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GENUS ZERO CORRELATION FUNCTIONS IN $c < 1$ STRING THEORY

Suresh Govindarajan, T. Jayaraman and Varghese John[1]

The Institute of Mathematical Sciences

C.I.T. Campus, Taramani

Madras 600113 INDIA

Abstract

We compute N-point correlation functions of pure vertex operator states(DK states) for minimal models coupled to gravity. We obtain agreement with the matrix model results on analytically continuing in the numbers of cosmological constant operators and matter screening operators. We illustrate this for the cases of the $(2k-1, 2)$ and $(p+1, p)$ models.

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[1]email: suresh, jayaram, john@imsc.ernet.in

1 Introduction

One of the important issues in the understanding of $c < 1$ models coupled to 2-d gravity has been the computation of the correlation functions in the continuum approach. These are well understood from the matrix model point of view. The most convenient method to calculate such correlation functions in the continuum approach is of course to use the free field representation of the $c < 1$ models. Three point functions on the sphere for states within the conformal grid were first calculated by this method in ref. [1] and then extended to 3-point functions with pure vertex operator states(DK states) with momenta outside the Kac table[2, 3, 4]. These calculations matched the matrix model results. N-point correlation functions(of states inside the Kac table) that did not need matter screening were calculated in ref.[5]. In particular the role of the screening operator was not clear. In this paper, we calculate arbitrary N-point correlation functions(with matter screening) involving DK states for the cases of $(2k-1, 2)$ and $(p+1, p)$ models. The general case will be dealt elsewhere. We shall show that these results agree with the corresponding matrix model results. This agreement requires an analytic continuation in the number of screening operators just as in refs. [1, 2, 7]. The free field representation of the matter part of the theory, involves the use of a screening operator as a BRST operator[6]. In the case of matter coupled to gravity, the physical states are given by the double cohomology of this BRST operator and the usual string BRST.

It was shown that DK states are also physical and are related to physical states of higher ghost number by means of descent equations in the two BRST operators[8]. It was also shown how correlation functions with states of higher ghost number could be converted into correlation functions involving DK states. Hence, calculation of correlation functions with DK states should provide a complete description of the theory. There is also a ring of ghost number zero operators in the theory[9] (similar to those found for $c = 1$ coupled to Liouville[10]). These operators prove useful in obtaining recursion relations involving DK states and enable us to convert integrals which are quite hard to compute into those for which the results of Dotsenko and Fateev[11] are sufficient. This enables us to calculate all correlation functions which require positive number of screening in the Liouville and matter sector. We then obtain all other correlation functions by analytically continuing these results to fractional number of screening operators.

2 Classification of DK States

The energy-momentum tensors of the matter and Liouville fields are given by

$$\begin{aligned} T^M &= -\tfrac{1}{4}\partial X \partial X + i\alpha_0 \partial^2 X \quad , \\ T^L &= -\tfrac{1}{4}\partial \phi \partial \phi + i\beta_0 \partial^2 \phi \quad , \end{aligned}$$

with central charges $c_M = 1 - 24\alpha_0^2$ and $c_L = 1 - 24\beta_0^2$. For the (p', p) model, $2\alpha_0 = \frac{(p'-p)}{\sqrt{p'p}}$ and $2\beta_0 = \frac{i(p'+p)}{\sqrt{p'p}}$. The physical vertex operators are of the form $exp(i\alpha X + i\beta\phi)$. In the matter part, there are two screening operators $Q_\pm$

$$Q_+ = \Delta(1 + \frac{p'}{p}) \int e^{i\alpha_+ X}, \quad Q_- = \Delta(1 + \frac{p}{p'}) \int e^{i\alpha_- X}$$

where $\alpha_+ = \frac{p'}{\sqrt{pp'}}$, $\alpha_- = \frac{-p}{\sqrt{pp'}}$ and $\Delta(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$. Note that the measure factor of $\frac{d^2z}{\pi}$ is implicit in all integrals. Both the screening operators furnish equally good BRST operators(in the sense of Felder[6]). One finds that physical operators can be represented by vertex operator states(DK states) with matter momenta both inside and outside the Kac table[3, 2, 8]. However, pure vertex operator states from outside the Kac table depend on the choice of screening operator as a BRST operator(we call this a choice of resolution). We shall deal only with operators which satisfy the condition $\beta < \beta_0$ in order that their scaling dimensions match with those obtained in the matrix models. This(choice of resolution) leads to two distinct sets of DK states. Further, one has to complete the set by including states belonging to the edge of the Kac table. For the case of the $(p+1, p)$, the DK states in the Q_- resolution are

$$V_n^\alpha = exp\frac{[p(n-2) + \alpha]\phi + [pn + \alpha + 2]iX}{2\sqrt{p(p+1)}} \quad , \tag{1}$$

where $\alpha = 0, \ldots, (p-2)$. This includes edge states of the form $(j(p+1), m)$. We have excluded the "wrong-edge" states which correspond to $\alpha = (p-1)$[9]. V_0^0 is the cosmological constant operator. Also, the other screening operator $Q_+(\sim \int V_2^0)$ is now a physical operator. Thus Q_+ cannot be used as a screening operator anymore. For momentum conservation in the matter sector, we shall restrict to using Q_- exclusively as a screening operator. Hence, we will need to use analytic continuation in the matter sector when necessary.

Similarly, the DK states in the Q_+ resolution are

$$V_n^\alpha = exp\frac{[(p+1)(n-2)+\alpha+2]\phi + [-(p+1)n-\alpha]iX}{2\sqrt{p(p+1)}} \tag{2}$$

where $\alpha = 0, \ldots, (p-1)$. This includes edge states of the form (m', jp). The other screening operator Q_- is now a physical operator. We normalise the pure vertex operators by $\Delta(n + \frac{(\alpha+1)}{p})$ for the Q_- resolution and $\Delta(n + \frac{(\alpha+1)}{(p+1)})$ for the Q_+ resolution. Note that this normalisation is not singular for the allowed values of α.

3 Correlation Functions of DK states

3.1 $(2k-1, 2)$ Models

The DK states for the $(2k-1, 2)$ models in the Q_- resolution are given by

$$V_n = \Delta(1+n+\frac{1}{2}) \;\; exp\frac{(n-k)\phi + (n+k-1)iX}{\sqrt{2(2k-1)}} \quad , \tag{3}$$

where $n = 0, 1, \ldots$. Note that the "physical" screening operator Q_+ is given by V_k. The ring elements are generated by[9]

$$a_- = -|bc + \frac{1}{2}\sqrt{\frac{(2k-1)}{2}}\partial(\phi - iX)|^2 \;\; exp\frac{(\phi+iX)}{\sqrt{2(2k-1)}} \quad . \tag{4}$$

The ring elements are $(a_-)^n$. The action of the ring element on the DK states is given by[12, 13, 14]

$$\lim_{z\to w} a_-(z)c\bar{c}V_n(w) \sim c\bar{c}V_{n+1}(w), \quad a_-(z)c\bar{c}e^{i\alpha_- X}(w) \sim 0 \tag{5}$$

Now consider a charge conserving correlation function with one a_- and DK states

$$F(w,\bar{w}) \equiv \langle a_-(w) c\bar{c}V_{n_1}(0) c\bar{c}V_{n_2}(1) c\bar{c}V_{n_3}(\infty) \prod_{i=4}^{N} \int V_{n_i} \frac{1}{R!}(Q_-)^R \rangle \tag{6}$$

One can check that $\partial_w F = \partial_{\bar{w}} F = 0$ using $\partial_w a_- = \{Q_B, b_{-1}a_-\}$ and then deforming the contour of Q_B. This implies that $F(w,\bar{w})$ is a constant independent of $w(\bar{w})$. Equating $F(0)$ with $F(1)$ and using eqn.(4), we obtain

$$\begin{aligned} &\langle c\bar{c}V_{n_1+1}(0) c\bar{c}V_{n_2}(1) c\bar{c}V_{n_3}(\infty) \prod_{i=4}^{N} \int V_{n_i} \frac{1}{R!}(Q_-)^R \rangle \\ &= \langle c\bar{c}V_{n_1}(0) c\bar{c}V_{n_2+1}(1) c\bar{c}V_{n_3}(\infty) \prod_{i=4}^{N} \int V_{n_i} \frac{1}{R!}(Q_-)^R \rangle \quad . \end{aligned} \tag{7}$$

This gives us the following recursion relation(similar to the one in ref. [15])

$$V_{n+1}(z)V_m(w) = V_n(z)V_{m+1}(w) \tag{8}$$

In particular, $V_nV_0 = V_{n-k}V_k$. This is sufficient to convert all correlation functions(with positive integer number of Liouville screening) to ones which are of the Dotsenko-Fateev type[11] making them computable. We shall now demonstrate this explicitly. Consider the correlation function

$$\langle\langle \prod_{i=1}^{L} \int V_{n_i} \rangle\rangle = \mu^S \Gamma(-S) \langle c\bar{c}V_{n_1}(0) c\bar{c}V_{n_2}(1) c\bar{c}V_{n_3}(\infty) \prod_{i=4}^{L} \int V_{n_i} (\int V_0)^S \frac{1}{R!}(Q_-)^R \rangle, \tag{9}$$

where $S = (2-L) + \frac{1}{k}(\sum_{i=1}^{L} n_i) + \frac{1}{k}$ and $2R = 2(\sum_{i=1}^{L} n_i) - (L+S-4)$. The expressions for S and R follow from the charge conservation relations in Liouville and matter respectively. The RHS of the above equation is obtained after completing the zero-mode integration of the Liouville mode[1] and introducing screening in the matter sector for charge conservation. We shall assume that both R and S are positive integers. We can now use the recursion relation, to obtain

$$\langle\langle \prod_{i=1}^{L} \int V_{n_i} \rangle\rangle = \mu^S \Gamma(-S) \langle c\bar{c}V_0(0) c\bar{c}V_0(1) c\bar{c}V_{k-1}(\infty) (\int V_k)^{L+S-3} \frac{1}{R!}(Q_-)^R \rangle \tag{10}$$

This correlator can be evaluated using formula (B.10) of ref.[11]. This gives

$$\langle\langle\prod_{i=1}^{L}\int V_{n_i}\rangle\rangle = \mu^S\Gamma(-S)(L+S-3)!\frac{\Gamma(1)}{\Gamma(0)}\rho = \mu^S\frac{\Gamma(L+S-2)}{\Gamma(S+1)}\rho \quad ,$$

where $\rho = \frac{2}{(2k-1)}$. The cases of non-integer S and R are obtained by analytically continuing the above result to non-integer values. Hence, one obtains

$$\langle\langle\prod_{i=1}^{L}\int V_{n_i}\rangle\rangle = \rho\mu^S\frac{\Gamma(\frac{(\sum_i n_i)+1}{k})}{\Gamma(S+1)} \quad , \tag{11}$$

This result is in complete agreement with matrix model results[16, 15]. This represents the generalisation of the 3 point function calculations given in [1, 2]. The use of the recursion relation has facilitated the calculation of arbitrary N-point functions.

3.2 $(p+1,p)$ models

The DK states in the Q_- resolution are

$$V_n^\alpha = \Delta(n+\frac{(\alpha+1)}{p})exp\frac{[p(n-2)+\alpha]\phi+[pn+\alpha+2]iX}{2\sqrt{p(p+1)}}$$

where $\alpha = 0,\ldots,(p-2)$. The ring elements for these models are generated by[13]

$$\begin{aligned} a_+ &= -|bc+\tfrac{1}{2}\sqrt{\tfrac{p}{p+1}}\partial(\phi+iX)|^2 \quad exp\tfrac{(p+1)(\phi-iX)}{2\sqrt{p(p+1)}}, \\ a_- &= -|bc+\tfrac{1}{2}\sqrt{\tfrac{p+1}{p}}\partial(\phi-iX)|^2 \quad exp\tfrac{p(\phi+iX)}{2\sqrt{p(p+1)}}. \end{aligned} \tag{12}$$

The ring elements in the Q_- resolution are[9]

$$(a_-)^n, \quad a_+(a_-)^n, \quad \ldots, \quad (a_+)^{p-1}(a_-)^n \quad . \tag{13}$$

with the equivalence relation $a_+^p \sim a_-^{p+1}$. The action of the ring on DK states is[12, 13, 14]

$$\begin{aligned} \lim_{w\to z} a_-(w)c\bar{c}V_n^\alpha(z) &\sim c\bar{c}V_{n+1}^\alpha(z) \quad , \\ \lim_{w\to z} a_+(w)c\bar{c}V_n^\alpha(z)\int V_m^\beta(t) &\sim c\bar{c}V_{n+m-1}^{\alpha+\beta+1}(z) \quad , \end{aligned} \tag{14}$$

where we have normalised all the DK states by their appropriate leg-factors. We shall now use the ring elements to calculate arbitrary N-point functions explicitly for the case of the

Ising model which is the $(4,3)$ model. We shall then present results for the general $(p+1,p)$ models. Details of the calculations will be given elsewhere. The case of pure gravity i.e., the $(3,2)$ model has already been obtained in the earlier section. In the Ising model, V_0^0 corresponds to the identity operator, $V_0^1=\sigma$ and $V_1^1=\epsilon$ correspond to the spin and energy operators respectively(with appropriate Liouville dressing). V_2^0 is the "physical" screening operator. One can prove the following shift recursion relation. The arguments are identical to the one used in deriving eqn.(8).

$$V_{n+1}^{\alpha}(z)V_m^{\beta}(w)=V_n^{\alpha}(z)V_{m+1}^{\beta}(w) \tag{15}$$

Now consider a charge conserving correlation function with one a_+ and DK states.

$$\langle a_+(w)c\bar{c}V_n^0(0)c\bar{c}V_m^1(1)c\bar{c}V_r^{\alpha}(\infty)\prod_{i=1}^{L}\int V_{n_i}^0\prod_{j=1}^{M}\int V_{m_i}^1\frac{1}{R!}(Q_-)^R\rangle \tag{16}$$

Using the w independence of the above correlation function, we equate the value of the correlator at $w=0$ and 1. This gives us after using the second equation in (14)

$$\begin{aligned}
&\sum_{k=1}^{L}\langle c\bar{c}V_{n+n_k-1}^1(0)c\bar{c}V_m^1(1)c\bar{c}V_r^{\alpha}(\infty)\prod_{i=1,i\neq k}^{L}\int V_{n_i}^0\prod_{j=1}^{M}\int V_{m_i}^1\rangle\\
&+\sum_{k=1}^{M}\langle c\bar{c}V_{n+m_k-1}^2(0)c\bar{c}V_m^1(1)c\bar{c}V_r^{\alpha}(\infty)\prod_{i=1}^{L}\int V_{n_i}^0\prod_{j=1,j\neq k}^{M}\int V_{m_i}^1\rangle\\
&=\sum_{k=1}^{L}\langle c\bar{c}V_n^0(0)c\bar{c}V_{m+n_i-1}^2(1)c\bar{c}V_r^{\alpha}(\infty)\prod_{i=1,i\neq k}^{L}\int V_{n_i}^0\prod_{j=1}^{M}\int V_{m_i}^1\rangle\\
&+\sum_{k=1}^{M}\langle c\bar{c}V_n^0(0)c\bar{c}V_{m+m_k}^0(1)c\bar{c}V_r^{\alpha}(\infty)\prod_{i=1}^{L}\int V_{n_i}^0\prod_{j=1,j\neq k}^{M}\int V_{m_i}^1\rangle
\end{aligned} \tag{17}$$

Using eqn. (15), one can see that every term inside each of the sums are the same. One can now see that the following operator relations provides a consistent solution to the above equality.

$$\begin{aligned}
V_m^2(z)V_n^1(w) &= V_m^0(z)V_{n+1}^0(w)\\
V_m^2(z)V_n^0(w) &= V_m^1(z)V_n^1(w)
\end{aligned} \tag{18}$$

However, V_m^2 belongs to the "wrong-edge" and is not physical. However by using this relation twice, we obtain the following recursion which involves only physical operators

$$V_m^1(z)V_n^1(w)V_r^1(t)=V_{m+1}^0(z)V_n^0(w)V_r^0(t)\quad . \tag{19}$$

For the case of the $(4,3)$ model, eqns. (15) and (19) are sufficient to convert all charge conserving correlation functions to those which are of the Dotsenko-Fateev type and hence are computable. We shall now demonstrate this. Consider the following correlation function

$$\begin{aligned}&\langle\langle\prod_{i=1}^{L}\int V_{n_i}^{\alpha_i}\rangle\rangle\\&=\mu^S\Gamma(-S)\quad\langle c\bar{c}V_{n_1}^{\alpha_1}(0)c\bar{c}V_{n_2}^{\alpha_2}(1)\ c\bar{c}V_{n_3}^{\alpha_3}(\infty)\textstyle\prod_{i=4}^{L}\int V_{n_i}^{\alpha_i}(\int V_0^0)^S\frac{1}{R!}(Q_-)^R\rangle\quad,\end{aligned}\qquad(20)$$

where $S=\frac{\sum_{i=1}n_i}{2}-L+2+\frac{(\sum_{i=1}\alpha_i)+2}{6}$ and $R=\sum_{i=1}(\frac{n_i}{2}+\frac{\alpha_i}{6})+\frac{S}{3}$. When, S and R are positive integers, using eqns. (15) and (19) in eqn.(20) we obtain

$$\langle\langle\prod_{i=1}^{L}\int V_{n_i}^{\alpha_i}\rangle\rangle=\mu^S\Gamma(-S)\langle c\bar{c}V_0^0(0)c\bar{c}V_0^0(1)\ c\bar{c}V_1^1(\infty)(\int V_2^0)^{(L+S-3)}\frac{1}{R!}(Q_-)^R\rangle\quad.\qquad(21)$$

The correlation function in the RHS can be explicitly computed using the formula of Dotsenko and Fateev. We obtain

$$\langle\langle\prod_{i=1}^{L}\int V_{n_i}^{\alpha_i}\rangle\rangle=\mu^S\frac{(L+S-3)!}{S!}\rho\quad,\qquad(22)$$

where $\rho=\frac{3}{4}$. This is in agreement with matrix model results. For the cases when S and R are not positive integers, the results are obtained by analytic continuation. Of course, one has to take care that the Z_2 invariance of the minimal models is not violated. One imposes this by setting all non-Z_2 invariant correlators to zero by hand. For the example of Ising model, the Z_2 charge of the operator V_n^α is $(-1)^{n+\alpha}$. So the correlation function in (20) is non-zero provided $\sum_i(n_i+\alpha_i)$ is even. For such cases, the result one obtains after analytic continuation in both S and R is

$$\langle\langle\prod_{i=1}^{L}\int V_{n_i}^{\alpha_i}\rangle\rangle=\rho\mu^S\frac{\Gamma(\frac{\sum_i n_i}{2}+\frac{(\sum_i\alpha_i)+2}{6})}{\Gamma(S+1)}\quad.\qquad(23)$$

This generalises for the $(p+1,p)$ model. For Z_2 invariant correlation functions one obtains

$$\langle\langle\prod_{i=1}^{L}V_{n_i}^{\alpha_i}\rangle\rangle=\rho\mu^S\frac{\Gamma(\frac{\sum_i n_i}{2}+\frac{(\sum_i\alpha_i)+2}{2p})}{\Gamma(S+1)}\quad,\qquad(24)$$

where $S=\frac{\sum_{i=1}n_i}{2}-L+2+\frac{(\sum_{i=1}\alpha_i)+2}{2p}$ and $\rho=\frac{p}{p+1}$. (Note that the Z_2 charge of the operator V_n^α is $(-1)^{(np+\alpha)}$).

4 Discussion

It is known that the DK states and ring elements of $c < 1$ string theory can be obtained by a target space Lorentz transformation of $c = 1$ string tachyons and ring elements[9]. The correlation functions we have evaluated here lead to integrals which are identical to those obtained in $c = 1$ string theory provided one chooses tachyons of specific momenta and chirality. Indeed, all DK states of a given resolution map on to tachyons of the same chirality while the screening operator is of the opposite chirality. However, the $\frac{\Gamma(1)}{\Gamma(0)}$ factor is compensated for by the $\Gamma(-S)$ coming from the zero-mode integration of the Liouville field. Evaluating the integrals, in particular requires some care. The consistent prescription appears to be the one where using the recursion relations the integrals appear in a form such that formula (B.10) of [11] is applicable. We also note that even for 3-point functions, it seems that the Liouville and matter sectors cannot be integrated separately as the results do not agree with the recursion relations.

The analytic continuation in the number of matter screening operators and the cosmological constant operators as used here appears essential to reproduce the matrix model results. The use of other physical operators as screening operators[9, 17] does not appear to be possible beyond three point functions. Analytic continuation in matter may seem surprising but is essential especially in the case of non-unitary models where the cosmological constant operator carries non-trivial matter momentum. An analytic continuation in the number of cosmological constant operators in these models forces an analytic continuation in the matter sector too. However, a deeper understanding of the origin of this analytic continuation still eludes us.

The well known $\alpha \to (2\alpha_0 - \alpha)$ duality symmetry of the free field representation of minimal models has to be given up for states outside the Kac table in order to be consistent with the choice of resolution. Though this is not true for states inside the Kac table, it is simpler to work with only the matter momenta given by the parametrisation of the resolution.

The string equation of the $(p+1, p)$ model in the matrix model approach can be obtained either from a $p-th$ order differential operator or a $(p+1)-th$ order differential operator. This ambiguity is reflected in the two possible parametrisations of the physical states in the continuum.

We thus appear to have a complete prescription to calculate correlation functions on the sphere for the arbitrary (p',p) minimal model coupled to gravity. It would be interesting to extend these techniques to higher genus where again the matrix model results are available.

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