb{R}^{n \times n}$  and a full rank left annihilator  $G^\perp(q) \in \mathbb{R}^{(n-m) \times n}$  of  $G$ , i.e.,  $G^\perp G = 0$  and  $\text{rank}(G^\perp) = n - m$ . Then, the system (2) in closed-loop with the IDA-PBC

$$u = (G^\top G)^{-1} G^\top (\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p) - K_v G^\top \nabla_p H_d \quad (5)$$

where  $K_v = K_v^\top > 0$ , takes the Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_v G^\top \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \quad (6)$$

where the new total energy function is

$$H_d(q, p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q). \quad (7)$$

Further, if  $M_d$  is positive definite in a neighborhood of  $q^*$  and

$$q^* = \arg \min V_d(q) \quad (8)$$

then  $(q^*, 0)$  is a stable equilibrium point of (6) with Lyapunov function  $H_d$ . This equilibrium is asymptotically stable if it is locally detectable from the output  $G^\top(q) M_d^{-1}(q) p$ . An estimate of the domain of attraction is given by  $\Omega_{\bar{c}}$  where  $\Omega_c \triangleq \{(q, p) \in \mathbb{R}^{2n} | H_d(q, p) < c\}$  and  $\bar{c}$  corresponds to the largest bounded sub-level set of  $H_d$ .

*Proof:* The first  $n$  equations of (6) are established noting from (2) that  $\dot{q} = M^{-1} p$  while  $\nabla_p H_d = M_d^{-1} p$ . On the other hand, multiplying the last  $n$  equations of (2) and (6) by the  $n \times n$  full-rank matrix  $\begin{bmatrix} G^\perp \\ G^\top \end{bmatrix}$  and setting them equal yields the equation shown at the bottom of the next page. Equating the second and the fourth identities, and using the fact that  $G^\top G$  is full rank, we obtain precisely equations (3)–(5).

To prove the stability claim we see, from (7), (8) and positivity of  $M_d$ , that  $H_d$  is a positive definite function in a neighborhood of  $(q^*, 0)$ . A straightforward calculation shows that, along trajectories of (6),  $\dot{H}_d$  satisfies

$$\dot{H}_d \leq -\lambda_{\min}\{K_v\} |G^\top M_d^{-1} p|^2 \leq 0$$

because  $J_2$  is skew-symmetric and  $K_v$  is positive definite—where  $|\cdot|$  is the Euclidean norm and  $\lambda_{\min}$  is the minimum eigenvalue. Hence,  $(q^*, 0)$  is a stable equilibrium. Furthermore, since (by definition)  $H_d$  is proper on its sub-level set  $\Omega_{\bar{c}}$ , all trajectories starting in  $\Omega_{\bar{c}}$  are bounded. Asymptotic stability, under the detectability assumption, is established invoking Barbashin–Krasovskii's theorem and the arguments

<sup>1</sup>All vectors in the paper are *column* vectors, even the gradient of a scalar function:  $\nabla_{(\cdot)} = (\partial/\partial(\cdot))$ —when clear from the context the subindex in  $\nabla$  will be omitted. We will also assume that all functions are sufficiently smooth and, whenever rank conditions are imposed, we assume that they hold uniformly with respect to their arguments.

used in the proof of [40, Lemma 3.2.8]. Finally, the estimate of the domain of attraction follows from the fact that  $\Omega_{\bar{c}}$  is the largest bounded sub-level set of  $H_d$  and LaSalle's Principle. ■

The main contribution of the present paper is the identification of a class of mechanical systems for which we can explicitly solve the PDEs (3), (4). In spite of the presence of the free matrix  $J_2$ , the kinetic energy PDE (3) is a complicated nonlinear matrix PDE. In order to solve it we propose in this paper to fix  $M_d$  transforming the PDE into an *algebraic equation* that we will solve for  $J_2$ . Toward this end, we make first the assumption that the inertia matrix  $M$  does not depend on the unactuated coordinates, thus eliminating the term  $G^\perp \nabla_q(p^\top M^{-1}p)$  of (3). Second, introducing suitable parameterizations for  $J_2$  and  $M_d$ , we will prove that—for the case of underactuation degree one—we have *enough degrees of freedom* in  $J_2$  to solve the algebraic equations. These developments are presented in Section III.

The potential energy PDE (4), even though linear, may also be difficult to solve analytically. To be able to provide an explicit solution we impose in Section IV the additional assumption that the unactuated component of the force induced by the potential energy, that is  $G^\perp \nabla V$ , is a function of *only one* of the actuated coordinates and make  $M_d$  a function of this coordinate as well. Stability will be established if we can assign a potential energy function  $V_d$  that satisfies (8). See Point 2 in Section I and Remark I.

*Remark 1:* It is clear that, for position regulation problems, our main objective is to shape the potential energy function hence we could leave  $M_d = M$  and (4) becomes  $G^\perp(\nabla V - \nabla V_d) = 0$ . If the system is underactuated our ability to modify  $V$  in this way is obviously limited, see Remark 4.3.18 of [40] and [23]. To overcome this obstacle it was proposed in IDA-PBC [31] to change also the kinetic energy term.<sup>2</sup> This is done through the modification of  $M$ —that introduces the “coupling term”  $M_d M^{-1}$  in the potential energy PDE. Our objective is then to find, among the set of positive definite  $M_d$  that solve (3), one that will allow us to shape  $V$ . The key player in this intertwined game is  $J_2$ , that we recall is free, thus providing degrees of freedom to assign  $M_d$ . See Remark 3 and [29] for additional discussions on the role of  $J_2$  for applications beyond the realm of mechanics.

<sup>2</sup>To the best of the authors' knowledge the first paper where shaping the *total* energy for stabilization of mechanical systems was proposed is [4], see [32, Ch. 3].

*Remark 2:* The class considered in the paper contains several practically relevant examples, with two of them given in Section VII. A particular case of this class has been studied in [3], and a complete characterization of all underactuation degree one mechanical systems which are feedback-equivalent to it is given in [2].

*Remark 3:* In the light of some recent misleading novelty claims reported in [41] we find necessary to clarify—again—the history of the term  $J_2$  and its role on stabilization. Already in the first publication concerning IDA-PBC [30] we indicated that, due precisely to the freedom in the choice of this term (that is intrinsic to IDA-PBC), the class of mechanical systems stabilized with IDA-PBC *strictly contains* the class stabilized via the controlled Lagrangian method of [12] or its extension [13]. It was shown that both methods coincide for a particular choice of  $J_2$ . This term was given an interpretation in terms of *gyroscopic forces* in a Lagrangian framework for the first time in [11], with a preliminary report widely distributed to the community as early as October 2 000. As openly recognized in the Introduction of [16], our work heavily inspired the modified controlled lagrangian method reported in [16], and utilized in [41]—that essentially mimics our derivations.

### III. SOLVING THE KINETIC ENERGY PDE

We now proceed to define the class of mechanical systems for which we can explicitly solve (3). Toward this end we introduce the following:

*Assumption A.1:* The system has underactuation degree one, that is,  $m = n - 1$ .

*Assumption A.2:* There exists a full rank left annihilator  $G^\perp$  of  $G$  such that

$$G^\perp \nabla_q(p^\top M^{-1}p) = 0. \quad (9)$$

Assumption A.2 essentially imposes that  $M$  does not depend on the unactuated coordinate. It is satisfied by many well-known physical examples, for instance, the Ball and Beam [20], the VTOL Aircraft [25] and the Acrobot [37]. It is easy to see that the assumption *may be satisfied*, taking (with some minor loss of generality)  $G = \begin{bmatrix} I_{n-1} \\ 0 \dots 0 \end{bmatrix}$  and introducing a partial feedback-linearization inner-loop [37]. Indeed, after some simple

$$\begin{aligned} \begin{bmatrix} G^\perp \\ G^\top \end{bmatrix} \dot{p} &= \begin{bmatrix} G^\perp \\ G^\top \end{bmatrix} (-\nabla_q H + Gu) && (\Leftarrow (2)) \\ &= \begin{bmatrix} G^\perp \\ G^\top \end{bmatrix} \left( -\frac{1}{2} \nabla_q(p^\top M^{-1}p) - \nabla V + Gu \right) && (\Leftarrow (1)) \\ &\equiv \begin{bmatrix} G^\perp \\ G^\top \end{bmatrix} (-M_d M^{-1} \nabla_q H_d + (J_2 - GK_v G^\top) \nabla_p H_d) && (\Leftarrow (6)) \\ &= \begin{bmatrix} G^\perp \\ G^\top \end{bmatrix} (-M_d M^{-1} \left[ \frac{1}{2} \nabla_q(p^\top M_d^{-1}p) + \nabla V_d \right] \right. \\ &\quad \left. + (J_2 - GK_v G^\top) M_d^{-1} p \right) && (\Leftarrow (7)) \end{aligned}$$

calculations we see that the partially feedback-linearized system takes the so-called Spong's normal form [19]

$$\ddot{q} = \begin{bmatrix} O \\ -\frac{1}{m_{nn}(q)}\psi_u(q, \dot{q}) \end{bmatrix} + \begin{bmatrix} I_{n-1} \\ -\frac{1}{m_{nn}(q)}m_u^\top(q) \end{bmatrix} u \quad (10)$$

where we have partitioned the inertia matrix and defined the function  $\psi_u \in \mathbb{R}$  as

$$M = \begin{bmatrix} \star & m_u \\ m_u^\top & m_{nn} \end{bmatrix} \\ \psi_u \triangleq e_n^\top \left[ \dot{M}\dot{q} - \frac{1}{2}\nabla_q(\dot{q}^\top M^{-1}\dot{q}) + \nabla V \right]$$

with  $m_u \in \mathbb{R}^{n-1}$ ,  $m_{nn} \in \mathbb{R}$  and  $e_n$  the  $n$ -th vector of the  $n$ -dimensional Euclidean basis. Under some conditions<sup>3</sup> this system may be written in the form (2) with a "new inertia matrix" equal identity, hence satisfying Assumption A.2. See the example in Subsection VII.A and [28] where a detailed study of the action of partial feedback linearization on the PDEs of IDA-PBC is carried out.

In the sequel we will impose some assumptions on  $M$ ,  $V$ , and  $G$  to define a class of mechanical systems for which we can solve the PDEs. These assumptions can be considerably simplified if we proceed from Spong's normal form. It is well-known that, in contrast to PBC, feedback-linearization is a fragile operation that requires exact knowledge of the systems parameters and states to ensure the "double integrator" structure. Therefore, we prefer to present the assumptions on the original system (2), stating as remarks their implication for the system in Spong's normal form.

#### A. Equivalent Representation of the PDE

We find convenient to first express (3) in an alternative equivalent form. For, we introduce a suitable parametrization of the free matrix  $J_2$ . It is clear from (3) that  $J_2$  should be *linear* in  $p$ . We make now the important observation that, without loss of generality (see Remark 4),  $J_2$  can be parameterized in the form

$$J_2 = \begin{bmatrix} 0 & \tilde{p}^\top \alpha_1 & \tilde{p}^\top \alpha_2 & \cdots & \tilde{p}^\top \alpha_{n-1} \\ -\tilde{p}^\top \alpha_1 & 0 & \tilde{p}^\top \alpha_n & \cdots & \tilde{p}^\top \alpha_{2n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\tilde{p}^\top \alpha_{n-1} & -\tilde{p}^\top \alpha_{2n-3} & \cdots & & 0 \end{bmatrix}$$

where the vector functions  $\alpha_i(q) \in \mathbb{R}^n$ ,  $i = 1, \dots, n_o$ ,  $n_o \triangleq (n/2)(n-1)$ , are free parameters and we have defined for notational convenience the (partial) coordinate

$$\tilde{p} \triangleq M_d^{-1}p. \quad (11)$$

Alternatively, we can write

$$J_2 = \sum_{i=1}^{n_o} \tilde{p}^\top \alpha_i W_i \quad (12)$$

<sup>3</sup>As explained in [28] partial feedback linearization does not necessarily preserve the Hamiltonian structure (2).

with the  $W_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, n_o$ , defined as follows. First, we construct  $n^2$  matrices of dimension  $n \times n$ , that we denote  $F^{kl} = \{f_{ij}^{kl}\}$ ,  $k, l \in \{1, 2, \dots, n\}$ , according to the rule

$$f_{ij}^{kl} = \begin{cases} 1 & \text{if } j > i, i = k \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

Notice that only  $n_o$  matrices are different from zero. Then, we define  $W^{kl} \triangleq F^{kl} - (F^{kl})^\top$ . Finally, we set (in an obvious way)

$$W_1 = W^{12}, W_2 = W^{13}, \dots, W_n = W^{1n} \\ W_{n+1} = W^{23}, \dots, W_{n_o} = W^{(n-1)n}.$$

For instance, for the case  $n = 3$ , for which also  $n_o = 3$ , we get

$$W_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad W_2 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ W_3 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Using this parameterization some simple calculations establish that the term  $G^\perp J_2$  that appears in (3) becomes

$$G^\perp(q)J_2(q, p) = \tilde{p}^\top \mathcal{J}(q)\mathcal{A}^\top(q) \quad (13)$$

where we defined

$$\mathcal{J} \triangleq [\alpha_1 : \alpha_2 : \dots : \alpha_{n_o}] \in \mathbb{R}^{n \times n_o}$$

which is a *free* matrix, and

$$\mathcal{A} \triangleq -[W_1(G^\perp)^\top, W_2(G^\perp)^\top, \dots, W_{n_o}(G^\perp)^\top] \in \mathbb{R}^{n \times n_o}. \quad (14)$$

*Proposition 2:* Under Assumptions A.1, A.2 the kinetic energy PDE (3) becomes

$$\sum_{i=1}^n \gamma_i(q) \frac{dM_d}{dq_i} = -[\mathcal{J}(q)\mathcal{A}^\top(q) + \mathcal{A}(q)\mathcal{J}^\top(q)] \quad (15)$$

where<sup>4</sup>

$$\gamma = \text{col}(\gamma_1, \dots, \gamma_n) \triangleq M^{-1}M_d(G^\perp)^\top \in \mathbb{R}^n \quad (16)$$

*Proof:* Using Assumption A.1, (13) and notation (16) the kinetic energy PDE (3) can be written as

$$\sum_{i=1}^n \gamma_i p^\top \frac{dM_d^{-1}}{dq_i} p - 2\tilde{p}^\top \mathcal{J}\mathcal{A}^\top \tilde{p} = 0.$$

That, using the relation

$$\frac{dM_d^{-1}}{dq_i} = -M_d^{-1} \frac{dM_d}{dq_i} M_d^{-1}$$

<sup>4</sup>Notice that, under Assumption A.1,  $G^\perp$  is a *row vector*.

and factoring  $\tilde{p}$ , becomes

$$\tilde{p}^\top \left[ \sum_{i=1}^n \gamma_i \frac{dM_d}{dq_i} + 2\mathcal{J}\mathcal{A}^\top \right] \tilde{p} = 0 \quad (17)$$

The proof is completed taking the symmetric part of the matrix  $\mathcal{J}\mathcal{A}^\top$  and setting the expression in brackets, which is independent of  $\tilde{p}$ , equal to zero. ■

*Remark 4:* An  $n \times n$  skew-symmetric matrix contains at most  $n_o$  nonzero different terms. Hence, the proposed  $J_2$  contains all skew-symmetric matrices which are linear in  $\tilde{p}$ , that is, all matrices of the form  $\sum_{i=1}^n \Omega_i \tilde{p}_i$ ,  $\Omega_i = -\Omega_i^\top$ , and the parametrization is done without loss of generality as claimed above.<sup>5</sup>

**B. A Parametrization of  $M_d$  That Solves the PDE**

In this section we present a parametrization of the desired inertia matrix for which there exists a  $\mathcal{J}$  that sets to zero the term in brackets of (17), that we write here for ease of reference as

$$\sum_{i=1}^n \gamma_i \frac{dM_d}{dq_i} = -2\mathcal{A}\mathcal{J}^\top \quad (18)$$

recalling that  $\gamma_i$ , as defined in (16), are functions of  $M_d$ . It is important to underscore that the set of  $M_d$  that satisfies (18) is strictly contained in the set that satisfies (15)—which, as stated in Proposition 2, characterizes all solutions of (3).<sup>6</sup> We decide to work with this smaller set because, as will be shown below, we can in this way give a simple explicit expression for  $M_d$ . Of course, all solutions of (18) are solutions of (3).

As explained in the introduction, we solve (18) as an algebraic equation in the unknown  $\mathcal{J}$  for a given  $M_d$ . Toward this end, we note from (14) and skew-symmetry of the matrices  $W_i$  that

$$G^\perp \mathcal{A} = 0. \quad (19)$$

The equation above indicates that  $\mathcal{A} \in \text{Im } G$  which, in view of (18), suggests to select  $M_d$  such that  $(dM_d/dq_i) \in \text{Im } G$  as well. The question on whether there will exist  $\mathcal{J}$  to solve (18) will depend on the rank of  $\mathcal{A}$  as shown in the following simple linear algebra lemma.

*Lemma 1:* Consider a matrix  $A \in \mathbb{R}^{n \times n_o}$  with  $n_o \geq n > 2$ ,  $\text{rank } A = n - 1$ , and such that  $w^\top A = 0$  for some  $w \in \mathbb{R}^n$ . Then, for all vectors  $x \in \mathbb{R}^n$  such that  $w^\top x = 0$  there exists a vector  $y \in \mathbb{R}^{n_o}$  such that  $x = Ay$ .<sup>7</sup>

*Proof:* First, recall that given  $A$  and  $x$ , there exists  $y$  such that  $x = Ay$  if and only if

$$\text{rank } A = \text{rank}[A|x].$$

Let us denote with  $\mathcal{S} \in \mathbb{R}^n$  the space of all  $n$ -dimensional vectors orthogonal to  $w$ , which is an  $n - 1$ -dimensional space. Now,  $w^\top A = 0$  implies that all columns of  $A$  are in  $\mathcal{S}$ . Also, from

<sup>5</sup>The space of skew-symmetric matrices, usually denoted  $so(n)$ , can be alternatively defined noting that  $so(n)$  is isomorphic to  $\mathbb{R}^{n_o}$  via the hat operator  $\hat{\cdot} : \mathbb{R}^{n_o} \rightarrow so(n)$ , and then use the basis  $\{\hat{e}_1, \dots, \hat{e}_{n_o}\}$ . We thank the reviewer for this remark.

<sup>6</sup>We thank the reviewer for this insightful observation.

<sup>7</sup>The proof of the lemma for the case  $n = 2$ , hence  $n_o = 1$ , follows from basic plane geometry considerations and is omitted for brevity.

$w^\top x = 0$  we have that  $x \in \mathcal{S}$ . Since the rank of  $A$  is  $n - 1$  there are  $n - 1$  linearly independent columns that span the whole space  $\mathcal{S}$ . Therefore, the rank of  $A$  cannot be increased by adding another vector in the same  $(n - 1)$ -dimensional space and the rank identity above holds. ■

In order to use Lemma 1 we now establish that  $\mathcal{A}$  satisfies the required rank condition.

*Lemma 2:* For the matrix  $\mathcal{A}$  defined in (14) we have

$$\text{rank } \mathcal{A} = n - 1.$$

*Proof:* We first recall (19). To establish the proof we will show that  $G^\perp$  spans the left kernel of  $\mathcal{A}$ —which will imply that  $\dim \ker \mathcal{A} = 1$ . To simplify the notation we define  $v \triangleq \text{col}(v_1, \dots, v_n) = (G^\perp)^\top$  and assume, without loss of generality, that  $v_1 \neq 0$ —see below.

From the construction of the matrices  $W^{kl}$  given in the previous section, we have

$$e_r^\top W^{kl} = \begin{cases} e_l^\top & \text{if } r = k \\ -e_k^\top & \text{if } r = l \\ 0 & \text{otherwise.} \end{cases}$$

Consider now a vector  $w = \sum_{i=1}^n a_i e_i$  in the left kernel of  $\mathcal{A}$ . We thus have

$$\begin{aligned} w^\top \mathcal{A} = 0 &\Rightarrow \sum_{i=1}^n a_i e_i^\top W^{1j} v = 0, & j = 2, \dots, n \\ &\Leftrightarrow a_1 e_j^\top v - a_j e_1^\top v, & j = 2, \dots, n \\ &\Leftrightarrow a_1 e_j^\top v - a_j v_1, & j = 2, \dots, n \end{aligned}$$

The latter is a set of  $n - 1$  equations with  $n$  unknowns (the coefficients  $a_j$ ) that, invoking the assumption of  $v_1 \neq 0$ , has the form

$$\begin{bmatrix} v_2 \\ v_1 \\ \vdots \\ v_n \\ v_1 \end{bmatrix} a_1 = \begin{bmatrix} a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Clearly, all solutions of this equation are co-linear with  $v$ , completing the proof. ■

To present the main result of this section—a parametrization of  $M_d$  such that (3) can be explicitly solved—we require the following.

*Assumption A.3:* The input matrix  $G$  is function of a single element of  $q$ , say  $q_r$ , with  $r$  an integer taking values in the set  $\{1, \dots, n\}$ .

Obviously, the assumption will be always satisfied if it is possible to (via an input change of coordinates and re-ordering of the variables  $q$ ) transform the input matrix into  $G = \begin{bmatrix} I_{n-1} \\ 0 \dots 0 \end{bmatrix}$ . On the other hand, referring to Spong’s Normal Form (10), we see that the assumption is satisfied for the partially-linearized system if the column of  $M$  corresponding to the unactuated coordinate depends only on  $q_r$ .

*Proposition 3:* Let Assumptions A.1–A.3 be satisfied. Under these conditions, for all desired (locally) positive definite inertia matrices of the form

$$M_d(q_r) = \int_{q_r^*}^{q_r} G(\mu)\Psi(\mu)G^\top(\mu)d\mu + M_d^0 \quad (20)$$

where the matrix function  $\Psi = \Psi^\top \in \mathbb{R}^{(n-1) \times (n-1)}$  and the constant matrix  $M_d^0 \in \mathbb{R}^{n \times n}$ ,  $M_d^0 = (M_d^0)^\top > 0$ , may be arbitrarily chosen, there exists a matrix  $J_2$  such that the kinetic energy PDE (3) holds in a neighborhood of  $q_r^*$ .

*Proof:* First, note that the integration limits have been chosen such that  $M_d(q_r^*) > 0$ . Therefore,  $M_d > 0$  on some neighborhood of  $q_r^*$ . Second, as  $M_d$  is only a function of  $q_r$  (18) becomes

$$\gamma_r(q) \frac{dM_d}{dq_r} = -2\mathcal{A}(q_r) \mathcal{J}^\top(q_r). \quad (21)$$

We have to prove that, for the proposed inertia matrix (20), there exists  $\mathcal{J}$  that solves this algebraic equation. This follows, invoking Lemma 1 and Lemma 2, from (19) and the fact that  $G^\perp(dM_d/dq_r) = 0$  by construction. ■

#### IV. SOLVING THE POTENTIAL ENERGY PDE

The potential energy PDE (4) can be written using (16) as

$$\gamma^\top(q) \nabla V_d = s(q) \quad (22)$$

where, to simplify the notation, we have defined the scalar function

$$s \triangleq G^\perp \nabla V. \quad (23)$$

This function, that is uniquely determined by the open-loop system, plays a critical role in the stabilization problem and we propose to take a brief pause to analyze it. First of all, notice that for all admissible equilibria  $\bar{q}$ , we have

$$s(\bar{q}) = 0. \quad (24)$$

This follows from the dynamic equations for momenta in (2), whose right hand side evaluated for  $p = 0$  becomes  $-\nabla V + Gu$ . Secondly, the vector  $\nabla V$  contains the forces induced by the potential energy, in particular,  $G^\perp \nabla V$  are those forces that cannot be (directly) affected by the control. Referring back to the original potential energy PDE (4), we recall that the mechanism to shape the potential energy is through the introduction of the term  $M_d M^{-1}$ . Since we have imposed that  $M_d$  depends on a single coordinate it is reasonable to require that  $s$  also depends only on  $q_r$ , as will be done below.

Once  $M_d$  is fixed,  $\gamma$  as given by (16) is also fixed, and (22) is a linear PDE that may be solved using, for instance, the techniques of [10]. See the examples worked out in [31]. Since our interest in this paper is to give a *constructive solution* to the stabilization problem we make two additional assumptions to be able to explicitly solve (22).

*Assumption A.4:* The vector  $\gamma$  and the function  $s$ , defined in (16), (23), respectively, are functions of  $q_r$  only, with  $q_r$  as in Assumption A.3.

*Assumption A.5:*  $\gamma_r(q_r^*) \neq 0$ .

Under Assumption A.3 and with  $M_d$  defined by (20)  $\gamma$  is a function of  $q_r$  if  $M$  is a function of  $q_r$ . Clearly, for systems in Spong's Normal Form, where the new inertia matrix is identity  $M = I_n$ , Assumption A.4 will be satisfied if  $\psi_u$  does not depend on  $p$ . Assumption A.5 is a generic condition that is imposed to ensure that the PDE (22) admits a well-defined solution

in a neighborhood of  $q_r^*$ . This stems from the fact that the  $\gamma_i$  are functions of  $q_r$  and, in view of (24),  $s$  vanishes at  $q_r^*$ .

We are in position to present our next result whose proof follows from the equivalence of (4) and (22) and some direct computations.

*Proposition 4:* Let Assumptions A.1–A.5 be satisfied and  $M_d$  be given by (20). Under these conditions, *all solutions* of the potential energy PDE (4) are given by

$$V_d(q) = \int_0^{q_r} \frac{s(\mu)}{\gamma_r(\mu)} d\mu + \Phi(z(q)) \quad (25)$$

with  $\gamma$ ,  $s$  given in (16), (23), respectively, and  $z \in \mathbb{R}^n$ ,<sup>8</sup> defined as

$$z(q) \triangleq q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)} d\mu \quad (26)$$

with  $\Phi$  an *arbitrary* differentiable function.

*Remarks 5:* Propositions 3 and 4 characterize a set of assignable energy functions of the form (1) in terms of the triplet  $\{\Psi, M_d^0, \Phi\}$ . The construction proposed for  $M_d$  ensures only  $M_d(q_r^*) > 0$ . To enlarge the domain of positivity of  $M_d$ —and consequently enlarge the domain of stability—suitable selections of  $\Psi$  and  $M_d^0$  must be found. The same comment applies to Assumption A.5 that should be satisfied in some (quantifiable, and hopefully big) neighborhood of  $q_r^*$ . We note that the functions  $(s/\gamma_r)$  and  $(\gamma_i/\gamma_r)$  appear explicitly in the control law (5) through the term  $\nabla V_d$  (implicit in  $\nabla H_d$ ).

#### V. MAIN STABILIZATION RESULT

In the previous section we proposed a parametrization of the assignable energy functions in terms of the triplet  $\{\Psi, M_d^0, \Phi\}$ . Here we will impose some additional constraints on these parameters to ensure asymptotic stability of the closed-loop. As expected, for stability we will require (besides positivity of  $M_d$ ) assignment of the desired minimum to  $V_d$ , i.e., (8). To articulate this condition we note first that the change of coordinates  $q \rightsquigarrow z + q_r e_r$  is a *diffeomorphism* that preserves the extrema—hence we analyze the potential energy function in these new coordinates, see [31] for a discussion on this issue. Now, from (25), and the fact that  $\Phi(z)$  is arbitrary, it is clear that restrictions will only be imposed on the term  $\int (s/\gamma_r)$ . Recalling (24) and Assumption A.5 we note that this function already has an extremum at  $q_r^*$ . To ensure that it is a minimum we verify that its second derivative, evaluated at  $q_r^*$ , is positive. Some simple calculations show that this condition is equivalent to the following.

*Assumption A.6:*

$$\gamma_r(q_r^*) \frac{ds}{dq_r}(q_r^*) > 0.$$

The assumption has the following interpretation. First, we recall from (23) that  $s$  represents the forces induced by the potential energy function that are unactuated. Second,  $q_r^*$  corresponds to an equilibrium that will, typically, be open-loop unstable therefore the open-loop potential energy function  $V$  will

<sup>8</sup> $z(q)$  is the, so-called, characteristic of the homogeneous part of the PDE [10]. Notice that  $z$  is an  $n$ -dimensional vector but  $z_r = 0$ . We have introduced this (awkward) definition for notational compactness.

have a *maximum* at this point and  $(ds/dq_r)(q_r^*) < 0$ . Finally, from (4) and (16) we see that  $\gamma_r$  is the element of the “coupling term,”  $G^\perp M^{-1} M_d$ , through which we can modify the (unactuated coordinates of the) open-loop potential energy (see Remark 1). In summary, Assumption A.6 reflects our ability to shape, for the purposes of stabilization, the potential energy through modification of the kinetic energy. See Examples in Section VII.

Interestingly, we will show in the proposition that the only additional condition imposed for *asymptotic* stability is as follows.

*Assumption A.7:*

$$|G^\top M^{-1} e_r(q_r^*)| \neq 0$$

Furthermore, for the particular case of quadratic  $\Phi$ , a very simple *explicit* expression for the control law is given.

*Proposition 5:* Consider the underactuated mechanical system (2) verifying Assumptions A.1–A.3. Assume there exists matrices  $\Psi$  and  $M_d^0$  such that Assumptions A.4–A.6 hold with  $M_d$  given by (20). Under these conditions, for all differentiable functions  $\Phi$  the IDA-PBC (5) ensures that the closed-loop dynamics is a Hamiltonian system of the form (6) with total energy function (7), with  $V_d$  defined in (25). Moreover,  $(q^*, 0)$  is a *locally stable* equilibrium with Lyapunov function  $H_d(q, p)$  provided the root  $q_r = q_r^*$  of  $s(q_r)$  is isolated, the function  $z(q)$  satisfies

$$z(q^*) = \arg \min \Phi(z) \quad (27)$$

and this minimum is isolated. It will be *asymptotically* stable if Assumption A.7 holds.

Furthermore, if we select

$$\Phi(z(q)) = \frac{1}{2} [z(q) - z(q^*)]^\top P [z(q) - z(q^*)]$$

with  $P = P^\top > 0$ , the control law is of the form

$$u = A_1(q) P S(q - q^*) + \begin{bmatrix} p^\top A_2(q_r) p \\ \vdots \\ p^\top A_n(q_r) p \end{bmatrix} + A_{n+1}(q_r) - K_v A_{n+2}(q_r) p \quad (28)$$

where  $K_v = K_v^\top > 0$  is free,  $S \in \mathbb{R}^{(n-1) \times n}$  is obtained removing the  $r$ -th row from the  $n$ -dimensional identity matrix, and the matrices  $A_i, i = 1, \dots, n+2$ , are of dimensions

$$A_1 \in \mathbb{R}^{(n-1) \times (n-1)}, A_2, \dots, A_n \in \mathbb{R}^{n \times n} \\ A_{n+1} \in \mathbb{R}^{(n-1) \times 1}, A_{n+2} \in \mathbb{R}^{(n-1) \times n}.$$

*Proof:* The first matching claim follows immediately from Propositions 3 and 4 and our previous derivations.

Some simple calculations establish that the *equilibrium set* of the closed-loop system is

$$\mathcal{E} \triangleq \{(\bar{q}, 0) \mid \nabla V_d(\bar{q}) = 0\}.$$

To prove stability of the desired equilibrium  $(q^*, 0)$  we note that  $M_d(q_r^*) > 0$  ensures  $H_d(q, p)$  is (locally) positive definite in  $p$ , therefore to qualify as a Lyapunov function candidate we only need to prove that  $V_d$  satisfies the minimum condition (8). As discussed above, we analyze the potential energy function in the coordinates  $z + q_r e_r$ . Condition (27) assures the required property for the  $z$  coordinates. In view of (24) and Assumption A.6—which pertain, respectively, to the first and second derivative of  $V_d$  with respect to  $q_r$  evaluated at  $q_r^*$ —the minimum is also at the desired equilibrium for  $q_r$ .

The control expression (28) is obtained, after some lengthy but straightforward calculation, from (5) selecting the free function  $\Phi(z(q))$  in (25) as indicated in the proposition.

It only remains to establish *asymptotic stability*. The derivative of  $H_d$  along the dynamics (6) is given by

$$\dot{H}_d = -p^\top M_d^{-1} G K_v G^\top M_d^{-1} p \leq 0.$$

From positivity of  $H_d$  and the expression above we conclude boundedness of all solutions starting sufficiently close to the equilibrium. We will prove that the dynamics restricted to the residual set

$$\Omega \triangleq \{(q, p) \mid G^\top(q_r) M_d^{-1}(q_r) p = 0\}$$

are described by a two-dimensional dynamical system subject to  $2n - 2$  algebraic constraints. We will show then that this planar system has no limit cycles close to the desired equilibrium. Since unbounded trajectories are ruled out by the stability property this establishes the claim of local asymptotic stability.

We proceed now to characterize the residual dynamics. First we prove that, for all trajectories in  $\Omega$ , the following holds:

$$\frac{dM_d}{dq_r} M_d^{-1} p = 0 \quad (29)$$

$$\nabla V_d = \frac{s}{\gamma_r} e_r \quad (30)$$

$$J_2 M_d^{-1} p = 0. \quad (31)$$

*Proof of (29):* Equation (29) follows from the fact that  $(dM_d/dq_r) \in \text{Im} G$  and the definition of  $\Omega$ .

*Proof of (30):* We note first the following chain of implications:

$$\begin{aligned} G^\top M_d^{-1} p = 0 &\Leftrightarrow \exists \nu : M_d^{-1} p = \nu (G^\perp)^\top \\ &\Rightarrow \dot{q} = M^{-1} p = \nu M^{-1} M_d (G^\perp)^\top = \nu \gamma \\ &\Rightarrow \dot{q}_r = e_r^\top \dot{q} = \nu \gamma_r \\ &\Leftrightarrow \nu = \frac{1}{\gamma_r} \dot{q}_r \\ &\Rightarrow \dot{q} = \frac{\gamma}{\gamma_r} \dot{q}_r \end{aligned} \quad (32)$$

where we have replaced the expression of  $\nu$  obtained in the fourth implication into the second one to get the last equation. Integrating the last equation in (32) we get the first  $n - 1$  algebraic equations that constrain the residual dynamics

$$q - \int_0^{q_r} \frac{\gamma(\mu)}{\gamma_r(\mu)} d\mu = z^0 \quad (33)$$

where  $z^0 = \text{col}(z_1^0, \dots, z_n^0) \in \mathbb{R}^n$  is a constant vector, with  $z_r^0 = 0$ .<sup>9</sup>

Computing the gradient of  $V_d$  from (25) we get

$$\nabla V_d = \frac{s}{\gamma_r} e_r + \left( I_n - \frac{1}{\gamma_r} e_r \gamma_r^\top \right) \nabla \Phi(z^0)$$

where we have used the fact that  $z = z^0$  to get  $\nabla \Phi(z) = \nabla \Phi(z^0)$ —a constant vector. Now, any point  $(\bar{q}, 0) \in \mathcal{E}$  is obviously in the residual dynamics, therefore, we can invoke (24) to get

$$\begin{aligned} s(\bar{q}_r) &= 0, \nabla V_d(\bar{q}) = 0 \\ \Rightarrow \left[ I_n - \frac{1}{\gamma_r(\bar{q}_r)} e_r \gamma_r^\top(\bar{q}_r) \right] \nabla \Phi(z^0) &= 0 \\ \Leftrightarrow \nabla \Phi(z^0) &= \kappa e_r, \kappa \in \mathbb{R}. \end{aligned} \quad (34)$$

Equation (30) is obtained replacing  $\nabla \Phi(z^0)$  in the expression on  $\nabla V_d$  above.

*Proof of (31):* From the first implication of (32) we have

$$p = M_d(G^\perp)^\top \nu \quad (35)$$

and, consequently

$$\begin{aligned} J_2 M_d^{-1} p &= J_2 (G^\perp)^\top \nu \\ &= -\mathcal{A} \mathcal{J}^\top M_d^{-1} p \nu \\ &= \frac{1}{2} \gamma_r \frac{dM_d}{dq_r} M_d^{-1} p \nu \end{aligned}$$

where we have used (13) and skew-symmetry of  $J_2$  to get the second identity, and (21) for the third. The proof is completed invoking (29).

To derive the residual dynamics let us repeat here, for convenience, the momentum equations of the closed-loop system (6), (7):

$$\dot{p} = -M_d M^{-1} \nabla_q H_d + (J_2 - GK_v G^\top) M_d^{-1} p.$$

Recalling that

$$\nabla_q H_d = \frac{1}{2} \nabla_q \left( p^\top M_d^{-1} \frac{dM_d}{dq_r} M_d^{-1} p \right) e_r + \nabla V_d$$

and replacing (29), (30) and (31) yields

$$\dot{p} = -M_d M^{-1} e_r \frac{s}{\gamma_r}. \quad (36)$$

Now, differentiating (35) we get

$$\begin{aligned} \dot{p} &= \dot{M}_d (G^\perp)^\top \nu + M_d \left( \frac{dG^\perp}{dt} \right)^\top \nu + M_d (G^\perp)^\top \dot{\nu} \\ &= M_d \left( \frac{dG^\perp}{dq_r} \right)^\top \gamma_r \nu^2 + M_d (G^\perp)^\top \dot{\nu} \end{aligned} \quad (37)$$

where we have used  $(dM_d/dq_r) \in \text{Im}G$  to set to zero the first right hand term of the first identity, and used the third line of (32), namely

$$\dot{q}_r = \gamma_r \nu \quad (38)$$

to obtain the right hand term of the second equation. Equating (36) and (37) and eliminating  $M_d$  yields

$$(G^\perp)^\top \dot{\nu} = -M^{-1} e_r \frac{s}{\gamma_r} - \left( \frac{dG^\perp}{dq_r} \right)^\top \gamma_r \nu^2.$$

Pre-multiplying by the  $n \times n$  full-rank matrix  $\begin{bmatrix} G^\perp \\ G^\top \end{bmatrix}$  it is possible to show that this equation is equivalent to the differential equation

$$\dot{\nu} = f_1(q_r) + f_2(q_r) \nu^2 \quad (39)$$

together with the  $n - 1$  algebraic constraints

$$G^\top \left[ M^{-1} e_r s + \gamma_r^2 \left( \frac{dG^\perp}{dq_r} \right)^\top \nu^2 \right] = 0 \quad (40)$$

where

$$f_1 \triangleq -\frac{G^\perp}{|G^\perp|^2} M^{-1} e_r \frac{s}{\gamma_r} \quad f_2 \triangleq -\frac{G^\perp}{|G^\perp|^2} \left( \frac{dG^\perp}{dq_r} \right)^\top \gamma_r.$$

Summarizing, we have established that the residual dynamics are described by the 2-dimensional dynamical system (38), (39) subject to the algebraic constraints (40).

To complete the proof of stability it only remains to show that the planar system  $(q_r, \nu)$  has no limit cycles (in a neighborhood of  $q_r^*$ ). For, we first note from (38)—and the fact that  $\gamma_r \neq 0$ —that on any half vertical line  $\dot{q}_r$  does not change sign. Therefore, if a limit cycle exists it has to encircle the point  $(q_r^*, 0)$ . Assume there exists one, then there is a time  $T$  such that  $(q_r(T), \nu(T)) = (q_r^0, 0)$  for some  $q_r^0 \in \mathbb{R}$ . On the other hand, the constraint (40)—which obviously holds along all trajectories—imposes  $G^\top M^{-1} e_r s(q_r^0) = 0$ . Now, Assumption A.7 ensures<sup>10</sup>

$\exists \delta > 0$ , such that

$$G^\top M^{-1} e_r s(q_r) \neq 0, \forall q_r \in [q_r^* - \delta, q_r^* + \delta], q_r \neq q_r^*.$$

Fix this  $\delta$  and select  $\epsilon > 0$  such that all trajectories starting in the ball  $|\text{col}(q_r, \nu)| \leq \epsilon$  remain in the ball of radius  $\delta$  for all time—the existence of  $\epsilon$  follows from the stability of the system. Since the trajectories cannot cross the plane  $\nu = 0$  through the interval  $[q_r^* - \delta, q_r^* + \delta]$  limit cycles cannot exist and trajectories have to converge to  $(q_r^*, 0)$  completing the proof. ■

*Remark 6:* To quantify the domain of attraction, e.g., to obtain an (almost) global version of the asymptotic stability claim, we need to rule out the existence of limit cycles in the whole space  $(q_r, \nu)$  as well as stable equilibria, different from the desired one. See the example of Section VII.B.

## VI. IMPLEMENTATION OF THE CONTROLLER VIA POSITION FEEDBACK

In this section, we prove that, using the recently introduced method of immersion and invariance [6], [22], we can design a speed estimator that allows the implementation of the proposed

<sup>9</sup>From  $z$ , we conclude that the characteristic  $z(q)$  is constant.

<sup>10</sup>We recall that  $s(q_r^*) = 0$  and this root is isolated.

controllers *measuring only position* for the following particular class of systems:

$$\begin{aligned}\dot{q} &= M^{-1}(q_r)p \\ \dot{p} &= \eta(q_r) + G(q_r)u\end{aligned}\quad (41)$$

that clearly satisfies Assumptions A.1–A.4 and contains the examples considered in Section VII. To ensure stability we will impose the (rather weak) additional assumption that the matrix  $\Psi$  (that defines  $M_d$ ) is *bounded*.

*Proposition 6:* Consider the system (41) assuming, without loss of generality, that  $G$  is bounded.<sup>11</sup> Select *bounded*  $\Psi$  and  $M_d^0$  in (20) such that Assumptions A.5 and A.6 hold. Define the *position feedback controller*

$$\begin{aligned}u &= A_1(q)PS(q - q^*) \\ &+ \begin{bmatrix} (\hat{p} + \lambda q)^\top A_2 (\hat{p} + \lambda q) \\ \vdots \\ (\hat{p} + \lambda q)^\top A_n (\hat{p} + \lambda q) \end{bmatrix} \\ &+ A_{n+1} - K_v A_{n+2} (\hat{p} + \lambda q)\end{aligned}\quad (42)$$

where  $\lambda > 0$ , and  $\hat{p}$  is an estimate of  $p - \lambda q$  generated via

$$\dot{\hat{p}} = \eta + Gu - \lambda M^{-1}(\hat{p} + \lambda q).\quad (43)$$

Then there exists a neighborhood of the point  $(q^*, 0, -\lambda q^*)$  such that all trajectories of the closed-loop system starting in this neighborhood are bounded and satisfy

$$\lim_{t \rightarrow \infty} (q(t), p(t), \hat{p}(t)) = (q^*, 0, -\lambda q^*).$$

Furthermore, if Assumption A.7 holds and the full state feedback controller (28) ensures global asymptotic stability then the neighborhood is the whole space  $\mathbb{R}^{3n}$ , thus boundedness and convergence are *global*.

*Proof:* To carry out the proof we follow *verbatim* the Immersion and Invariance procedure of [6], [22]. For, we define the partial coordinate

$$\zeta = \hat{p} - p + \lambda q$$

whose derivative, upon replacement of the system dynamics (41) and the estimator above, takes the simple form

$$\dot{\zeta} = -\lambda M^{-1}\zeta.$$

From boundedness and positivity of  $M$  we immediately conclude that  $\zeta(t) \rightarrow 0$  exponentially fast—for instance, evaluating the derivative of  $|\zeta|^2$ .

We will show now that the proposed position-feedback control law can be expressed as the sum of the full-state feedback control plus a perturbation term that depends on  $\zeta$ , that as shown above, exponentially goes to zero. Indeed, using  $\hat{p} + \lambda q = \zeta + p$ , the controller (42) can be written as  $u = u_*(q, p) + \chi(q_r, p, \zeta)$ ,

<sup>11</sup>This assumption is without loss of generality, because we can always re-define the control signal with a scalar normalizing factor without affecting the stabilizability properties.

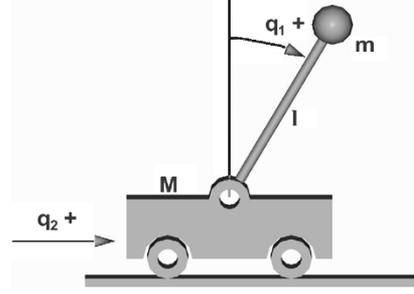


Fig. 1. Pendulum on a cart.

where we use  $u_*(q, p)$  to denote the full state feedback controller (28), and we have defined

$$\chi(q_r, p, \zeta) \triangleq \begin{bmatrix} \zeta^\top A_2 \zeta + \zeta^\top [A_2 + A_2^\top] p \\ \vdots \\ \zeta^\top A_n \zeta + \zeta^\top [A_n + A_n^\top] p \end{bmatrix} - K_v A_{n+2} \zeta.\quad (44)$$

Replacing (44) in (41) and denoting  $x = [q^\top, p^\top]^\top$ , we have that the closed-loop system can be written in the perturbed form

$$\dot{x} = f_*(x) + \begin{bmatrix} 0 \\ G\chi(q_r, p, \zeta) \end{bmatrix}$$

where  $\dot{x} = f_*(x)$  are the dynamics of the system in closed-loop with the full state feedback controller. From Proposition 4, we have that the latter is asymptotically stable. Furthermore, the disturbance term is such that  $G\chi(q_r, p, 0) = 0$ . Invoking well-known results of asymptotic stability of cascaded systems [36] completes the proof of local asymptotic stability.

To complete the global claim we invoke the recent result of [38], and see that the proof will be completed if we can establish boundedness of the trajectories  $x(t)$ . Toward this end, we proceed as follows. As shown in Proposition 5 the desired total energy qualifies as Lyapunov function for the unperturbed system  $\dot{x} = f_*(x)$ . Computing its time derivative for the complete system we get the bounds

$$\begin{aligned}\dot{H}_d &\leq -\lambda_{\min}\{K_v\}[G^\top \hat{p}]^2 + \hat{p}^\top G\chi(q_r, p, \zeta) \\ &\leq \kappa |\hat{p}| |\chi(q_r, p, \zeta)|\end{aligned}$$

where the second bound has been obtained using the assumption of bounded  $G$ ,  $\kappa$  is a positive constant, and we recall that  $\hat{p}$  is defined in (11). From the expression above it is clear that the key step to prove boundedness of trajectories is to establish a suitable bound for  $\chi(q_r, p, \zeta)$ . The third right hand term of (44) is an exponentially decaying disturbance whose effect on the inequality above can be dominated invoking standard (Young's inequality) arguments. The second right hand term stems from the quadratic term in  $p$  of  $\nabla_q H_d$ , more precisely from the term  $-(1/2)\hat{p}^\top (dM_d/dq_r)\hat{p}e_r$ . Replacing (20) we see, after some simple calculations, that it has the form

$$\begin{aligned}-\frac{1}{2}(G^\top G)^{-1}G^\top [\zeta^\top M_d^{-1}G\Psi G^\top M_d^{-1}\zeta \\ + 2\zeta^\top M_d^{-1}G\Psi G^\top \hat{p}]e_r.\end{aligned}$$

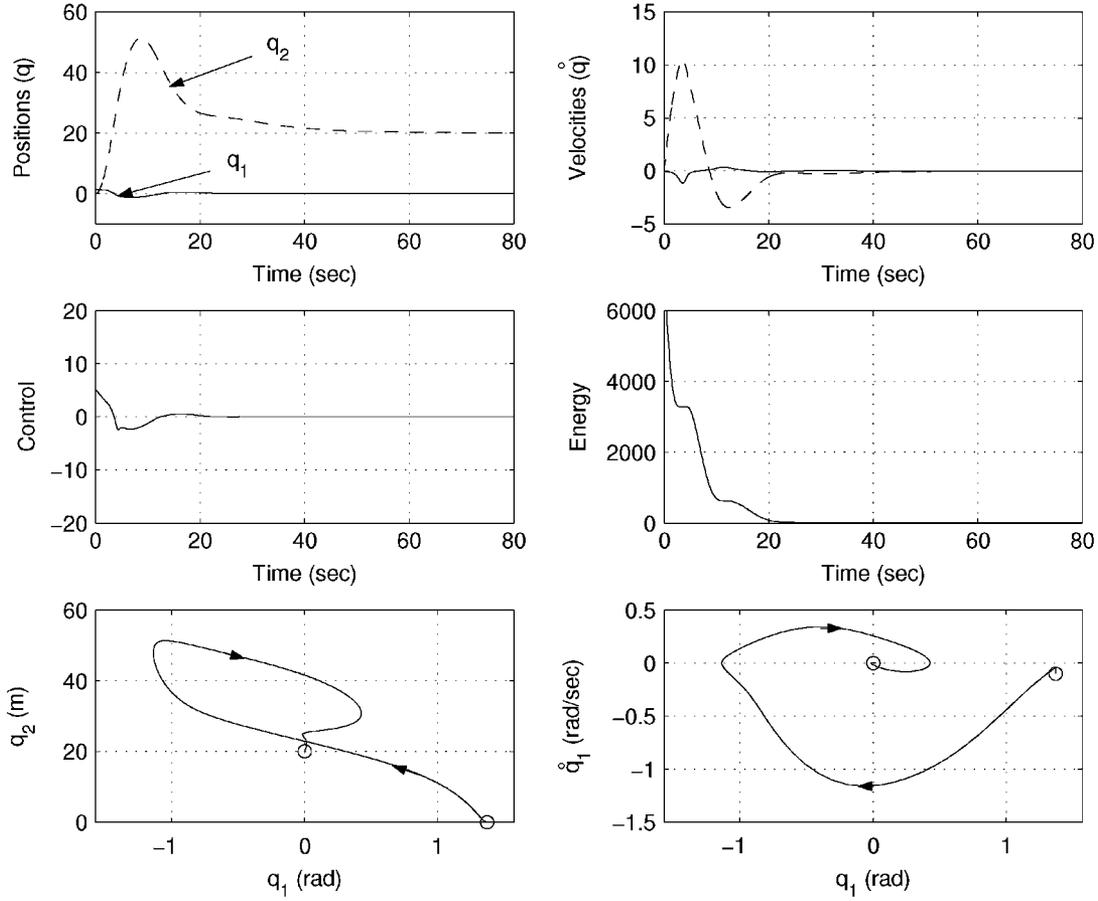


Fig. 2. Trajectories with the pendulum starting near the horizontal  $(q(0), p(0)) = (\pi/2 - 0.2, -0.1, 0.1, 0)$ , (full state feedback.)

If  $\Psi$  is bounded—hence the need for the additional assumption—this term is (linearly) bounded by  $(|G^\top \tilde{p}| + 1)\epsilon_t$  where  $\epsilon_t$  is an exponentially decaying term.

From the bound  $H_d \geq (1/2)\tilde{p}^\top M_d \tilde{p}$  and the remarks made above we can prove the existence of an *integrable* function  $k(t)$  such that  $\dot{H}_d \leq k(t)H_d$ , from which, invoking the Comparison Lemma [40] we immediately conclude boundedness of trajectories and complete the proof. ■

## VII. EXAMPLES

In this section we apply the preceding design methodology to the problem of stabilizing the positions of the pendulum on a cart, and an arbitrary position with zero roll angle and zero speed of the vertical takeoff and landing aircraft. For other applications we refer the reader to [1]–[3].

### A. The Pendulum on a Cart

The dynamic equations of the pendulum on a cart depicted in Fig. 1 are given by (2) with  $n = 2, m = 1$ , hence satisfying Assumption A.1, and

$$M(q_1) = \begin{bmatrix} 1 & b \cos q_1 \\ b \cos q_1 & m_3 \end{bmatrix}, V(q_1) = a \cos q_1 \\ G = e_2 \quad a = \frac{g}{\ell}, b = \frac{1}{\ell}, m_3 = \frac{M+m}{m\ell^2}$$

where  $q_1, q_2$  denote the cart position and the pendulum angle with the upright vertical,  $m$  and  $\ell$  are, respectively, the mass and the length of the pendulum,  $M$  is the mass of the cart and  $g$  is the gravity acceleration. The equilibrium to be stabilized is the upward position of the pendulum with the cart placed in *any desired location*, which corresponds to  $q_1^* = 0$  and an arbitrary  $q_2^*$ .

Since  $G^\perp = e_1^\top$  this system clearly *does not* satisfy Assumption A.2. However, applying partial feedback linearization, as done in [39], we can rewrite the system as

$$\ddot{q} = a \sin q_1 e_1 + \begin{bmatrix} -b \cos q_1 \\ 1 \end{bmatrix} u \quad (45)$$

where  $u$  is the new control input. This system is still a mechanical system of the form (2) with the same potential energy  $V$ , but with  $M = I$  and  $G = \text{col}(-b \cos q_1, 1)$ . Hence, satisfies Assumptions A.2, A.3 with  $r = 1$ . Notice that  $G^\perp = [1, b \cos q_1]$ .

Let us denote

$$M_d(q_1) = \begin{bmatrix} m_{11}(q_1) & m_{12}(q_1) \\ m_{12}(q_1) & m_{22}(q_1) \end{bmatrix}.$$

Then

$$\gamma = \begin{bmatrix} m_{11} + m_{12}b \cos q_1 \\ m_{12} + m_{22}b \cos q_1 \end{bmatrix} \quad s = -a \sin q_1$$

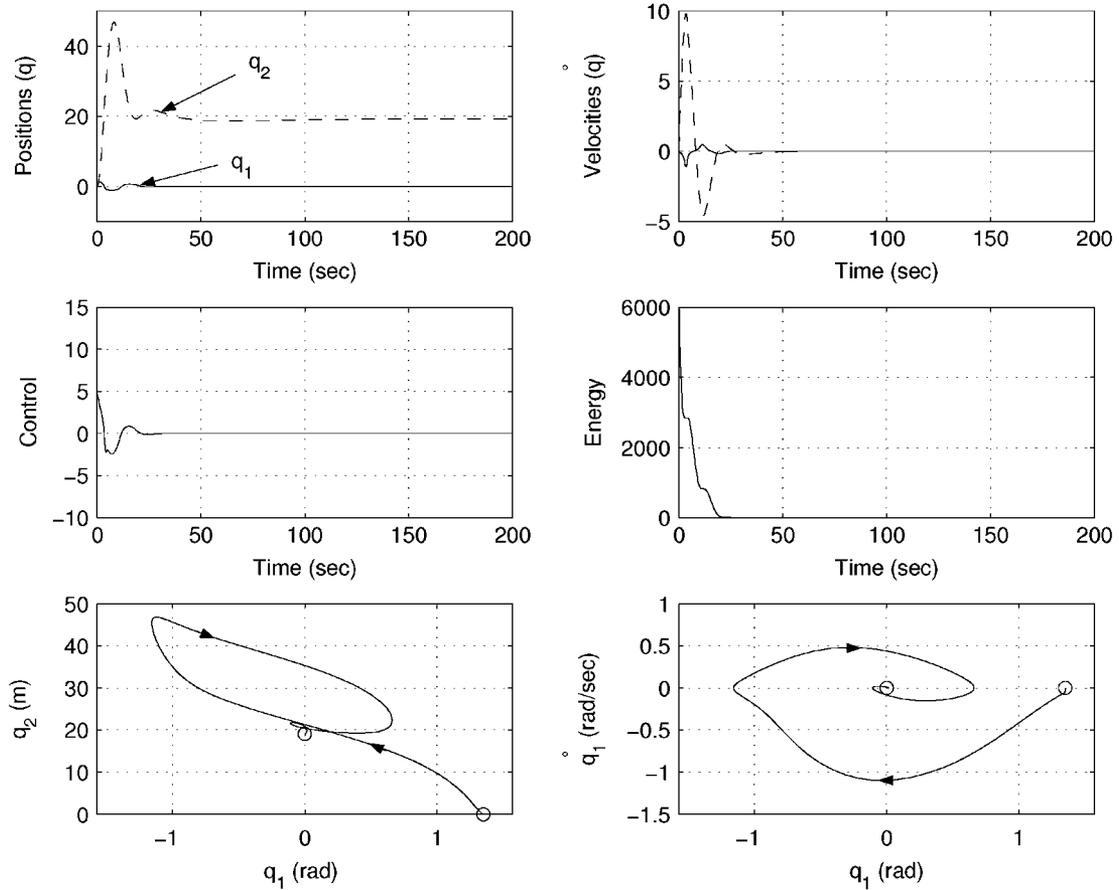


Fig. 3. Trajectories with the pendulum starting near the horizontal  $(q(0), p(0)) = (\pi/2 - 0.2, -0.1, 0.1, 0)$ , (position feedback.)

thus verifying Assumption A.4. We will construct the matrix as suggested in Proposition 3. For simplicity we take  $\Psi$  to be a scalar function and compute from (20)

$$M_d = \int_{q_1^*}^{q_1} \Psi(\mu) \begin{bmatrix} b^2 \cos^2 \mu & -b \cos \mu \\ -b \cos \mu & 1 \end{bmatrix} d\mu + M_d^0.$$

Unfortunately, the function  $\Psi$  cannot be taken to be a constant, say  $\Psi^0$ , because this leads to

$$m_{11} = \Psi^0 b^2 \left( \frac{1}{2} \cos q_1 \sin q_1 + q_1 \right) - \Psi^0 b^2 \left( \frac{1}{2} \cos q_1^* \sin q_1^* + q_1^* \right) + m_{11}^0$$

that contains a linear term in  $q_1$  that is clearly unbounded. To select this function we look at the stability condition of Assumption A.6, which imposes  $\gamma_1(0) < 0$ . We propose then  $\Psi(q_1) = -k \sin q_1$ , with  $k > 0$  a free parameter that yields, on one hand,  $\gamma_1 = -(kb^2/6) \cos^3 q_1$ —that satisfies Assumption A.6. On the other hand, we get

$$M_d = \begin{bmatrix} \frac{kb^2}{3} \cos^3 q_1 & -\frac{kb}{2} \cos^2 q_1 \\ -\frac{kb}{2} \cos^2 q_1 & k \cos q_1 + m_{22}^0 \end{bmatrix} \quad (46)$$

with  $m_{22}^0 \geq 0$  arbitrary. Furthermore, it is easy to prove that this matrix is positive definite and bounded for all  $q_1 \in ((-\pi/2), (\pi/2))$ —a domain where Assumption A.5 holds and to which we will restrict our system to operate.

Finally, we compute  $G^\top M^{-1} e_1 = -b \cos q_1$  that clearly verifies Assumption A.7.

We have the following result.

**Proposition 7:** A set of energy functions of the form (7) assignable via IDA-PBC to the (partially feedback-linearized) pendulum on a cart system (45) is characterized by the *locally* positive definite and bounded inertia matrix (46), for all  $m_{22}^0 \geq 0$ ,  $k > 0$ , and the potential energy function

$$V_d = \frac{3a}{kb^2 \cos^2 q_1} + \frac{P}{2} \left[ q_2 - q_2^* + \frac{3}{b} \ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb} \tan q_1 \right]^2$$

that satisfies (8) for all constants  $P > 0$ .

Moreover, the IDA-PBC

$$u = A_1(q_1)P(q_2 - q_2^*) + p^\top A_2(q_1)p + A_3(q_1) - K_v A_4(q_1)p \quad (47)$$

where the matrices  $A_i(q_1)$  are given in the Appendix, ensures *asymptotic stability* of the desired equilibrium  $(0, q_2^*, 0, 0)$  with a domain of attraction containing the set  $(-\pi/2, \pi/2) \times \mathbb{R}^3$ .

*Proof:* The expressions for the control and the potential energy function are obtained, from Proposition 5, with a quadratic  $\Phi$ .

Taking into account Proposition 5 it only remains to prove the claim regarding the estimate of the domain of attraction.

For, we note that  $H_d$  is a radially unbounded function on the set  $(-\pi/2, \pi/2) \times \mathbb{R}^3$ , hence any trajectory that starts inside this set will remain in it—eventually converging to the desired equilibrium. This completes the proof. ■

Simulations were made with the normalized values  $a = b = 1$ , the constant for the damping injection was fixed to  $K_v = 0.01$  and the other parameters given by  $m_{22}^0 = k = 0.01$  and  $P = 1$ . We tested a set of “limiting” initial conditions with the pendulum starting near the horizontal  $(q(0), p(0)) = (\pi/2 - 0.2, -0.1, 0.1, 0)$  and the desired position for the cart  $q_2^* = 20$ , that is, very far away from the origin. The result for *full state feedback* is shown in Fig. 2 where an excellent performance is observed. We should underscore that, in contrast with the proposed scheme, most of the existing controllers for this problem can stabilize the upward position of the pendulum with zero cart velocity, but the cart *position* cannot be arbitrarily fixed. Also, we would like to bring to the readers attention the shape of the control action, which is a smooth low amplitude signal that moves the cart at the right time instants in the right direction. Again, this should be compared with other controllers, e.g., those stabilizing the homoclinic orbit, where the control action is essentially bang-bang—even when the initial conditions of the pendulum are in the upper half plane.

We also have made simulations of the proposed *position feedback* controller. The result is shown in Fig. 3. As expected, a slower performance is observed, due to the time needed by the nonlinear speed estimator to converge. To improve the transient performance the gain  $\lambda$  of the nonlinear speed estimator (43) was taken as a diagonal matrix with values  $\lambda = \text{diag}\{0.02, 0.01\}$ .

*Remark 7:* There exists an obstacle for swinging up the pendulum with the proposed technique because Assumptions A.5 and A.6 are impossible to satisfy—with a positive definite  $M_d$ —outside the interval  $[-(\pi/2), (\pi/2)]$ . Indeed, Assumption A.6 requires  $\gamma_1$  to be negative at zero, while Assumption A.5 hampers the function to cross through zero, consequently  $\gamma_1$  should always be negative. Unfortunately, this is in contradiction with  $M_d > 0$ , that requires  $m_{11} > 0$ , because

$$\gamma_1 \left( \frac{\pi}{2} \right) = m_{11} \left( \frac{\pi}{2} \right).$$

The obstacle, that comes from the use of partial feedback linearization, obviously prevails *independently* of the choice of the free functions  $\{\Psi, \Phi, M_d^0\}$ . It is interesting to compare this example with the Inertia Wheel Pendulum, that was (almost) globally stabilized via IDA-PBC in [31], and which differs from (45) only in the input matrix— $G = [-1 \ 1]^T$  in the latter.

*Remark 8:* As explained in Subsection III-B the proposed  $M_d$  is a particular case of all possible solutions of the kinetic energy PDE (3), or equivalently (15). (This is true even if we restrict  $M_d$  to be function of  $q_1$  only.) Some simple calculations for the Cart on the Pendulum example show that the general solution is given by any triplet  $\{m_{11}, m_{12}, m_{22}\}$  satisfying

$$\frac{dm_{11}}{dq_1} + \left( 2 \frac{dm_{12}}{dq_1} + b \cos q_1 \frac{dm_{22}}{dq_1} \right) b \cos q_1 = 0.$$

Obviously, all matrices of the form (20) satisfy this equation but not *vice versa*.

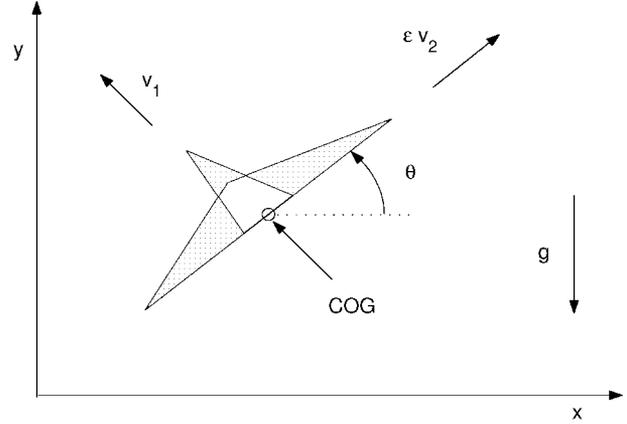


Fig. 4. Schematic picture for the VTOL problem.

### B. Strongly Coupled Vertical Takeoff and Landing Aircraft

Our second example is the vertical takeoff and landing (VTOL) aircraft depicted in Fig. 4 whose dynamics are given by [20], [35]

$$\begin{aligned} \ddot{x} &= -\sin \theta v_1 + \epsilon \cos \theta v_2 \\ \ddot{y} &= \cos \theta v_1 + \epsilon \sin \theta v_2 - g \\ \ddot{\theta} &= v_2 \end{aligned} \quad (48)$$

where  $\theta$  is the roll angle, the VTOL moves in the  $(x, y)$  plane,  $g$  is the gravity acceleration,  $v_1, v_2$  are the control actions and  $\epsilon$  is a parameter that captures the effect of the “sloped” wings—and clearly induces a coupling between the vertical and the roll dynamics. Control requirements for the VTOL are typically expressed in terms of asymptotic regulation from any initial condition to an arbitrary position with zero roll angle and zero speed, that is, the asymptotic stabilization of all equilibria of the form  $\text{col}(x^*, y^*, 0, 0, 0, 0)$ .

To apply the theory developed in this paper we introduce the (globally defined) change of input [35]

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{g}{\epsilon} \begin{bmatrix} -\cos \theta \\ \sin \theta \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} -\sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{bmatrix} u$$

where  $u = [u_1, u_2]^T$  is the new control vector. This transformation yields the VTOL dynamics

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= \frac{g}{\epsilon} \sin q_3 e_3 + Gu \end{aligned} \quad (49)$$

where we have introduced the notation  $q = [x, y, \theta]^T, p = [\dot{x}, \dot{y}, \dot{\theta}]^T$  and defined the matrix

$$G \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\epsilon} \cos q_3 & \frac{1}{\epsilon} \sin q_3 \end{bmatrix}.$$

The system is in the form (2) with  $n = 3, m = 2, M = I_3, V = g/\epsilon \cos q_3$ —hence, satisfying Assumption A.1–A.3 with  $r = 3$ . Notice that  $G^\perp = [\cos q_3, \sin q_3, -\epsilon]$ , thus Assumption A.4, with  $s = g \sin q_3$ , also holds.

In Proposition 3, we fixed  $M_d$ —according to (20). Our interest with this example is to provide an alternative parametrization of the set of assignable energy functions that satisfy (18),

still keeping  $M_d$  a function of  $q_3$  only. We recall that the  $3 \times 3$  matrix  $\mathcal{J}^\top$  is free, hence, (18) can be alternatively expressed as  $(dM_d/dq_3)e_i \in \text{Im}\mathcal{A}, i = 1, 2, 3$ , where

$$\mathcal{A} = \begin{bmatrix} -\sin q_3 & \epsilon & 0 \\ \cos q_3 & 0 & \epsilon \\ 0 & \cos q_3 & \sin q_3 \end{bmatrix}.$$

To verify the key stability Assumption A.6, and at the same time Assumption A.5, we fix  $\gamma_3 = \gamma_3^0$ , a positive constant. Since  $\gamma_3 = G^\perp M_d e_3$  this restricts the third column of  $M_d$  to have a constant projection along  $G^\perp$ . Then, define the two remaining columns of  $M_d$  to obtain a positive definite matrix, with derivatives also living in the range space of  $\mathcal{A}$ .

The lemma below characterizes all admissible 3-dimensional vectors  $f = M_d e_3$ .<sup>12</sup>

*Lemma 3:* Fix a constant  $\gamma_3^0 \neq 0$  and define the set

$$\left\{ f \triangleq \text{col}(f_1, f_2, f_3) : \mathbb{R} \rightarrow \mathbb{R}^3 | G^\perp f = \gamma_3^0 \text{ and } \frac{df}{dq_3} \in \text{Im}\mathcal{A} \right\}.$$

All elements of the set are generated as

$$f_1 = [\gamma_3^0 + \epsilon f_3] \cos q_3 \quad f_2 = [\gamma_3^0 + \epsilon f_3] \sin q_3$$

where  $f_3$  is an arbitrary differentiable function.

*Proof:* Express  $f_1, f_2$  in polar coordinates as

$$f_1 = r \cos[\phi] \quad f_2 = r \sin[\phi].$$

Using the trigonometric identity  $\cos(A - B) = \cos A \cos B + \sin A \sin B$ , and the definition of  $G^\perp$  we see that  $G^\perp f = \gamma_3^0$  if and only if

$$r \cos[q_3 - \phi] = \gamma_3^0 + \epsilon f_3.$$

This equation has a solution for some  $r$  and  $\phi$  if and only if

$$r \geq \gamma_3^0 + \epsilon f_3. \quad (50)$$

Extracting  $\phi$  from the expression above, replacing it in  $f_1, f_2$  and using the identity  $\sin[\arccos A] = \sqrt{1 - A^2}$  we conclude that all functions which satisfy  $G^\perp f = \gamma_3^0$  are generated as

$$\begin{aligned} f_1 &= [\gamma_3^0 + \epsilon f_3] \cos q_3 + \sqrt{r^2 - [\gamma_3^0 + \epsilon f_3]^2} \sin q_3 \\ f_2 &= [\gamma_3^0 + \epsilon f_3] \sin q_3 - \sqrt{r^2 - [\gamma_3^0 + \epsilon f_3]^2} \cos q_3 \end{aligned}$$

where  $f_3$  is arbitrary and  $r$  is any function verifying (50).

Now, since  $\text{rank}\mathcal{A} = 2$  and  $G^\perp \in \ker \mathcal{A}$  then  $(df/dq_3) \in \text{Im}\mathcal{A}$  if and only if  $G^\perp(df/dq_3) = 0$ . Some simple calculations show that this is true if and only if (50) holds with the equality sign. ■

In the sequel, we pick one element of the class characterized in the lemma above and—for the sake of simplicity—choose

the function that parameterizes the set to be a constant, that is  $f_3 = k_2$ , this yields

$$M_d e_3 = \begin{bmatrix} k_1 \cos q_3 \\ k_1 \sin q_3 \\ k_2 \end{bmatrix} \quad (51)$$

where, for ease of notation, we have defined

$$k_1 \triangleq \gamma_3^0 + \epsilon k_2. \quad (52)$$

Notice that, since Assumption A.6 imposes  $\gamma_3^0 > 0$  we require  $k_1 - \epsilon k_2 > 0$ . We still have to decide the two remaining columns of the inertia matrix, that is (using the notation  $M_d = \{m_{ij}\}$ ) we look for functions  $m_{ij}, i, j = 1, 2$  such that

$$\begin{bmatrix} \frac{dm_{11}}{dq_3} \\ \frac{dm_{12}}{dq_3} \\ -k_1 \sin q_3 \end{bmatrix}, \begin{bmatrix} \frac{dm_{12}}{dq_3} \\ \frac{dm_{22}}{dq_3} \\ k_1 \cos q_3 \end{bmatrix} \in \text{Im}\mathcal{A}. \quad (53)$$

Our first observation is that, since  $e_3$  is not in the range space of  $\mathcal{A}$ , we cannot take the  $m_{ij}$  to be constant.

Computing for the first column, we get

$$\begin{aligned} \frac{dM_d}{dq_3} e_1 \in \text{Im}\mathcal{A} &\Leftrightarrow \frac{dm_{11}}{dq_3} = -b(q_3) \sin q_3 \\ &\frac{dm_{12}}{dq_3} = b(q_3) \cos q_3 - k_1 \epsilon \end{aligned}$$

for some function  $b$ . Similarly for the second column

$$\begin{aligned} \frac{dM_d}{dq_3} e_2 \in \text{Im}\mathcal{A} &\Leftrightarrow \frac{dm_{22}}{dq_3} = c(q_3) \cos q_3 \\ &\frac{dm_{12}}{dq_3} = -c(q_3) \sin q_3 + k_1 \epsilon \end{aligned}$$

for some function  $c$ . Equating both expressions of  $(dm_{12}/dq_3)$  we get  $b \cos q_3 + c \sin q_3 = 2k_1 \epsilon$ . To satisfy this equation we pick

$$b = 2k_1 \epsilon \cos q_3, c = 2k_1 \epsilon \sin q_3.$$

Replacing these functions into the expressions above and integrating we finally get

$$M_d = \begin{bmatrix} k_1 \epsilon \cos^2 q_3 + k_3 & k_1 \epsilon \cos q_3 \sin q_3 & k_1 \cos q_3 \\ k_1 \epsilon \cos q_3 \sin q_3 & -k_1 \epsilon \cos^2 q_3 + k_3 & k_1 \sin q_3 \\ k_1 \cos q_3 & k_1 \sin q_3 & k_2 \end{bmatrix} \quad (54)$$

where  $k_3 > 0$  is an integration constant added to ensure positivity of  $M_d$ .

Finally, we compute

$$G^\top M^{-1} e_3 = (1/\epsilon) \text{col}(-\cos q_3, \sin q_3)$$

that clearly verifies Assumption A.7.

We are in position to present the following proposition.

*Proposition 8:* A set of energy functions of the form (7) assignable via IDA-PBC to the VTOL system (49) is characterized

<sup>12</sup>The second author thanks W. Pasillas for help with the proof of this lemma.

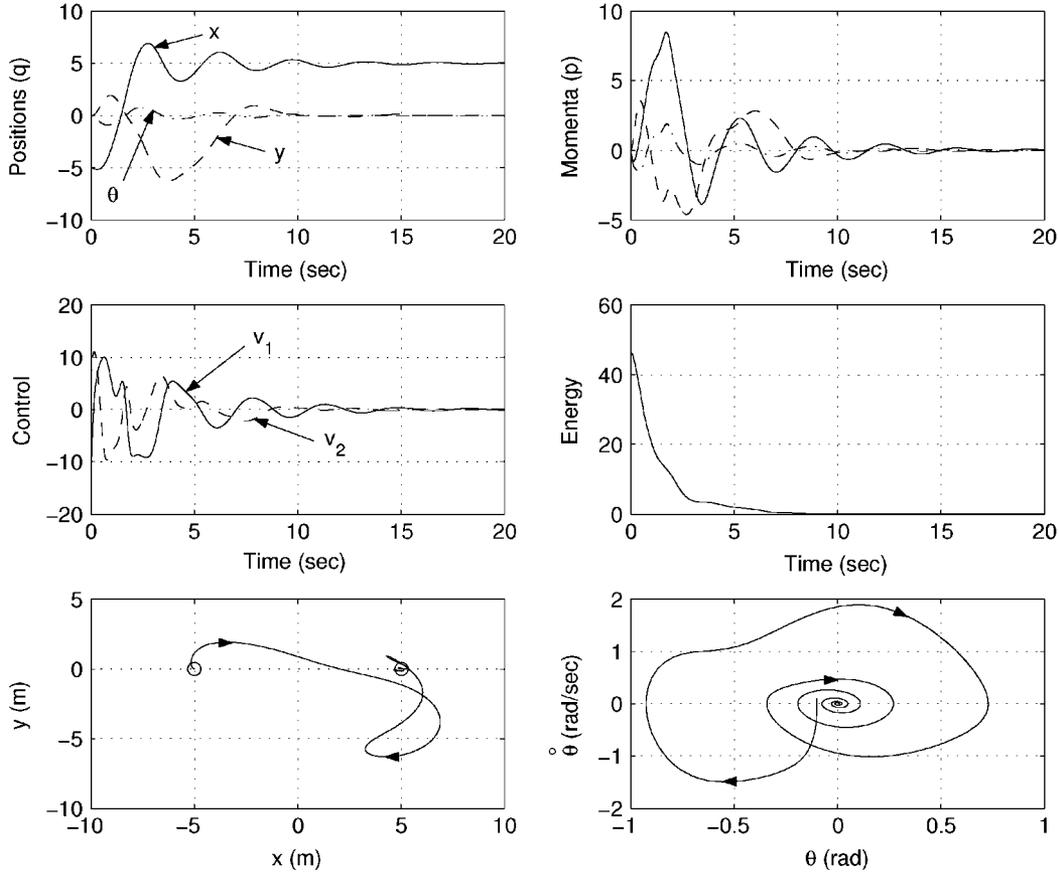


Fig. 5. “Badly” tuned controller for VTOL;  $q = (x, y, \theta)$ . Initials conditions  $(q(0), p(0)) = (-5, 0, 0.1, -0.1, -0.1, 0.1)$  and references  $q^* = (5, 0, 0)$ .

by the *globally* positive definite and bounded inertia matrix (54) with  $k_1$  an arbitrary positive number and  $k_2, k_3$  verifying

$$k_3 > 5k_1\epsilon \quad \frac{k_1}{\epsilon} > k_2 > \frac{k_1}{2\epsilon}$$

and the potential energy function

$$V_d = -\frac{g}{k_1 - k_2\epsilon} \cos q_3 + \frac{1}{2}[z(q) - z(q^*)]^\top P[z(q) - z(q^*)] \quad (55)$$

with

$$z(q) - z(q^*) = \begin{bmatrix} q_1 - q_1^* - \frac{k_2}{k_1 - k_2\epsilon} \sin q_3 \\ q_2 - q_2^* + \frac{k_3 - k_1\epsilon}{k_1 - k_2\epsilon} (\cos q_3 - 1) \end{bmatrix}$$

that satisfies (8) for all  $P = P^\top > 0$ .

Moreover, the IDA-PBC law

$$u = A_1 P \begin{bmatrix} q_1 - q_1^* \\ q_2 - q_2^* \end{bmatrix} + \begin{bmatrix} p^\top A_2 p \\ p^\top A_3 p \end{bmatrix} + A_4 - K_v A_5 p \quad (56)$$

where the matrices  $A_i$  are given in the Appendix, ensures *almost global asymptotic stability* of the desired equilibrium  $(q_1^*, q_2^*, 0, 0, 0, 0)$ .<sup>13</sup>

*Proof:* Since we have already verified all Assumptions A.1–A.6 of Proposition 5, it only remains to prove positivity of the inertia matrix. For, we note that  $k_2 > 0$ . We will prove

positivity decomposing  $M_d$  into the sum of two matrices. For, we write  $k_3 = a + d$  and start with the matrix

$$\begin{bmatrix} d & 0 & k_1 \cos q_3 \\ 0 & d & k_1 \sin q_3 \\ k_1 \cos q_3 & k_1 \sin q_3 & k_2 \end{bmatrix}.$$

It is easy to show that  $d > 4k_1\epsilon, k_2 > (k_1/2\epsilon)$  ensures positive definiteness, and is consistent with the additional requirement  $k_1 > \epsilon k_2$ , imposed by the potential energy shaping. On the other hand, the matrix

$$\begin{bmatrix} k_1\epsilon \cos^2 q_3 + a & k_1\epsilon \cos q_3 \sin q_3 & 0 \\ k_1\epsilon \cos q_3 \sin q_3 & -k_1\epsilon \cos^2 q_3 + a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is positive semidefinite if and only if  $a > k_1\epsilon$ . Adding the lower bounds on  $a$  and  $d$  completes the proof of positivity of the inertia matrix.

The expression for the potential energy and the control law are immediately obtained replacing (54) in (25) and (26) and doing some simple calculations.

Finally, to prove the almost global claim we note that the system lives in the set  $\mathbb{R}^2 \times [-\pi, \pi] \times \mathbb{R}^3$ , and that the energy function  $H_d$  is *positive definite and proper* throughout this set. Then, since  $\dot{H}_d \leq 0$ , we have that *all solutions* are bounded. From the previous analysis, we know that the desired equilibrium is asymptotically stable. Invoking the argument used for

<sup>13</sup>That is, the domain of attraction is the whole state space minus a set of Lebesgue measure zero, see e.g., [31].

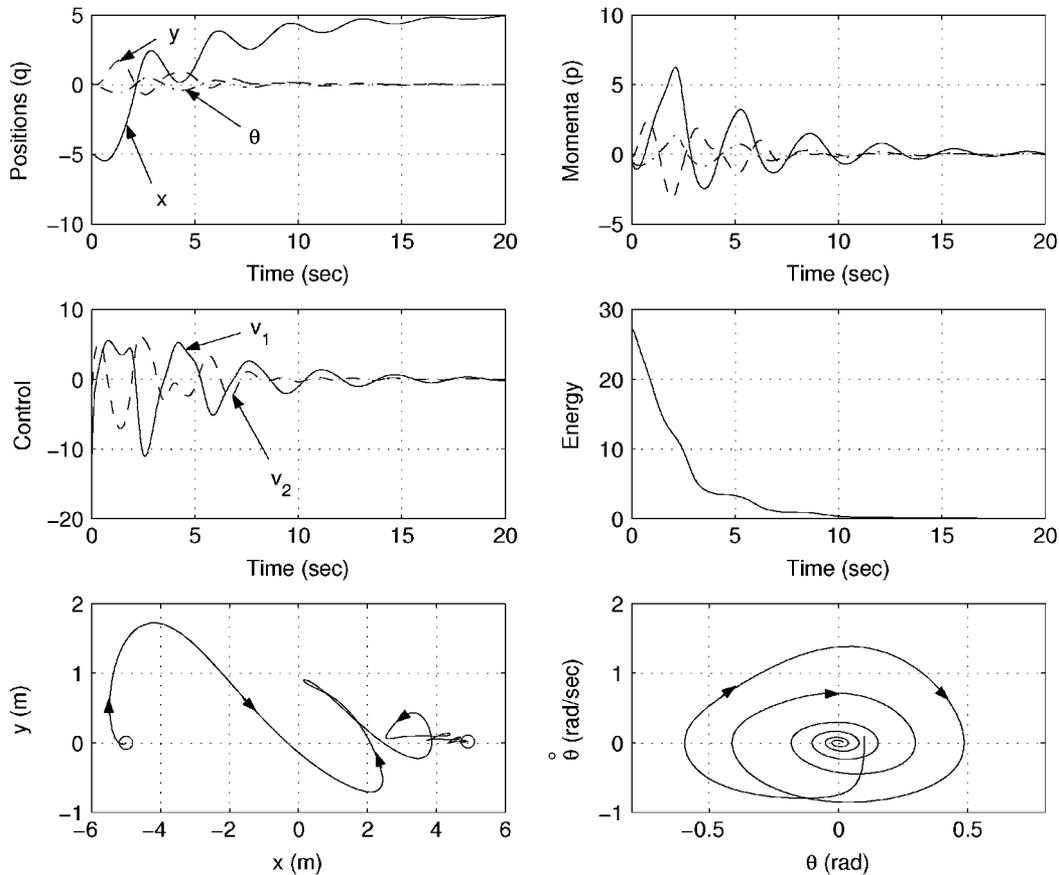


Fig. 6. “Well” tuned controller for VTOL;  $q = (x, y, \theta)$ . Initial conditions  $(q(0), p(0)) = (-5, 0, 0.1, -0.1, -0.1, 0.1)$  and references  $q^* = (5, 0, 0)$ .

the proof of asymptotic stability in Proposition 5, we rule out the existence of limit cycles. We will now show that the other equilibria are unstable. Indeed, the linearization of the closed-loop system at these equilibria has eigenvalues with strictly positive real part and at least one eigenvalue with strictly negative real part. Associated to the latter there is a stable manifold, and trajectories starting in this manifold will converge to these equilibria. However, it is well-known that an  $s$ -dimensional invariant manifold of an  $n$ -dimensional system has Lebesgue measure zero if  $s < n$ . Consequently, the set of initial conditions that converges to the “bad” equilibrium has zero measure. ■

Simulations were carried out with a twofold objective, first to show how the energy shaping controller proposed in this paper ensures a satisfactory response for strong coupling coefficients  $\epsilon > 0$ , and second to illustrate the tuning flexibility provided by the design parameters. All simulations are made with a strong value of coupling, i.e.,  $\epsilon = 1$ . The damping injection matrix was fixed to  $K_v = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$ .

The normal conditions of maneuvering for the VTOL aircraft is to keep an accurate lateral motion near the ground. This problem has been normally solved in two steps (see for instance [35]): Decoupling the altitude output from the lateral motion and rolling moment by means of a prefeedback control law and then, designing a control law for the new decoupled system; this procedure renders satisfactory results for small enough  $\epsilon$ . With the energy shaping controller, independently of the value of  $\epsilon$ , it is possible to “virtually decouple” the outputs using the weighting matrix  $P$  in the potential energy (25). To illustrate this point

two simulations were made, first with a “bad” potential energy taking  $P$  diagonal and the weights equal to 1 and 0.1. This simulation for a lateral motion is shown in Fig. 5. The same simulation was made for a “good” potential energy taking the  $P$  again diagonal but with the weights now 1/2 and 1, with the response shown in Fig. 6—notice the different scales in the graphs. The posture of the VTOL aircraft along the trajectory for both cases is shown (at the same scale) in Fig. 7. It can be seen that, for the first case, the altitude ( $q_2 = y$ ) makes very large excursions to drive the VTOL to rest, while in the second one a simple slow amplitude rocking motion achieves the objective.

The third simulation, depicted in Fig. 8, shows the behavior of the controlled system in an aggressive maneuver, from a limit position, upside down, for the roll angle ( $q_3$ ) and a great step on the lateral motion ( $q_1$ ) and altitude ( $q_2$ ). The high performance of the controlled system is clearly seen from the figure. The posture of the VTOL aircraft along the trajectory is shown in Fig. 9.

We also have made the third simulation using *only position* measurements. The result is shown in Fig. 10. A slower performance is observed again, as in the pendulum example. A matrix gain,  $\lambda = \text{diag}\{0.005, 0.005, 0.01\}$ , was used for the nonlinear speed estimator.

## VIII. CONCLUSION AND FUTURE RESEARCH

In this paper, we have identified a class of underactuated mechanical systems for which the IDA-PBC design methodology gives a complete *constructive* solution to the stabiliza-

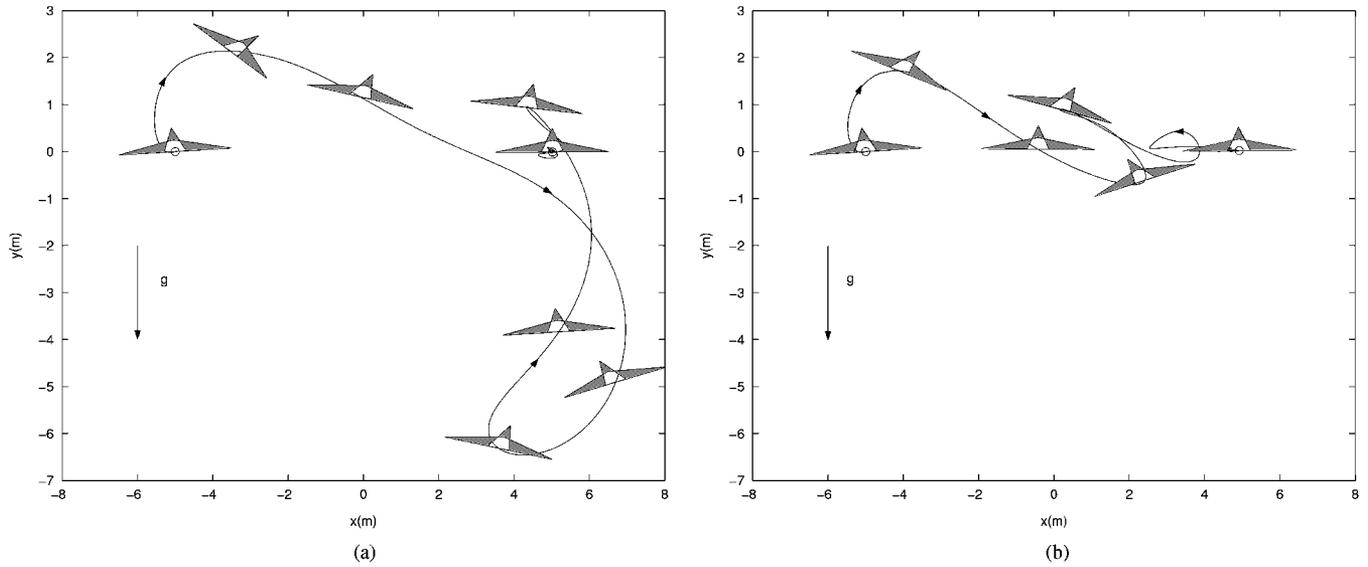


Fig. 7. Posture of the VTOL along the trajectory;  $q = (x, y, \theta)$ . (a) “Badly” tuned controller. (b) “Well” tuned controller.

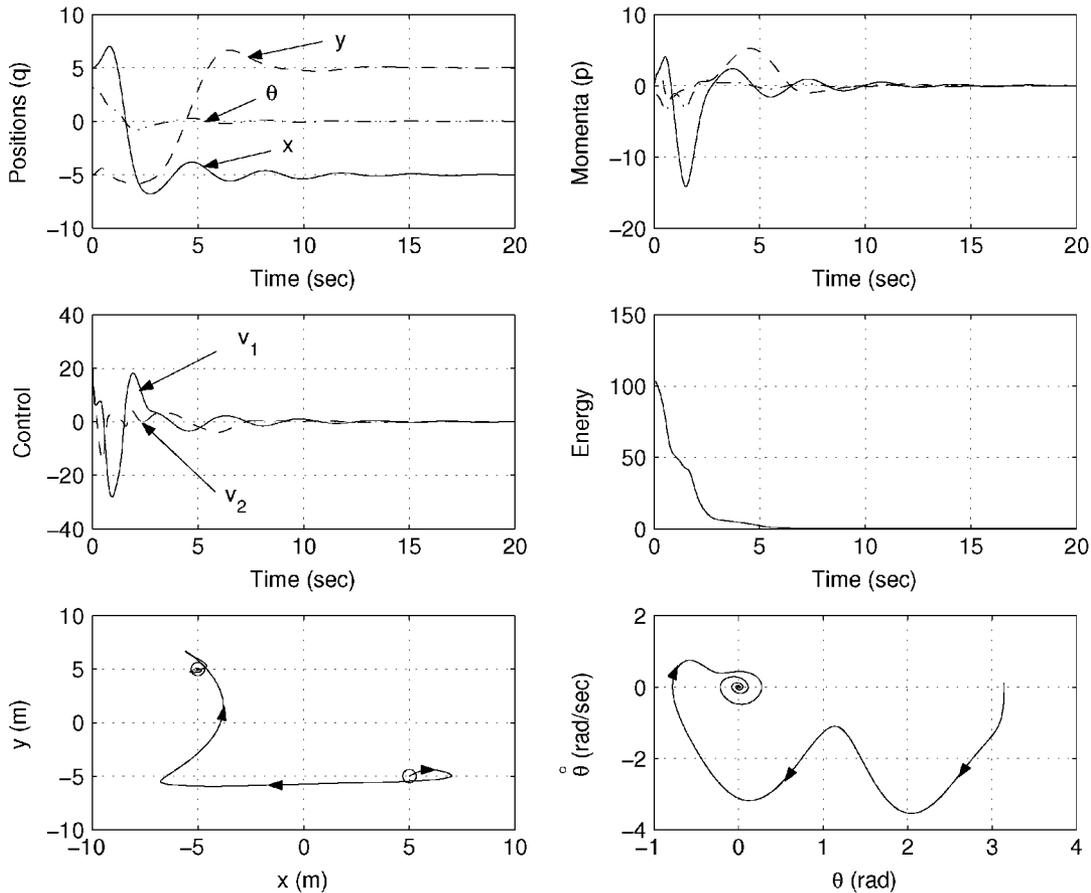


Fig. 8. Upside down simulation;  $q = (x, y, \theta)$ . Initials conditions  $(q(0), p(0)) = (5, -5, \pi, 0.1, -0.1, 0.1)$  and references  $q^* = (-5, 5, 0)$ .

tion problem—without the need to solve any PDE. The main assumptions made on the system are that it has underactuation degree one and that, roughly speaking, the dynamics that are not directly affected by the control, e.g., “in  $\ker G$ ,” can be modified through the action of one actuated coordinate  $q_r$ . The underactuation degree Assumption A.1 is needed to ensure there are enough degrees of freedom in the free IDA-PBC parameter  $J_2$  to solve the kinetic energy PDE as an algebraic equation.

Assumptions A.2 and A.3 ensure that we can construct the solution choosing  $(dM_d/dq_r) \in \text{Im}G$ . Assumptions A.4 and A.5, needed to solve the potential energy PDE, specify the role of  $q_r$ . Finally, Assumption A.6 measures our ability to affect the potential energy function through the modification of  $M_d$ .

We have also presented a *position feedback* implementation—with provable stability properties—for a subclass of the class considered in the paper. (In [2] a characterization

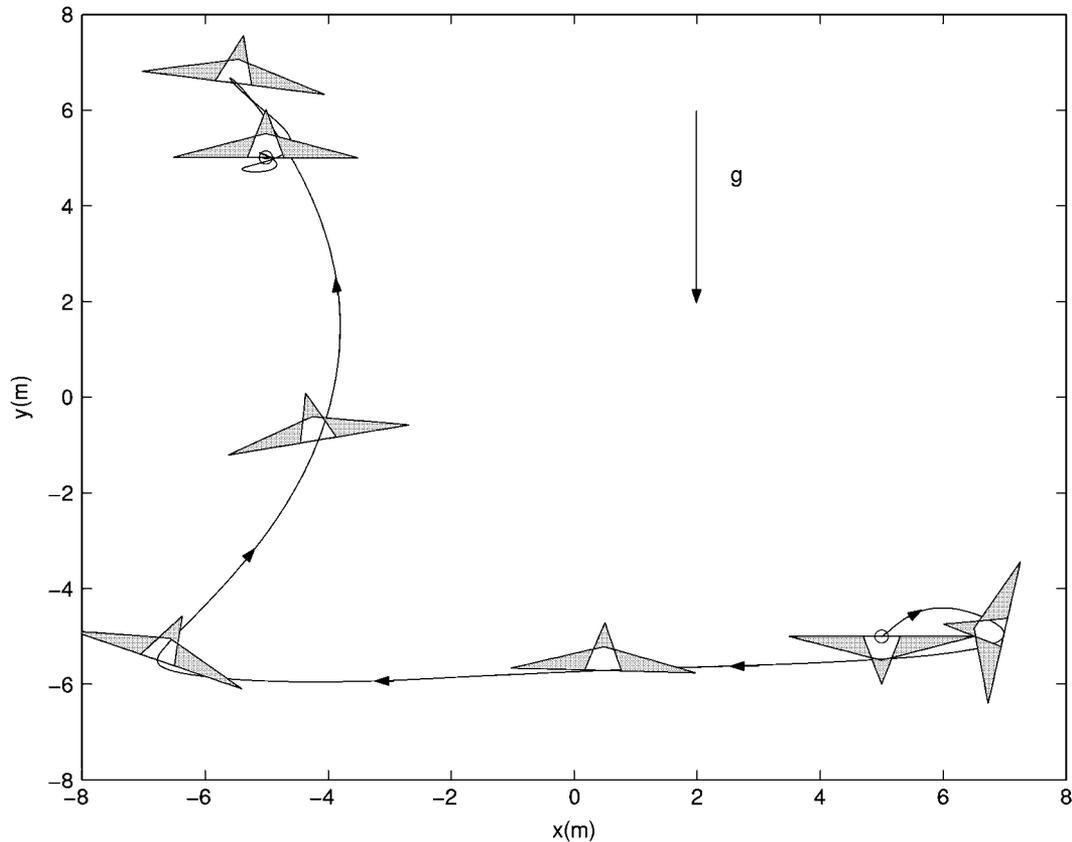


Fig. 9. Upside down simulation. Posture of the VTOL along the trajectory for the simulation of Fig. 8.

of all mechanical systems that are feedback-equivalent to this subclass is given in terms of solvability of a set of PDEs with algebraic constraints.) This class contains several practically interesting benchmark examples, some of which are studied in the paper. In particular, we present an almost globally stabilizing controller for the VTOL aircraft that ensures asymptotic regulation from any initial condition to an arbitrary position with zero roll angle and zero speed; and a controller for the pendulum on the cart that can swing-up the pendulum from any position in the (open) upper half plane and stop the cart at any desired location.

Besides ensuring asymptotic stability the IDA-PBC methodology provides the designer with some degrees of freedom to improve transient performance and robustness. These degrees of freedom are given in terms of parameterized expressions for the assignable energy functions. More precisely, the total energy function can be effectively shaped via the selection of the scaling matrix  $\Psi$ , the constant matrix  $M_d^0$  in the inertia matrix (20) and the choice of the function  $\Phi$  in the potential energy (25). An additional tuning parameter is the damping injection gain  $K_v$  that may be any positive definite (possibly state-dependent) matrix.

For simplicity we have chosen in our simulations a quadratic function  $\Phi$  for the potential energy, but motivated by other considerations, e.g., input constraints or rate saturations, we could have also taken other (logarithmic or saturated) functions. An advantage of a quadratic function is that the control law takes a very nice expression (28), which consists of the sum of three types of terms that are modulated by functions of the distinguished coordinate  $q_r$ .

- (“proportional-like”) linear terms on the additional coordinate error  $S(q - q^*)$  that contribute to the potential energy shaping;<sup>14</sup>
- (“derivative-like”) linear terms in  $p$  due to the damping injection that enforce asymptotic stability;
- (“gyroscopic-like”) quadratic terms in  $p$  that come from the interconnection matrix  $J_2$ . These terms, which serve to propagate the damping through the well-known mechanism of feedback interconnection of passive and strictly passive systems [29], are essential for the solution of the present problem. See Remark 3.

Current research is under way to extend the present work in the following directions.

- In [24], we worked out two examples, the Acrobot and the Furuta’s Pendulum, that do not satisfy Assumptions A.2 nor A.4. The term,  $G^\perp \nabla_q (p^\top M^{-1} p)$  introduces a quadratic term in  $M_d$  in the kinetic energy PDE, but it can still be solved with a suitable choice of  $J_2$ . Similarly, even though Assumption A.4 does not hold, we can solve the potential energy PDE with a machinery specifically tailored for these examples. Developing a general theory for a well-identified class of systems containing these examples is currently under investigation.
- In the proof of asymptotic stability in Proposition A.7, we have established that in the residual set  $\Omega$  the char-

<sup>14</sup>We have shown with examples the importance of a suitable selection of the relative weights (the matrix  $P$ ) of the configuration coordinates.

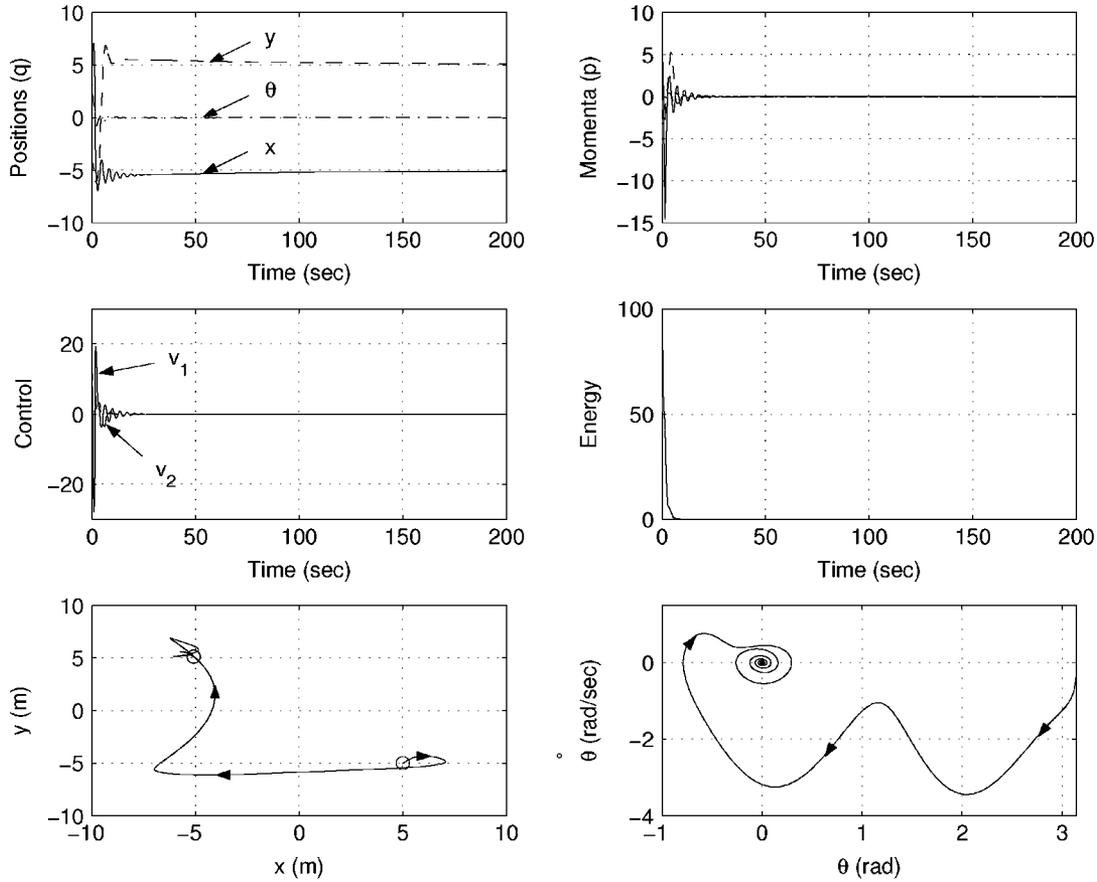


Fig. 10. Upside down simulation;  $q = (x, y, \theta)$ . Initials conditions  $(q(0), p(0)) = (5, -5, \pi, 0.1, -0.1, 0.1)$  and references  $q^* = (-5, 5, 0)$ , (position feedback.)

acteristic of the potential energy PDE is constant. This seems to be a geometric property of the PDEs that needs to be further clarified. In particular, it would be desirable to use it to simplify the proof and remove the, rather awkward, Assumption A.7. (We point out that this property of  $z(q)$  holds for other classes of mechanical systems—for instance, the Ball-and-Beam and the Acrobot systems which do not satisfy Assumptions A.2 nor A.4.)

- To relax Assumptions A.3 and A.4 we need to explore the complete set of solutions for  $M_d$  defined by (3), or equivalently (15). See Remark 8. In particular, it seems necessary to make  $M_d$  function of all coordinates.
- Working out a general theory without Assumption A.1 seems a difficult task. On one hand, we cannot transform the kinetic energy PDE into an algebraic equation. On the other hand, as indicated in [23], some geometric obstacles that hamper our ability to shape  $V_d$  may appear in this case.
- Comparison of the class studied here with the one identified, via elegant geometric conditions, in [12]. See also [11]. Also, it would be interesting to explore the connections with the recent work [19], where the authors consider underactuation degree one mechanical systems with a cyclic coordinate.
- The examples presented in the paper are transformed into Spong's Normal Form via partial feedback lin-

earization. It has been argued in this paper that this operation is fragile so it would be interesting to avoid it.<sup>15</sup> In [28] it is shown that this is indeed possible, as the PDEs are invariant to partial feedback linearization.

- The proposed controllers should be tested experimentally and confronted with other existing schemes. The outcome of this research will be reported elsewhere.

#### APPENDIX

In this appendix, the matrices  $A_i, i = 1, \dots, n + 2$ , for the controllers of both examples, are given explicitly. The elements of the vectors  $\alpha_i$  are denoted  $\alpha_{ij}$ .

#### PENDULUM ON A CART

The matrices  $A_i, i = 1, \dots, 4$ , in the controller (47) of Proposition 7 are

$$A_1 = - \left( m_{12} \frac{dF}{dq_1} + m_{22} \right)$$

$$A_2 = - \frac{1}{2} m_{12} M_d^{-1} \begin{bmatrix} \frac{dm_{11}}{dq_1} - \alpha_{11} & \frac{dm_{12}}{dq_1} \\ \frac{dm_{12}}{dq_1} - \alpha_{12} & \frac{dm_{22}}{dq_1} \end{bmatrix} M_d^{-1}$$

$$A_3 = - m_{12} \frac{6a \sin q_1}{kb^2 \cos^3 q_1} + PF(q_1) A_1$$

<sup>15</sup>This extension is also of interest if a true position feedback controller on the actual system is to be realized. Toward this end, the result of Section VI should be extended to a broader class of systems.

$$\begin{aligned}
A_1 &= - \begin{bmatrix} P_{11}(m_{11} + m_{13}) + P_{12}m_{12} & P_{22}(m_{12} + m_{13}) + P_{12}m_{11} \\ P_{11}(m_{12} + m_{23}) + P_{12}m_{22} & P_{22}(m_{22} + m_{23}) + P_{12}m_{12} \end{bmatrix}, \\
A_2 &= \frac{1}{2}M_d^{-1} \begin{bmatrix} m_{13} \frac{dm_{11}}{dq_3} & m_{13} \frac{dm_{12}}{dq_3} + 2\alpha_{11} & m_{13} \frac{dm_{13}}{dq_3} + 2\alpha_{21} \\ m_{13} \frac{dm_{12}}{dq_3} & m_{13} \frac{dm_{22}}{dq_3} + 2\alpha_{12} & m_{13} \frac{dm_{23}}{dq_3} + 2\alpha_{22} \\ m_{13} \frac{dm_{13}}{dq_3} & m_{13} \frac{dm_{23}}{dq_3} + 2\alpha_{13} & 2\alpha_{23} \end{bmatrix} M_d^{-1}, \\
A_3 &= \frac{1}{2}M_d^{-1} \begin{bmatrix} m_{23} \frac{dm_{11}}{dq_3} - 2\alpha_{11} & m_{23} \frac{dm_{12}}{dq_3} & m_{23} \frac{dm_{13}}{dq_3} + 2\alpha_{31} \\ m_{23} \frac{dm_{12}}{dq_3} - 2\alpha_{12} & m_{23} \frac{dm_{22}}{dq_3} & m_{23} \frac{dm_{23}}{dq_3} + 2\alpha_{32} \\ m_{23} \frac{dm_{13}}{dq_3} - 2\alpha_{13} & m_{23} \frac{dm_{23}}{dq_3} & 2\alpha_{33} \end{bmatrix} M_d^{-1}, \\
A_4 &= - \begin{bmatrix} m_{11}(P_{11}F_1 + P_{12}F_2) + m_{12}(P_{12}F_1 + P_{22}F_2) + m_{13}(P_{11}F_1 + P_{22}F_2 + P_{12}(F_2 \frac{dF_1}{dq_3} + F_1 \frac{dF_2}{dq_3})) - g \sin q_3 / \gamma_3^0 \\ m_{12}(P_{11}F_1 + P_{12}F_2) + m_{22}(P_{12}F_1 + P_{22}F_2) + m_{23}(P_{11}F_1 + P_{22}F_2 + P_{12}(F_2 \frac{dF_1}{dq_3} + F_1 \frac{dF_2}{dq_3})) - g \sin q_3 / \gamma_3^0 \end{bmatrix}, \\
A_5 &= G^T M_d^{-1} = \begin{bmatrix} 1 & 0 & \cos q_3 / \epsilon \\ 0 & 1 & \sin q_3 / \epsilon \end{bmatrix} M_d^{-1}.
\end{aligned}$$

Fig. 11. Matrices  $A_i, i = 1, \dots, 5$  for the VTOL controller given by (56).

$$A_4 = G^T M_d^{-1} = [-b \cos q_1 \ 1] M_d^{-1}$$

where

$$\begin{aligned}
F(q_1) &\triangleq \frac{3}{b} \ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb} \tan q_1 \\
\alpha_1 &= \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \end{bmatrix} \triangleq \frac{\gamma_1}{2} k \sin q_1 \begin{bmatrix} b \cos q_1 \\ -1 \end{bmatrix}.
\end{aligned}$$

#### STRONGLY COUPLED VTOL AIRCRAFT

The matrices  $A_i, i = 1, \dots, 5$  for the controller (56) of Proposition 8 are given in Fig. 11, where

$$\begin{aligned}
\alpha_1 &\triangleq -\frac{k_1 \gamma_3^0}{2} \begin{bmatrix} -2\epsilon \cos q_3 \\ 2\epsilon \sin q_3 \\ 1 \end{bmatrix} \\
\alpha_2 &\triangleq -\frac{k_1 \gamma_3^0}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\alpha_3 &\triangleq -\frac{k_1 \gamma_3^0}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\
F_1 &\triangleq \frac{k_3}{k_1 - k_2 \epsilon} \sin q_3 \\
F_2 &\triangleq \frac{k_3 - k_1 \epsilon}{k_1 - k_2 \epsilon} (\cos q_3 - 1).
\end{aligned}$$

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