GKM THEORY FOR ORBIFOLD STRATIFIED SPACES AND APPLICATION TO SINGULAR TORIC VARIETIES

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Abstract. We study the GKM theory for a equivariant stratified space having orbifold structures in tis successive quotients. Then, we introduce the notion of an *almost simple polytope*, as well as a *divisive toric variety* generalizing the concept of a divisive weighted projective space. We employ the GKM theory to compute the generalized equivariant cohomology theories of toric varieties associated to almost simple polytopes and divisive toric varieties.

1. Introduction

A toric variety of complex dimension n is a complex algebraic variety with an action of the algebraic torus $(\mathbb{C}^*)^n$ having an open dense orbit. It is equipped with a natural action of compact n-dimensional torus T^n. The category of toric varieties has been one of the main attractions in algebraic and symplectic geometry from the beginning of 1970s. One of the reasons is their rich interaction with different fields of mathematics, such as representation theory and combinatorics. For instance, one may get a lattice polytope P from a projective toric variety X by the convexity theorem [Ati82, GS82] and vice versa by Delzant's construction [Del88, Gui94]. Moreover, the lattice points in P give a weight decomposition of $H^0(X,\mathcal{L})$ as a torus representation, where $\mathcal{L}$ is a very ample line bundle over X.

From the topological point of view, such a correspondence leads us to ask how to extract topological invariants for a toric variety from the associated combinatorics, i.e., lattice polytopes or their normal fans. Indeed, there is a rich and vast literature dealing with this question for several invariants. For example, we refer to [Dan78, DJ91, Jur85], [Mor93], [BB00], [BR98] for non-equivariant cohomology theories, and [Bag07, VV03] for equivariant cohomology theories. However, most of the computations are focused on the category of smooth toric varieties.

Now, we change gears to singular toric varieties. Simplicial toric varieties, namely toric varieties having at worst orbifold singularities, may be the mildest class of singular toric varieties. Their ordinary cohomology and Borel equivariant cohomology over rational coefficients behave in a similar manner to smooth toric varieties (see [CLS11, Section 12.3]), while their integral cohomology theories are known only for particular classes under some hypothesis [Kaw73, AA97, BSS17, BNSS19].

For even worse singular toric varieties, their topological invariants are far away from the situation of smooth toric varieties. For example, their ordinary cohomology may not vanish in odd degrees in general, which complicates the computation

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of some spectral sequences such as Leray–Serre or Atiyah–Hirzebruch spectral sequences.

In this paper, we introduce the concept of an *orbifold stratified G-space X* for some topological group G, i.e., X is a finite G-CW complex with an equivariant stratification

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\ell = X \tag{1.1}$$

such that each of the successive quotients X_j/X_{j-1} is homeomorphic to the Thom space of an orbifold G-vector bundle and $X_j - X_{j-1}$ is equipped with an effective orbifold structure. We note that the total space X may have arbitrary singularities. For instance, toric varieties associated to polytopes illustrated in Examples 3.2, 3.3 and 3.4 are singular, but not orbifolds. A relevant concept of an orbifold stratified space is studied in [CLW16, Definition 1.1] in the language of Lie groupoid.

The main purpose of this paper is to study the generalized GKM theory for the category of orbifold G-stratified spaces as in (1.1). Over this category, we give a concrete description of complex-oriented generalized equivariant cohomologies with rationals, namely, we consider $E^*_G(-) \otimes_{\mathbb{Z}} \mathbb{Q}$. For example, E^*_G can be Borel equivariant cohomology H^*_G or complex equivariant K-theory K^*_G following [Seg68]. In particular, if the orbifold singularity of $X_j - X_{j-1}$ is trivial for all j, we recover the main results of [HHH05, HHRW16].

This paper is organized as follows. In Section 2, we discuss the notion of a simple orbifold G-bundle and the equivariant Thom isomorphism as a foundation of the GKM theory for orbifold stratified spaces. Then, we introduce the definition of an orbifold G-stratification and verify how the generalized GKM theory of [HHH05] can be extended to the category of orbifold G-stratifications.

Section 3 is devoted to a combinatorial characterization of toric varieties for which our main results hold. Such a class of toric varieties may have arbitrary toric singularities beyond orbifold singularities. Here, we bring the idea of *retraction sequence* [BSS17] of a simple polytope and extend this to the category of general convex polytopes. This allows us to give an orbifold torus-equivariant stratification on the corresponding toric variety, see Theorem 3.6. For these toric varieties, we give the GKM theoretic description of generalized cohomology theories in Proposition 3.9.

In Section 4, we summarize the concept of a piecewise algebra associated to a fan, which is studied in [HHRW16, Section 4]. Then, we establish Theorem 4.3 describing $E^*_{T^n}(X) \otimes \mathbb{Q}$ where X is a singular toric variety discussed in Section 3 and T^n is the compact torus acting on X.

Finally, generalizing the idea of a divisive weighted projective space, we introduce the notion of a *divisive toric variety* in Section 5 to compute generalized equivariant cohomologies with integers. The conclusion is stated in Proposition 5.4.

We close the introduction with some previous works relevant to the study of this article. The author of [Gon14] considers '$\mathbb{Q}$-filterable spaces' and if they are projective T-varieties then they have a stratification similar to (1.1) where $X_j - X_{j-1}$ is a 'rational cell' which may not be an orbifold. Under the assumption of 'T-skeletal', he studies GKM-theory to obtain the Borel equivariant cohomology of those spaces. Nevertheless we are also interested in other generalized equivariant cohomology theories.

The authors of [HW19] studies equivariant K-theory of toric varieties associated to 'fans with distant singular cones', where they use Mayer–Vietories sequence to show their main theorem [HW19, Theorem 7.2]. There are many singular toric varieties beyond their consideration with stratification as in (1.1). For instance, simplicial toric varieties such that all fixed points are singular are excluded from their study.

We also note that [DKU19], [SU] and [Sar20] discuss integral equivariant cohomology theories for (quasi)toric manifolds, toric orbifolds [DJ91] and locally standard torus orbifolds [HM03], respectively. We emphasize that there are many interesting toric varieties which are not orbifolds, such as the Gelfand–Zetlin toric variety in Example 5.5 or see [KV19], whose integral generalized equivariant cohomology theories (H^*_T, K^*_T and MU^*_T) can be described by using Proposition 5.4.

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2. GKM theory for orbifold stratified spaces

The goal of this section is to apply the GKM theory studied in [HHH05] to the category of equivariant stratified G-spaces, where certain orbifold structures are involved in their successive quotients. By a G-space we mean a finite G-CW complex for a topological group G. In this paper, we are interested in G-equivariant cohomology theory E^*_G associated to a ring G-spectrum E as defined in [May96, Chapter XIII]. We note that $E^*_G(X)$ is a commutative ring together with the structure of $E^*_G(pt)$-algebra induced from the equivariant collapsing map $X \to \{pt\}$. In particular, we study $E^*_G(X) \otimes \mathbb{Q}$ for a G-space X. For simplicity, we use $E^*_G(X)$ in place of $E^*_G(X) \otimes \mathbb{Q}$. Note that for Borel equivariant cohomology H^*_G, we have $H^*_G(X) \otimes \mathbb{Q} \cong H^*_G(X; \mathbb{Q})$.

To establish the structure of stratification of a G-space, we begin with a complex E-orientable G-vector bundle $\xi \colon V \to B$ and a finite group A acting linearly each fiber of ξ, which commutes with G-action on V and preserves E-orientation of ξ. (See [May96, Chapter XVI, Definition 9.1] for the definition of E-orientation.) Then, one may consider the induced fiber bundle

$$\xi^A \colon V/A \to B,$$

which we call a *simple orbifold G-bundle*. The associated disc bundle $D(V) \to B$ and the sphere bundle $S(V) \to B$ of ξ are invariant under A-action, as A acts linearly on each fiber. Hence, one can define a $\mathbf{q}$-*disc bundle* $D(V/A)(= D(V)/A) \to B$ and a $\mathbf{q}$-*sphere bundle* $S(V/A)(= S(V)/A) \to B$ in the usual manner, which yields the Thom space $\mathrm{Th}(V/A) := D((V/A)/S(V/A))$ of ξ^A and the map

$$\tilde{\xi}^A \colon \mathrm{Th}(V/A) \to B.$$

We refer to [PS10, Section 4] and [BNSS19, Section 2] for the notions of a $\mathbf{q}$-disc and a $\mathbf{q}$-sphere.

When the cohomology theory E^*_G is Borel equivariant cohomology H^*_G or equivariant K-theory K^*_G (see [Seg68]), then we have the equivariant Thom isomorphism for a simple orbifold G-bundle.

Proposition 2.1 (Thom isomorphism)**.** *Let $\xi^A \colon V/A \to B$ be a simple orbifold G-bundle of rank n. For cohomology theories $E^*_G = H^*_G$ or K^*_G, there exists a cohomology class $\eta_A \in E^n_G(\mathrm{Th}(V/A))$ such that*

$$\cup\eta_A \colon E^*_G(X) \to E^{*+n}_G(\mathrm{Th}(V/A))$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Th}(V) & \longrightarrow & EG \times_G \mathrm{Th}(V) & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow{=} \\ \mathrm{Th}(V/A) & \longrightarrow & EG \times_G \mathrm{Th}(V/A) & \longrightarrow & BG, \end{array}$$

where vertical maps are projections induced from the A-action and two horizontal compositions are Borel fibrations. Applying Leray–Serre spectral sequence for two Borel fibrations, we get the isomorphisms

$$H^*_G(\mathrm{Th}(V)) \cong H^*(\mathrm{Th}(V)) \otimes_{\mathbb{Q}} H^*(BG) \tag{2.1}$$

$$H^*_G(\mathrm{Th}(V/A)) \cong H^*(\mathrm{Th}(V/A)) \otimes_{\mathbb{Q}} H^*(BG). \tag{2.2}$$

We note that $H^*(V) \cong H^*(V/A)$ and $H^*(V_0) \cong H^*(V_0/A)$ with rational coefficients, where V_0 denotes the complement of the zero section. Now, the long exact sequence of the pair (V, V_0) together with the Five Lemma shows that $H^*(\mathrm{Th}(V)) \cong H^*(\mathrm{Th}(V/A))$. We refer to [PS10, Section 5.1] for more details. Hence, the left-hand sides of (2.1), (2.2) and $H^*_G(\mathrm{Th}(V/A))$ are isomorphic.

Consider the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V/A \\ \downarrow{\xi} & & \downarrow{\xi^A} \\ B & \xrightarrow{=} & B, \end{array}$$

where ρ is the canonical projection given by the action of A on V. This induces the diagram

$$\begin{array}{ccc} H^{*+n}_G(\mathrm{Th}(V)) & \xleftarrow{\rho^*} & H^{*+n}_G(\mathrm{Th}(V/A)) \\ {\scriptstyle\cup\eta}\uparrow & & \uparrow \\ H^*_G(B) & \xleftarrow[=]{} & H^*_G(B), \end{array}$$

where the left vertical map $\cup\eta$ is the equivariant Thom isomorphism (see [May96, Theorem 9.2, Chapter XVI]) given by the cup product of the equivariant Thom class $\eta \in H^{\alpha}_G(\mathrm{Th}(V))$. Therefore, the right vertical map is an isomorphism given by the cup product of the pull back $\eta_A := (\rho^*)^{-1}(\eta) \in H^n_G(\mathrm{Th}(V/A))$.

Next, to show the claim for the equivariant K-theory, we consider the composition $ch \circ \psi$, where ch denotes the *Chern character* from Borel equivariant K-theory to equivariant cohomology and ψ is the canonical monomorphism from $K_G(\mathrm{Th}(V))$ to $K(EG \times_G \mathrm{Th}(V))$ defined by assigning the vector bundle $EG \times_G \xi$ to each G-equivariant vector bundle ξ, we refer to [AS69]. Applying the same composition to $\mathrm{Th}(V/A)$ and the usual Thom isomorphism theorem for a genuine vector bundle, we have the commutative diagram:

$$\begin{array}{ccccccc}
K_G^*(B) & \xrightarrow{\cong} & K_G^{*+n}(\mathrm{Th}(V)) & \xrightarrow{\psi} & K^{*+n}(EG\times_G \mathrm{Th}(V)) & \xrightarrow{ch} & H_G^{*+n}(\mathrm{Th}(V)) \\
{\scriptstyle =}\uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow & & {\scriptstyle \cong}\uparrow \\
K_G^*(B) & \xrightarrow{\phi} & K_G^{*+n}(\mathrm{Th}(V/A)) & \xrightarrow{\psi_A} & K^{*+n}(EG\times_G \mathrm{Th}(V/A)) & \xrightarrow{ch} & H_G^{*+n}(\mathrm{Th}(V/A)),
\end{array}$$

where we claim that ϕ is an isomorphism. Indeed, Chern characters are injective as we are working with rationals. Hence, the surjectivity and the injectivity of f^* follow from the commutativity of the left most square and right two squares of the diagram, respectively. Therefore, we have $K_G^*(B) \cong K_G^{*+n}(\mathrm{Th}(V/A))$, induced from the Thom isomorphism $K_G^*(B) \cong K_G^{*+n}(\mathrm{Th}(V))$ for a genuine vector bundle $V \to B$. □

Throughout this paper, we consider a simple orbifold G-bundle with rank n over a topological space B and assume that a complex-oriented G-equivariant cohomology theory E_G^* has the Thom isomorphism

$$E_G^*(B) \cong E_G^{*+n}(\mathrm{Th}(V/A))$$

for a simple orbifold G-bundle given by the cup product of the equivariant Thom class $\eta_A \in E_G^*(\mathrm{Th}(V/A))$. Then, one can define the equivariant Euler class $e_G(\xi^A) \in E_G^*(B)$ of a simple orbifold G-bundle ξ^A by the restriction of η_A to the zero section.

Now we consider an equivariant stratification

$$\{pt\} = X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{\ell-1} \subseteq X_\ell = X \tag{2.3}$$

of a G-space X such that each of the successive quotients X_j/X_{j-1} is homeomorphic to the Thom space $\mathrm{Th}(V_j/A_j)$ of a simple orbifold G-bundle $\xi^{A_j}\colon V_j/A_j \to B_j$. Therefore, X can be built from X_1 inductively by attaching **q**-disc bundles $D(V_j/A_j)$ to X_{j-1} via some G-equivariant maps

$$\phi_j : S(V_j/A_j) \to X_{j-1},$$

which gives us a cofibrations

$$X_{j-1} \to X_j \xrightarrow{q} \mathrm{Th}(V_j/A_j) \tag{2.4}$$

for $j = 2, \ldots, \ell$. Now, one gets the following proposition by the induction on the stratification (2.3).

Proposition 2.2. *Let X be an orbifold stratified G-space as in* (2.3)*. If each equivariant Euler class $e_G(\xi^{A_j}) \in E_G^*(B_j)$ of the associated simple orbifold G-bundle ξ^{A_j} is not a zero divisor, then the map*

$$\iota^*\colon E_G^*(X) \to \prod_j E_G^*(B_j). \tag{2.5}$$

induced from the inclusion $\iota\colon \bigsqcup B_j \hookrightarrow X$ is injective.

Proof. Essentially, the argument is similar to the proof of [HHH05, Theorem 2.3]. Here we briefly make the foundation for the orbifold stratification (2.3). If the stratification (2.3) has length 1, then $X_1 = B_1$ is a point. Therefore, (2.5) is an isomorphism.

Let the stratification (2.3) have length ℓ and assume that (2.5) is an injective map for any stratification (2.3) of length less than ℓ. By the assumption on the stratification, we have the cofiber sequence (2.4) for each $j < \ell$. Since equivariant

Euler classes are not zero divisors and a cohomology theory E_G^* is considered to have the equivariant Thom isomorphism, we get a short exact sequence

$$0 \to E_G^*(\mathrm{Th}(V_j/A_j)) \xrightarrow{q^*} E_G^*(X_j) \to E_G^*(X_{j-1}) \to 0.$$

Hence, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_G^*(\mathrm{Th}(V_j/A_j)) & \xrightarrow{q^*} & E_G^*(X_j) & \longrightarrow & E_G^*(X_{j-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_G^*(B_j) & \longrightarrow & \prod_{i\leq j} E_G^*(B_i) & \longrightarrow & \prod_{i<j} E_G^*(B_i) & \longrightarrow & 0, \end{array}$$

where the left vertical map is injective, as $e_G(\xi^{A_j})$ is not a zero divisor. The right vertical map is also injective by the induction hypothesis. Now the Five Lemma completes the proof. □

To describe the image of ι^* in (2.5), we set up the following assumptions on a stratified G-space X.

(**A**1) Simple orbifold bundles $\xi^{A_j} \colon V_j/A_j \to B_j$ for $j = 2, \dots \ell$ are E-orientable and have decompositions

$$(\xi^{A_j} \colon V_j/A_j \to B_j) \cong \bigoplus_{s<j} (\xi^{A_{j,s}} \colon V_{j,s}/A_{j,s} \to B_j) \tag{2.6}$$

into simple orbifold bundles $\xi^{A_{j,s}}$, possibly $V_{j,s}$ can be trivial. We note that (2.6) is inherited from the decomposition

$$(\xi \colon V_j \to B_j) \cong \bigoplus_{s<j} (\xi \colon V_{j,s} \to B_j)$$

of vector bundles and $A_{j,s}$'s are the quotients of A_j by non-effective kernels.

(**A**2) The restriction $\phi_j|_{S(V_{j,s}/A_{j,s})}$ of the attaching map $\phi_j \colon S(V_j/A_j) \to X_{j-1}$ to $S(V_{j,s}/A_{j,s})$ satisfies

$$\phi_j|_{S(V_{j,s}/A_{j,s})} = f_{j,s} \circ \xi^{A_{j,s}}$$

for some G-equivariant map $f_{j,s} \colon B_j \to B_s$, identifying B_s with its image in X_{j-1} for each $s < j$.

(**A**3) The equivariant Euler classes $e_G(\xi^{A_{j,s}})$ are not zero divisors and pairwise relatively prime in $E_G^*(B_j)$.

We remark that the G-invariant stratifications with trivial A_j's are studied in [HHH05]. Nevertheless, under the above assumptions on a G-space X with the property as in (2.3), one may obtain the following proposition.

Proposition 2.3. *Let X be an orbifold stratified G-space as in* (2.3) *and satisfy assumptions* (**A**1) *to* (**A**3)*. Then the image of $\iota^* \colon E_G^*(X) \to \prod_j E_G^*(B_j)$ is*

$$\Gamma_X := \Big\{ (x_j) \in \prod_{1\leq j\leq \ell} E_G^*(B_j) \;\Big|\; e_G(\xi^{A_{j,s}}) \mid x_j - f_{j,s}^*(x_s) \text{ for } s<j \Big\}.$$

Proof. The proof can be obtained from proof of [HHH05, Theorem 3.1] by replacing genuine G-vector bundles $V_{j,s} \to B_j$ and their equivariant Euler classes into simple orbifold G-bundles $\xi^{A_{j,s}} \colon V_{j,s}/A_{j,s} \to B_j$ and corresponding equivariant Euler classes $e_G(\xi^{A_{j,s}})$. For the reader's convenience, we briefly outline the argument here.

The proof goes by the induction on the filtration. For $X_1 = \{pt\}$, the claim holds as $e_G(\xi^{A_1}) = 1$. Now, we suppose that the claim holds for X_{j-1}. From assumptions (**A**1) – (**A**3) on the filtration, we have subspaces $X_{j,s}$ of X for $1 \leq s < j$ such that

$$X_{j,s} = B_s \cup_{f_{j,s} \circ \xi^{A_{j,s}}} D(V_{j,s}/A_{j,s}).$$

Then, following the proof of [HHH05, Lemma 3.5] and Proposition 2.1, one can show $e_G(\xi^{A_{j,s}})$ divides $x_j - f_{j,s}^*(x_s)$ where x_j and x_s are pull-backs of a class $x \in E_G^*(X)$ under $B_j \hookrightarrow X$ and $B_s \hookrightarrow X$, respectively.

Now we consider the natural restriction map $\gamma_j \colon \Gamma_j \to \Gamma_{j-1}$, where

$$\Gamma_i := \left\{ (x_k) \in \prod_{k \leq i} E_G^*(B_k) \;\middle|\; e_G(\xi^{A_{k,r}}) | x_k - f_{k,r}^*(x_r) \text{ for } r < k \right\}.$$

Then (**A**3) verifies that $\ker(\gamma_j) \cong e_G(\xi^{A_j}) E_G^*(B_j)$. Moreover, one can obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_G^*(X_j, X_{j-1}) & \longrightarrow & E_G^*(X_j) & \longrightarrow & E_G^*(X_{j-1}) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & \ker(\gamma_j) & \longrightarrow & \Gamma_j & \longrightarrow & \Gamma_{j-1} & \longrightarrow & 0, \end{array} \tag{2.7}$$

where the exposition about the validity of (2.7) is given in [HHH05]. Finally, one can complete the proof by the Five Lemma. □

Remark 2.4. If all A_j's associated to the stratification (2.3) are trivial, then $\mathrm{Th}(V_j/A_j) = \mathrm{Th}(V_j)$ which is the Thom space for a genuine vector bundle for each $j = 1, \ldots, \ell$. A class of examples satisfying this condition will be discussed in Section 5. With this assumption, Proposition 2.2 and Proposition 2.3 agree with the first part of Theorem 2.3 and Theorem 3.1 in [HHH05], respectively. In this case, the cohomology theory E_G^* can also be the complex cobordism MU_G^*.

3. Toric varieties over almost simple polytopes

In this section, we give a combinatorial characterization of toric varieties which is essential for the main results of this paper. Let Σ be a full dimensional rational polytopal fan in $\mathbb{R}^n$ and P the lattice polytope whose normal fan is Σ. The corresponding toric variety X_Σ is equipped with an action of compact torus $T^n \subset (\mathbb{C}^*)^n$. Here, we identify $\mathbb{R}^n$ with the Lie algebra of T^n.

Following the result of [Jur81] (we also refer to [CLS11, Theorem 12.2.5]), there is a T^n-equivariant homeomorphism

$$f \colon X_\Sigma \xrightarrow{\cong} (T^n \times P)/_\sim,$$

where $(t,p) \sim (s,q)$ whenever $p = q$ and $t^{-1}s$ is an element of the subtorus $T_{F(p)} \subseteq T^n$ whose Lie algebra is generated by the outward normal vectors of the codimension-1 faces of P which contain p if p is not in the interior of P. When p is in the interior of P, we consider $T_{F(p)}$ to be trivial.

Here, we notice that T^n-action on $(T^n \times P)/_\sim$ is induced from the multiplication on the first factor of $T^n \times P$ and the corresponding orbit map,

$$\pi \colon (T^n \times P)/_\sim \to P, \tag{3.1}$$

is given by $[t, p]_\sim \mapsto p$, where $[t, p]_\sim$ denotes the equivalence class of (t, p). Therefore, the topology of a toric variety can be studied by the combinatorics of the orbit space P and its geometric data, namely, outward normal vectors of codimension-1 faces of P.

We discuss the combinatorics of P in Subsection 3.1 and study some topological information of X_P obtained from the geometry of P in Subsection 3.2.

3.1. **Retraction sequence of a convex polytope.** The goal of this subsection is to introduce a combinatorial characterization of certain convex polytopes which was initiated in [BSS17].

Let P be a convex polytope of dimension n. Regarding P as a polytopal complex [Zie95, Definition 5.1], i.e., P is the set of all its faces, we consider a finite sequence of triples

$$(P_1, Q_1, v_1) \to (P_2, Q_2, v_2) \to \cdots ,$$

where P_j is a polytopal subcomplex of P, Q_j is a face of P_j and v_j is a vertex of Q_j, which are defined inductively as follows.

We set the initial term (P_1, Q_1, v_1) such that $P_1 = P$, v_1 is a vertex of P_1 having a neighborhood homeomorphic to $\mathbb{R}^n_\geq$ as manifold with corners and $Q_1 = P_1$ as an element of polytopal complex P_1. Given (P_j, Q_j, v_j), the next term $(P_{j+1}, Q_{j+1}, v_{j+1})$ is defined by setting

$$P_{j+1} = \bigcup \{Q \in P_j \mid v_j \notin V(Q)\},$$

where $V(Q)$ is the set of vertices of Q. Next we choose a vertex v_{j+1} of P_{j+1} such that v_{j+1} has a neighborhood homeomorphic to $\mathbb{R}^d_\geq$ as manifold with corners for some $1 \leq d \leq n$. We call v_{j+1} a *free vertex* of P_{j+1}. A face Q_{j+1} is defined to be the unique maximal face of P_{j+1} containing v_{j+1}. Note that a free vertex may not exist in general, see Remark 3.5. Hence, we proceed to define a sequence if a free vertex exists. A sequence as defined above is called a *retraction sequence* of P if the sequence ends up with (P_ℓ, Q_ℓ, v_ℓ) such that $P_\ell = Q_\ell = v_\ell$ for some vertex v_ℓ of P, where ℓ denotes the cardinality of $V(P)$.

Definition 3.1. A convex polytope is called *almost simple* if it admits at least one retraction sequence.

A simple convex polytope is almost simple. Indeed, for a simple convex polytope P in $\mathbb{R}^n$, a height function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ which is *generic* in the sense that each vertex of P has different height defines a retraction sequence of P. We refer to [BSS17, Proposition 2.3] for the details. Notice that not every retraction sequence can be obtained from a height function. Below, we introduce several examples of non-simple polytopes which are almost simple.

Example 3.2. A cone $C(P)$ on a simple polytope P has a retraction sequence. Indeed, let

$$(P_1, Q_1, v_1) \to \cdots \to (P_\ell, Q_\ell, v_\ell)$$

be a retraction sequence of P. Then,

$$(C(P_1), C(Q_1), v_1) \to \cdots \to (C(P_\ell), C(Q_\ell), v_\ell) \to (*, *, *)$$

is a retraction sequence for $C(P)$, where $*$ is the apex of $C(P)$. Notice that $C(P)$ is not a simple polytope unless P is a simplex.

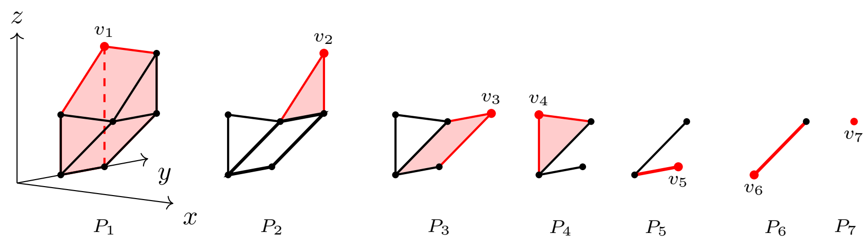

FIGURE 1. A retraction sequence of 3-dimensional Gelfand–Zetlin polytope.

Example 3.3. Let P be a 3-dimensional polytope given by the system of inequalities described as follows:

$$\begin{array}{ccccccccc} 0 & & & & 1 & & & & 2 \\ & \leq & & \leq & & \leq & & \leq & \\ & & x & & & & y & & \\ & & & \leq & & \leq & & & \\ & & & & z & & & & \end{array}$$

It is a 3-dimensional example of *Gelfand–Zetlin* polytopes which plays an important role particularly in the algebro-geometric study of flag varieties. See Figure 1 for a pictorial description of a retraction sequence of P. One can also construct different retraction sequences beginning with other vertices except for v_7.

Example 3.4. Some retraction sequences for 3-dimensional Bruhat interval polytopes [TW15] are illustrated in [LM20, Figures 25, 27], which are not simple polytopes.

Remark 3.5. Not every convex polytope has a retraction sequence, for instance the convex hull of $\{\pm(1,0,0), \pm(0,1,0), \pm(0,0,1)\}$. It is a 3-dimensional convex polytope with 6 vertices and each of the vertices does not have any neighborhood homeomorphic to $\mathbb{R}^3_{\geq}$.

In terms of toric varieties, above examples shows that the category of toric varieties over almost simple polytopes is strictly larger than the category of all simplicial toric varieties.

3.2. **Torus-equivariant stratifications.** From now on, we consider a toric variety X whose orbit space, via the orbit map $\eta\colon X \to P$ defined in (3.1), is an almost simple polytope P. Such a toric variety have the following property, which is one of the main observations in this paper.

Theorem 3.6. *A retraction sequence $(P_1, Q_1, v_1) \to \cdots \to (P_\ell, Q_\ell, v_\ell)$ of P yields a T^n-equivariant stratification of X*

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\ell = X$$

such that the quotient X_j/X_{j-1} is homeomorphic to the Thom space $\mathrm{Th}(\xi^{A_j})$ of the simple orbifold T^n-bundle

$$\xi^{A_j}\colon \mathbb{C}^{k_j}/A_j \to \eta^{-1}(v_{\ell-j+1}), \tag{3.2}$$

for some $k_j \in \mathbb{N}$ and finite abelian group A_j, where $v_i \in V(P)$ denotes the free vertex of P_i to define P_{i+1}, for $i = 1, \ldots, \ell-1$.

Proof. We define $X_j := \pi^{-1}(P_{\ell-j+1})$ for $j = 1, \dots, \ell$. Then, $X_{j+1} \subset X_j$ as $P_j \supset P_{j+1}$, see the diagram

$$\begin{array}{ccccccccccccc} X_1 & \subseteq & \cdots & \subseteq & X_{j-1} & \subseteq & X_j & \subseteq & X_{j+1} & \subseteq & \cdots & \subseteq & X_\ell \\ \big\downarrow{\scriptstyle \pi} & & & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & & & \big\downarrow{\scriptstyle \pi} \\ P_\ell & \subseteq & \cdots & \subseteq & P_{\ell-j+2} & \subseteq & P_{\ell-j+1} & \subseteq & P_{\ell-j} & \subseteq & \cdots & \subseteq & P_1. \end{array} \tag{3.3}$$

Since π is the orbit map with respect to T^n-action, (3.3) is T^n-equivariant.

To prove the second assertion, we consider the unique maximal face $Q_{\ell-j+1}$ of $P_{\ell-j+1}$ which contains the free vertex $v_{\ell-j+1}$ and denote by $U_{\ell-j+1} \subset Q_{\ell-j+1}$ the union of all relative interiors of faces in $Q_{\ell-j+1}$ containing $v_{\ell-j+1}$. For instance, the colored faces in Figure 1 are $U_{\ell-j+1}$ for $\ell = 7$ and $2 \leq j \leq 7$. Then, one can see from the property of a free vertex that $U_{\ell-j+1}$ is homeomorphic to $\mathbb{R}^{k_j}_{\geq}$ as manifolds with corners, where $k_j := \dim U_{\ell-j+1} = \dim Q_{\ell-j+1}$. Also, we note that

$$X_j - X_{j-1} = \eta^{-1}(U_{\ell-j+1}).$$

Let $\mathbb{R}(Q_{\ell-j+1})$ be the subspace of $\mathbb{R}^n$ generated by the normal vectors of facets of P intersecting $Q_{\ell-j+1}$. Notice that $\mathbb{R}(Q_{\ell-k_j+1})$ is of dimension $n-k_j$ as $Q_{\ell-j+1}$ is k_j-dimensional face of P. Since $v_{\ell-j+1}$ is a free vertex of $P_{\ell-j+1}$, there are k_j-many facets, say $F_1, \dots, F_{k_j}$, such that $v_{\ell-j+1} = \bigcap_{i=1}^{k_j}(Q_{\ell-j+1} \cap F_i)$. Consider the projection

$$\mathbb{Z}^n \to \mathbb{Z}^n/(\mathbb{Z}^n \cap \mathbb{R}(Q_{\ell-j+1})) \cong \mathbb{Z}^{k_j} \tag{3.4}$$

and the images $\mu_1, \dots, \mu_{k_j}$ of primitive outward normal vectors of $F_1, \dots, F_{k_j}$ via (3.4), respectively.

Now, the result of [BNSS19, Proposition 4.4] shows

$$\pi^{-1}(U_{\ell-j+1}) \cong D^{2k_j}/A_j, \tag{3.5}$$

where

$$A_j = \ker(\exp\left[\mu_1 \mid \cdots \mid \mu_{k_j}\right] : T^{k_j} \to T^{k_j}). \tag{3.6}$$

Here, one can regard the space (3.5) as a **q**-disc bundle of a simple orbifold T^{k_j}-bundle

$$\xi^{A_j} : \mathbb{C}^{k_j}/A_j \to \pi^{-1}(v_{\ell-j+1}), \tag{3.7}$$

where the standard T^{k_j}-action on $\mathbb{C}^{k_j}$ induces an action on $\mathbb{C}^{k_j}/A_j$ and T^{k_j} acts on the fixed point $\pi^{-1}(v_{\ell-j+1})$ trivially. Note that T^n acts on this bundle via the projection of $T^n \to T^{k_j}$ determined by (3.4). Hence we have T^n-equivariant homeomorphisms

$$X_j/X_{j-1} \cong \pi^{-1}(Q_{\ell-j+1})/\pi^{-1}(Q_{\ell-j+1} \cap P_{\ell-j+2}) \cong \mathrm{Th}(\xi^{A_j}).$$

□

We refer to [BNSS19, Proposition 4.4] for a relevant interpretation of a retraction sequence from the viewpoint of **q**-CW complexes.

Next corollary extends the result of [CLS11, Theorem 12.3.11] from the category of simplicial toric varieties to the category of toric varieties over almost simple polytopes.

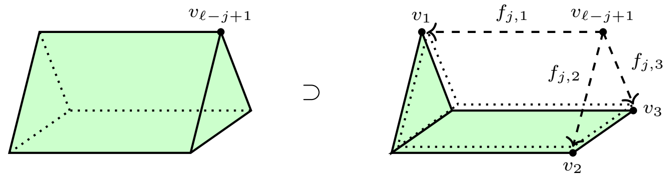

Figure 2. An attaching map.

Corollary 3.7. *The ordinary cohomology with rational coefficients of a toric variety X over an almost simple polytope vanishes in odd degrees, i.e., $H^{2k+1}(X;\mathbb{Q}) = 0$ for all k.*

Proof. The identifications (3.5) for each j realizes a *building sequence* defined in [BNSS19, Definition 2.4] of X. Since each (3.5) is even dimensional, the result directly follows from [BNSS19, Theorem 1.1]. □

Proposition 3.8. *The T^n-equivariant stratification in* (3.3) *satisfies assumptions* (**A**1), (**A**2) *and* (**A**3) *in Section 2.*

Proof. Recall that $\pi^{-1}(v_{\ell-j+1})$ in (3.2) is a fixed point. Observe that the total space of (3.7) is a quotient of a T^n-representation on $\mathbb{C}^{k_j}$ by a finite subgroup A_j of T^n. Therefore, $\mathbb{C}^{k_j}$ can be decomposed into 1-dimensional representations as

$$\mathbb{C}^{k_j} \cong \mathbb{C}(\alpha_1) \oplus \cdots \oplus \mathbb{C}(\alpha_{k_j})$$

for some characters $\alpha_s \colon T^n \to S^1$. Since each $\mathbb{C}(\alpha_s)$ is invariant under A_j, we have

$$(\xi^{A_j} \colon \mathbb{C}^{k_j}/A_j \to \pi^{-1}(v_{\ell-j+1})) \cong \bigoplus_{s=1}^{k_j} \big(\xi^{A_{j,s}} \colon \mathbb{C}(\alpha_s)/A_{j,s} \to \pi^{-1}(v_{\ell-j+1})\big)$$

for some finite groups $A_{j,1}, \dots, A_{j,k_j}$. This proves assumption (**A**1).

The quotient of the 1-dimensional representation $\mathbb{C}(\alpha_s)$ by T^n-action is identical to $\mathbb{R}_{\geq 0}$ which corresponds to an edge, say e_s, of $U_{\ell-j+1}$. Indeed, since $\pi^{-1}(U_{\ell-j+1}) \to \pi^{-1}(v_{\ell-j+1})$ is the **q**-disc bundle associated with $\xi^{A_j} \colon \mathbb{C}^{k_j}/A_j \to \pi^{-1}(v_{\ell-j+1})$, one can see that two projections $\mathbb{C}(\alpha_s)/A_{j,s} \to \pi^{-1}(v_{\ell-j+1})$ and $\pi^{-1}(e_s) \to \pi^{-1}(v_{\ell-j+1})$ are identical. Note that one can write the attaching map ϕ_j explicitly by the proof of [BSS17, Theorem 4.1]. Therefore, the image of $\phi_j|_{S(\mathbb{C}(\alpha_s))}$ is a vertex v_s of e_s which is opposite to $v_{\ell-j+1}$. A pictorial explanation is given in Figure 2. Considering

$$f_{j,s} \colon \pi^{-1}(v_{\ell-j+1}) \to \pi^{-1}(v_s)$$

as a map between two fixed points, we conclude Assumption (**A**2).

Assumption (**A**3) follows from [HHH05, Lemma 5.2], as the vectors $\mu_1, \dots, \mu_{k_j}$ defined by (3.4) are linearly independent. □

The following is an application of Proposition 2.3 to the category of toric varieties over almost simple polytopes.

Proposition 3.9. *Let X be a toric variety over an almost simple polytope with an orbifold G-equivariant stratification as in Theorem 3.6. Let $E^*_{T^n}$ be a generalized*

T^n-equivariant cohomology theory discussed in Section 2. Then,

$$E^*_{T^n}(X) = \Big\{(x_i) \in \prod_{1\leq i\leq \ell} E^*_{T^n}(pt) \;\Big|\; e_{T^n}(\xi^{A_{j,s}}) \mid x_{\ell-j+1} - f^*_{j,s}(x_s) \text{ for } s < \ell - j + 1\Big\}.$$

We note that $H^*_{T^n}(pt)$ and $K^*_{T^n}(pt)$ are isomorphic to the ring of polynomials and the ring of Laurant polynomials with n-variables, respectively. For $MU^*_{T^n}(pt)$, though its structure is unknown, it is referred as the ring of *T^n-cobordism forms* in [HHRW16].

4. Piecewise algebras and Applications

We begin this section with a summary of the concept of some piecewise algebras associated to a fan, studied in [HHRW16, Section 4]. The authors apply those algebras to weighted projective spaces to get a description of generalized equivariant cohomology theories. Here, we generalize their several results to a wider class of singular toric varieties discussed in Section 3.

Recall that if σ is a cone in a fan Σ, then all of the faces of σ belong to Σ. This leads us to form a small category $\text{CAT}(\Sigma)$ whose objects are elements of Σ and morphisms are face inclusions. The zero cone $\{0\}$ is the initial object of this category.

Let Σ be an n-dimensional rational fan in $\mathbb{R}^n$, namely, one-dimensional cones are generated by rational vectors in $\mathbb{R}^n$. Here we may identify $\mathbb{R}^n$ with the Lie algebra of T^n. Given a $k(\leq n)$-dimensional cone $\sigma \in \Sigma$, we consider a subtorus T_σ generated by primitive vectors spanning 1-dimensional cones in σ. For the category T^n-TOP of T^n-spaces, we define a diagram

$$\mathcal{V}\colon \text{CAT}(\Sigma) \to T^n\text{-TOP} \tag{4.1}$$

by $\mathcal{V}(\sigma) := T^n/T_\sigma$ and $\mathcal{V}(\sigma \subseteq \tau) = (T^n/T_\sigma \twoheadrightarrow T^n/T_\tau)$, where the projection $T^n/T_\sigma \twoheadrightarrow T^n/T_\tau$ is induced from the natural inclusion $T_\sigma \subseteq T_\tau$. Then, the toric variety X_Σ associated to Σ is homotopy equivalent to the homotopy colimit $\text{hocolim}\mathcal{V}$ of (4.1), see for instance [HHRW16, Section 4] as well as [Fra10, WZZ99]. Next, regarding $E^*_{T^n}$ as a functor from T^n-TOP to the category GCALG_E of graded commutative $E^*_{T^n}$-algebras, we consider the composition

$$\mathcal{EV}\colon \text{CAT}(\Sigma) \xrightarrow{\mathcal{V}} T^n\text{-TOP} \xrightarrow{E^*_{T^n}} \text{GCALG}_E, \tag{4.2}$$

which leads us to the following definition.

Definition 4.1. [HHRW16, Definition 4.6] Let Σ be an n-dimensional rational fan in $\mathbb{R}^n$. We call $\lim \mathcal{EV}$ the *piecewise algebra* over $E^*_{T^n}$.

We note that the object $\mathcal{EV}(\sigma)$ can be calculated explicitly as follows. The natural action of T^n on $\mathcal{V}(\sigma) = T^n/T_\sigma$ yields a T^n-representation η_σ on which T_σ acts trivially. Since T^n is abelian, η_σ can be decomposed into 1-dimensional representations, say $\eta_\sigma \cong \bigoplus_{i=1}^{n-k} \eta_\sigma(i)$, where $k = \dim T_\sigma$. We denote by $S^1_{\eta_\sigma(i)}$ the corresponding circle for each $i = 1, \ldots, n-k$. The inclusion of $S^1_{\eta_\sigma(i)}$ into the unit disc $D_{\eta_\sigma(i)}$ gives an equivariant cofiber sequence

$$S^1_{\eta_\sigma(i)} \hookrightarrow D_{\eta_\sigma(i)} \to D_{\eta_\sigma(i)}/S^1_{\eta_\sigma(i)}. \tag{4.3}$$

Regarding each term of (4.3) as an S^1-bundle, disc bundle over a point and the associated Thom space, respectively, we may consider the equivariant Euler class $e_{T^n}(\eta_\sigma(i)) \in E^*_{T^n}$ for each $i = 1, \ldots, n-k$.

Proposition 4.2. [HHRW16, Section 4, (4.11)]
$$\mathcal{EV}(\sigma) \cong E^*_{T^n}(T^n/T_\sigma) \cong E^*_{T^n}(pt)/(e_{T^n}\big(\eta_\sigma(1)), \dots, e_{T^n}(\eta_\sigma(n-k))\big).$$

The proof of the following theorem is almost same as the proof of [HHRW16, Theorem 5.5] with very few modifications in notation. To be more precise, one needs to replace the equivariant Euler classes [HHRW16, (3.8)] corresponding to the filtration of a divisive weighted projective space by the equivariant Euler classes for simple orbifold bundles defined in Section 2.

Theorem 4.3. *Let X_P be a toric variety over an almost simple polytope P and Σ_P the normal fan of P. Let $E^*_{T^n}$ be a T^n-equivariant cohomology theory discussed in Section 2. Then, $E^*_{T^n}(X_P)$ is isomorphic to the piecewise algebra $\lim \mathcal{EV}$, as $E_{T^n}(pt)$-algebras.*

Remark 4.4. Compact symplectic toric orbifolds are toric varieties with fans defined by their moment polytope which are simple, see [LT97, Section 9]. Therefore they satisfy the hypotheses of Theorem 3.6 and Theorem 4.3.

5. Divisive toric varieties

In this section, we introduce the notion of *divisive toric varieties* motivated by divisive weighted projective spaces [BFR09]. They are singular toric varieties which may have singularities beyond orbifold singularities, whose generalized equivariant cohomologies can be obtained over integers. We follow the same arguments as we discussed in Section 4. Here, the cohomology theory $E^*_{T^n}$ in this section can also be $MU^*_{T^n}$ without taking tensor with $\mathbb{Q}$.

Definition 5.1. Let X be a toric variety satisfying the hypothesis of Theorem 3.6. If the finite groups A_j's in (3.7) are trivial, then we call X a *divisive toric variety.*

Example 5.2. Recall the 3-dimensional Gelfand–Zetlin polytope P described in Example 3.3. The outward normal vectors of facets intersecting $v_1 = (0, 2, 2)$ in P_1 of Figure 1 are $(-1, 0, 0)$, $(0, -1, 1)$ and $(0, 1, 0)$, which form an integral basis of $\mathbb{Z}^3$. See Figure 3 for the vertices and primitive outward normal vectors of P. Hence, the finite group A_7 defined in (3.6) is trivial. To compute A_6, we consider the facet given by $\{x = 1\}$ whose primitive outward normal vector is $(1, 0, 0)$. In this case, the map (3.4) yields the projection $T^3 \twoheadrightarrow T^2$ onto the last two coordinates. Hence, $A_6 = \ker(\rho\colon T^2 \to T^2)$, where $\rho(t_1, t_2) = (t_1^{-1}t_2, t_1)$, which is trivial. We refer to [BSS17, Proposition 4.3] for the general statement about this computation. Finally, one can conclude by similar computations for the other vertices $v_2, \dots, v_6$ that the associated toric variety X_P is divisive.

The following proposition is straightforward from Corollary 3.7 and the definition of a divisive toric variety.

Proposition 5.3. *The ordinary cohomology with integer coefficients of a divisive toric variety X is torsion free and vanishes in vanishes in odd degrees.*

Note that [HHRW16] defines the piecewise algebras of a fan with integers. To be more precise, the map $\mathcal{EV}$ in (4.2) is a composition of $\mathcal{V}$ and the T^n-equivariant cohomology theories without taking tensor with $\mathbb{Q}$. In particular, we denote the associated piecewise algebras by

- $PP[\Sigma]$, the algebra of piecewise polynomials if $E^*_{T^n} = H^*_{T^n}$;

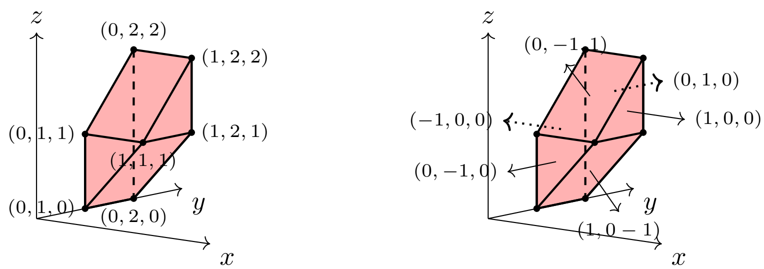

Figure 3. A 3-dimensional Gelfand–Zetlin Polytope.

- $PL[\Sigma]$, the algebra of piecewise Laurant polynomials if $E^*_{T^n} = K^*_{T^n}$;
- $PC[\Sigma]$, the algebra of piecewise T^n-cobordism forms if $E^*_{T^n} = MU^*_{T^n}$.

If a toric variety is divisive, then it is equipped with a T^n-equivariant stratification in the sense of [HHH05, Section 2]. So one can apply their results to divisive toric varieties, which yields the following proposition over integers.

Proposition 5.4. *Let X be a divisive toric variety over an almost simple polytope P and Σ_P the normal fan of P. Then,*

(1) *$H^*_{T^n}(X_P;\mathbb{Z})$ is isomorphic to $PP[\Sigma_P]$ as an $H^*_{T^n}(pt;\mathbb{Z})$-algebra;*
(2) *$K^*_{T^n}(X_P;\mathbb{Z})$ is isomorphic to $PL[\Sigma_P]$ as a $K^*_{T^n}(pt;\mathbb{Z})$-algebra;*
(3) *$MU^*_{T^n}(X_P;\mathbb{Z})$ is isomorphic to $PC[\Sigma_P]$ as an $MU^*_{T^n}(pt;\mathbb{Z})$-algebra.*

To exhibit an example of piecewise algebra, we revisit the 3-dimensional Gelfand–Zetlin polytope discussed in Example 3.3 and Example 5.2.

Example 5.5. Let P be the 3-dimensional Gelfand–Zetlin polytope and Σ_P its normal fan. For each face Q in P of dimension k $(0 \leq k \leq 3)$, we denote by σ_Q the associated $(3-k)$-dimensional cone in Σ_P, i.e., σ_Q is the cone generated by normal vectors of facets intersecting the relative interior of Q. For example, $\sigma_{(1,1,1)}$ is the cone generated by $(0,-1,1), (0,-1,0), (1,0,-1)$ and $(1,0,0)$. Particularly when $E^*_{T^3} = H^*_{T^3}$,

$$\mathcal{EV}(\sigma_v) \cong H^*_{T^3}(pt) \cong \mathbb{Z}[u_1,u_2,u_3]$$

for each vertex v of P. Hence, the ring $PP[\Sigma_P]$ of piecewise polynomials with rational coefficients is the set of tuples

$$(f_{\sigma_v})_{v\in V(P)} \in \bigoplus_{v\in V(P)} \mathbb{Z}[u_1,u_2,u_3]$$

such that $f_{\sigma_v}|_{\sigma_e} = f_{\sigma_w}|_{\sigma_e}$ whenever v and w are connected by an edge e. Here, we list some of its elements as follows:
See Figure 4 for the description of those elements on the original polytope P.

Remark 5.6. The first element in Table 1 or Figure 4 is the multiplicative identity of $PP[\Sigma_P]$. The faces or the unions of faces in Figure 4 whose vertices have nontrivial elements in $\mathbb{Z}[u_1,u_2,u_3]$ are *dual Kogan faces* [KM05] and polynomials are related to Thom classes defined in [MMP07, MP06].

$\sigma_{(1,1,1)}$	$\sigma_{(0,1,0)}$	$\sigma_{(0,2,0)}$	$\sigma_{(0,1,1)}$	$\sigma_{(1,2,1)}$	$\sigma_{(1,2,2)}$	$\sigma_{(0,2,2)}$
1	1	1	1	1	1	1
0	0	u_2	0	u_2	u_2+u_3	u_2+u_3
u_1+u_3	0	0	u_3	u_1+u_3	u_1+u_3	u_3
0	0	0	0	$u_2(u_1+u_3)$	$u_1(u_2+u_3)$	0
0	0	0	0	0	$u_3(u_2+u_3)$	$u_3(u_2+u_3)$
0	0	0	0	0	$u_1u_3(u_2+u_3)$	0

TABLE 1. Some elements in $PP[\Sigma_P]$.

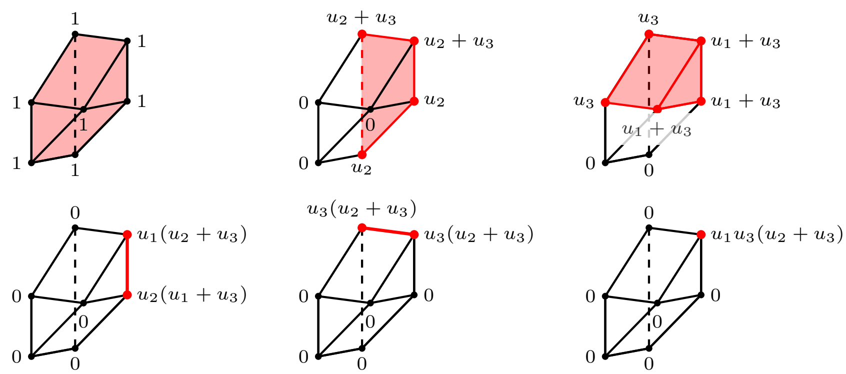

FIGURE 4. Dual description of Table 1.

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