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Generalized trigonometric interpolation

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ABSTRACT

This article proposes a generalization of the Fourier interpolation formula, where a wider range of the basic trigonometric functions is considered. The extension of the procedure is done in two ways: adding an exponent to the maps involved, and considering a family of fractal functions that contains the standard case. The studied interpolation converges for every continuous function, for a large range of the nodal mappings chosen. The error of interpolation is bounded in two ways: one theorem studies the convergence for Hölder continuous functions and other develops the case of merely continuous maps. The stability of the approximation procedure is proved as well.

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1. Introduction

Trigonometric interpolation is a useful method for data processing. It is specially suitable for data which are periodic of known period or variable on the circle, although in case of non-periodicity some manipulations may be made in order to deal with a periodic function.

In the case of equidistant nodes, the interpolation can be transformed into the problem of finding a phase polynomial:

$$p(x) = \beta_0 + \beta_1 e^{ix} + \cdots + \beta_{n-1} e^{i(n-1)x}$$

such that

$$p(x_k) = y_k,$$

for $k = 0, 1, \ldots, n-1$. The interpolation of equidistant support points (x_k, y_k), where $x_k = 2\pi k/n$, leads to expressions of the form:

$$\beta_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k e^{-\frac{2\pi ijk}{n}}, \; j = 0, \ldots, n-1.$$

The computational evaluation of these expressions is expensive and induced the discovery of Cooley & Tukey Fast Fourier Transform algorithm, that opened up a wide range of new areas of application like image, audio and signal processing, resolution of partial differential equations, noise reduction, etc.

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The properties of the ordinary trigonometric interpolation in terms of convergence are nice if one considers the error in $\mathcal{L}^p$-sense. A theorem of Marcinkiewicz and Zygmund [1] proves that if f is a continuous periodic function on $[0, 2\pi]$ and $S_n(f)$ is the trigonometric interpolation of order n, then

$$\|S_n(f) - f\|_p \leq K d_n^*(f) \leq K' \omega_f \left(\frac{1}{n}\right),$$

where $\|\cdot\|_p$ is the norm in the space $\mathcal{L}^p$ and K, K' are positive constants. The quantity $d_n^*(f)$ is the minimum uniform distance between f and the set of trigonometric polynomials of degree n, and ω_f is the modulus of continuity of f. This inequality proves the convergence in $\mathcal{L}^p$-sense for a continuous function (since $\omega_f(1/n)$ goes to zero as n tends to infinity). However the continuity is not a necessary condition for this type of convergence. The Riemann integrability of f is sufficient to provide p-convergence [1]. The article of Prestin and Xu [2] deals with mild sufficient conditions for this kind of convergence as well.

All these nice results fail in the case of pointwise and uniform error and convergence. Marcinkiewicz [3] proved the existence of continuous functions for which the sequence of trigonometric interpolating polynomials (with equidistant nodes) diverges everywhere. Gosselin [4] resumed the subject, considering partitions where the points are defined as:

$$\gamma + \frac{2i}{2n+1}\pi,$$

where γ is any real number and $i = 0, \pm 1, \pm 2, \ldots$. In his Ref. [4], the author proves the strong dependence of the convergence behavior for certain functions on the number γ. For some choices of γ the behavior may be worst possible, diverging for any x non-null, whereas for the same function a different choice would provide uniform convergence.

There are classical results on this topic due to Dunham Jackson. In [5], he proves the following inequality for a continuous periodic function and its trigonometric interpolant at equidistant points:

$$\|f - S_n\|_\infty \leq 7\|f\|_\infty \log(n).$$

Of course this inequality does not ensure convergence when the nodes increase indefinitely, but later in same article [5], the author proves the following result:

Theorem 1. *If f satisfies a Dini–Lipschitz condition, then S_n converges to f uniformly as n tends to infinity. For instance, if f satisfies the Lipschitz condition:*

$$|f(x') - f(x'')| \leq \lambda |x' - x''|,$$

for all x', x'', then for $n \geq 2$,

$$\|f - S_n\|_\infty \leq \frac{21\lambda \log(n)}{n}.$$

If f is Hölder continuous, the uniform convergence is sure as well since f satisfies a Dini–Lipschitz condition ($\omega_f(\delta) \leq K\delta^q$).

There are indeed a large amount of results concerning this topic. For instance, in Ref. [6], it is proved that the error of the trigonometric interpolation on an even partition goes to zero if the second derivative is piecewise continuous. Moreover, the rate of this convergence takes into account the smoothness of the function f, that is to say, the rate improves for the functions that have more derivatives. Specifically, if $M_{r+1} = \|f^{(r+1)}\|_\infty$ exists, then

$$E_n = \mathcal{O}\left(\frac{M_{r+1}}{n^{r-1/2}}\right),$$

where E_n is the maximum error of the interpolation on the interval. Another important result states that if $f \in \mathcal{L}^2$ then

$$|E_n(x)| \leq \frac{\psi_n}{n^{r-1/2}},$$

where $\psi_n = o(1)$, as n tends to infinity. Thus, trigonometric interpolation is not characterized by the undesirable property of "saturation of smoothness": the rate of convergence does not react to additional smoothness of the function.

The Lebesgue constants of the associated partition play an essential role in all the interpolation problems. They are defined as:

$$\Lambda_n = \max_x \sum_{j=1}^{n} |\varphi_j(x)|, \tag{1}$$

where φ_j are the nodal basic functions used to define the interpolation (for instance, the Lagrange polynomials in case of polynomial interpolation). The Lebesgue constants of the trigonometric interpolation behave better than the corresponding in the polynomial case. In Ref. [6], the authors prove the next theorem (in case of equidistant nodes):

Theorem 2. *The Lebesgue constant of the trigonometric interpolation satisfies the upper estimates:*

$$\Lambda_n \leq 2(n+1).$$

Moreover,

$$|a_k| \leq 2\max|y_m|,$$

$$|b_k| \leq 2\max|y_m|,$$

where a_k, b_k are the coefficient of the trigonometric interpolating polynomial and y_m are the value of the function at the node points.

In the case of polynomial interpolation, the equidistant Lebesgue constants increase exponentially [6], although in case of choosing the Chebyshev nodes, the growth is logarithmic (in terms of the number of points). The problem is that one cannot always choose the position of the nodes.

Interpolation theory plays an important role in many fields of science and engineering. In many practical situations, we do not know if a variable providing a set of samples is differentiable or not, we only know its values at some nodes. However, most of the current methods make use of smooth functions. To palliate this issue, Barnsley (see [7,8]) introduced the concept of fractal interpolation function (FIF) using the theory of Iterated Function System (IFS). FIF are defined as fixed points of maps between spaces of functions. FIF have some advantages over the classical interpolation functions such as (i) FIF provide a method to render smooth or non-smooth approximants depending on the choice of scaling parameters [9] (ii) FIF can have local or global dependence on data points based on the choice of scaling factors (iii) FIF retain the self-referentiality (iv) the interpolant or a certain derivative of it may have a non-integer box-counting dimension depending on the magnitude of scaling factors.

Here we address the problem with the help of a classical interpolant (as explained in Section 2). If an Iterated Function System is chosen suitably in terms of a given continuous function f, then a family of fractal functions $\{f^\alpha\}$ can be generated by taking interpolation data from f on a compact interval [10]. It is possible to preserve smoothness and fundamental shape properties of the original function f when we impose appropriate restrictions on the parameter α. The notion of fractal function provides new exciting fields of research and applications.

This paper is devoted to study the interpolation of periodic functions using a procedure that generalize the ordinary trigonometric interpolation [5,11]. We also define non-smooth fractal versions of the standard case. The uniform error between the original function in $\mathcal{C}[-\pi,\pi]$ and its fractal interpolants are bounded for a wide range of an exponential parameter. The limits proposed prove the convergence of interpolants with very weak conditions as the sampling frequency is indefinitely increased. Unlike the classical trigonometric case, the only hypothesis of continuity is sufficient to provide the uniform convergence when the number of terms tends to infinity.

The rest of the paper is organized as follows. Section 2 describes some preliminary results needed for the subsequent parts. Section 3 is devoted to the construction of a new type of trigonometric interpolants and then its fractal extension. Section 4 deals with error bounds of the formulas of the fractal trigonometric interpolation. Convergence and stability results of interpolatory process are studied in Section 5. In Section 6 we have given application of our method to temperature recordings of India.

2. Preliminaries

In this section, we review the construction of FIF and α-fractal function that are needed in the sequel. For more details, the reader can refer to [7,10,12].

2.1. Fractal interpolation functions

Let Δ be a partition of a real compact interval $I=[a,b]$, i.e, $\Delta=\{x_0,x_1,\ldots,x_N\}$ satisfying $a=x_0<x_1<\cdots<x_N=b$. Let a set of data points $\{(x_i,y_i), i\in\mathbb{N}_N^0\}$ be given, where $\mathbb{N}_k^0=\{0,1,\ldots,k\}$, and $I_i=[x_{i-1},x_i]$. Let $L_i: I\to I_i, i\in\mathbb{N}_N$ be contractive homeomorphisms such that

$$L_i(x_0)=x_{i-1},\ \ L_i(x_N)=x_i, \tag{2}$$

where $\mathbb{N}_k$ is the set of the first k natural numbers. Let $K=I\times\mathbb{R}$ and N continuous mappings, $F_i: K\to\mathbb{R}$ be satisfying

$$F_i(x_0,y_0)=y_{i-1}, F_i(x_N,y_N)=y_i,\ \ |F_i(x,y)-F_i(x,y')|\leq|\alpha_i||y-y'|, \tag{3}$$

where $(x,y),(x,y')\in K,\quad \alpha_i\in(-1,1),\quad i\in\mathbb{N}_N$. Now define functions $w_i: K\to K$ as $w_i(x,y)=(L_i(x),F_i(x,y))\ \forall\ i\in\mathbb{N}_N$. The following is a fundamental theorem in the subject of fractal functions.

Theorem 3 (*Barnsley [7]*). *Let $\mathcal{C}(I)$, the space of all real-valued continuous functions on a compact interval I, be endowed with the Chebyshev or supremum norm $\|g\|_\infty:=\max\{|g(x)|: x\in I\}$ and consider the closed metric subspace*

$$\mathcal{C}_{y_0,y_N}(I):=\{g\in\mathcal{C}(I):\ \ g(x_0)=y_0,\ \ g(x_N)=y_N\}.$$

The following hold.

1. *The Iterated Function System* $\{K; w_i, i = 1, 2, \ldots, N\}$ *has a unique attractor* G *which is the graph of a continuous function* $f^* : I \to \mathbb{R}$ *satisfying* $f^*(x_i) = y_i$ *for* $i = 0, 1, \ldots, N$.
2. *The function* f^* *is the fixed point of the Read–Bajraktarević (RB) operator* $T : \mathcal{C}_{y_0,y_N}(I) \to \mathcal{C}_{y_0,y_N}(I)$ *defined via*

$$(Tg)(x) = F_i\big(L_i^{-1}(x), g \circ L_i^{-1}(x)\big), \ x \in I_i, \ i \in \mathbb{N}_N.$$

The function f^* appearing in the foregoing theorem is called fractal interpolation function (FIF) corresponding to the IFS $\{w_i(x, y)\}_{i=1}^N$ and it is unique satisfying the functional equation

$$f^*(x) = F_i(L_i^{-1}(x), f^* \circ L_i^{-1}(x)) \ \ \forall x \in [x_{i-1}, x_i], \quad i \in \mathbb{N}_{N-1}. \tag{4}$$

The most used fractal interpolation functions so far are defined by the IFS

$$L_i(x) = a_i x + b_i, \ \ F_i(x, y) = \alpha_i y + q_i(x), \tag{5}$$

where a_i and b_i are determined by Eqs. (2), and admit the following expressions:

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0},$$
$$b_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0};$$

$\alpha_i \in (-1, 1)$ is called vertical scaling factor of the transformation w_i and $q_i : I \to \mathbb{R}$ are continuous functions satisfying

$$q_i(x_0) = y_{i-1} - \alpha_i y_0, \quad q_i(x_N) = y_i - \alpha_i y_N$$

due to the condition (3). The parameter $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1, 1)^N$ is called the scale vector of the FIF f^*.

2.2. α-fractal functions

The next definition was given in Ref. [10]. Let $f \in \mathcal{C}(I)$ be a continuous function. Choose a partition $\{a = x_0, x_1, \ldots, x_N = b\}$ of I, and consider the case $q_i(x) = f \circ L_i(x) - \alpha_i b(x)$, $i \in \mathbb{N}_N$, where b is continuous and such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$.

Definition 1. Let f^α be the continuous function defined by the IFS (4)–(5). f^α is the α-fractal function associated with f with respect to b, the partition Δ and the scale vector α.

According to (4) and (5), f^α satisfies the fixed point equation

$$f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b) \circ L_i^{-1}(x), \ x \in I_i, \ i \in \mathbb{N}_N. \tag{6}$$

The uniform distance between f^α and f is bounded as (see for instance [10])

$$\|f^\alpha - f\|_\infty \le \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty, \tag{7}$$

where $|\alpha|_\infty = \max\{|\alpha_i|; i \in \mathbb{N}_N\}$.

Remark 1. According to inequality (7), if $\alpha = 0$ or $f = b$, then $f^\alpha = f$.

Remark 2. The functions f^α and f agree at the nodes of the partition:

$$f^\alpha(x_i) = f(x_i),$$

for $i = 0, 1, \ldots, N$.

3. New type of trigonometric interpolant and its fractal

In this section, we consider a generalization of the trigonometric interpolation formula [5,13]. For abscissae x_i such that $x_{i+1} - x_i = \frac{2\pi}{2n+1}$ and $i = 1, 2, \ldots, 2n$, we define the new kernels of nodal interpolation with a positive exponent β as

$$Q_{n,i,\beta}(x) = \left| \frac{\sin\left((n + \frac{1}{2})(x_i - x)\right)}{\sin\left(\frac{1}{2}(x_i - x)\right)} \right|^\beta \quad \text{for } x \ne x_i,$$

and

$$Q_{n,i,\beta}(x_i) = (2n + 1)^\beta.$$

Table 1
Approximation errors of different function evaluations for the ordinary method and the procedure proposed, for several choices of n and β.

	Function values	Ordinary	$\beta = 2.5$	$\beta = 3$	$\beta = 3.5$
$n = 6$	$\sqrt{\lvert \sin(\frac{\pi}{2})\rvert}$	0.0335504	0.0142490	0.0209276	0.0054145
$n = 6$	$\sqrt{\lvert \sin(1)\rvert}$	0.0365604	0.0174729	0.0065050	0.0033678
$n = 8$	$\sqrt{\lvert \sin(\frac{\pi}{3})\rvert} + \sqrt{\lvert \cos(\frac{\pi}{3})\rvert}$	0.0462932	0.0015128	0.0095281	0.0166085
$n = 10$	$\frac{\pi}{5}\sin(\frac{7\pi}{5})$	0.2528410	0.1919620	0.1362740	0.0970759
$n = 10$	$\min(\sin(\frac{\pi}{10}), \cos(\frac{\pi}{10}))$	0.2840350	0.0712920	0.2407090	0.0693152

If f is continuous and periodic with period 2π, we propose the generalized trigonometric formula:

$$I_{n,\beta}(f)(x) = K_{n,\beta}(x) \sum_{i=1}^{2n+1} f(x_i) Q_{n,i,\beta}(x), \tag{8}$$

where $\beta > 0$, and

$$K_{n,\beta}^{-1}(x) = \sum_{i=1}^{2n+1} Q_{n,i,\beta}(x). \tag{9}$$

The choice of general exponent in the formula (8) provides a wider range of functions for approximation of periodic data sets.

Proposition 1. *The formula proposed in the expression (8) is interpolatory.*

Proof. Let us prove that

$$Q_{n,i,\beta}(x_j) = (2n+1)^{\beta} \delta_{ij},$$

where δ_{ij} is the Kronecker delta. For $j = i$, it is clear from the definition of $Q_{n,i,\beta}$. If $j \neq i$, let $r \in \mathbb{N}$ be such that $j = i + r$, then

$$x_j - x_i = \frac{2\pi r}{2n+1},$$

and thus

$$\sin\left(\left(n + \frac{1}{2}\right)(x_i - x_j)\right) = \sin(\pi r) = 0,$$

and hence $Q_{n,i,\beta}(x_j) = 0$.

As a consequence

$$K_{n,\beta}(x_j)^{-1} = Q_{n,j,\beta}(x_j) = (2n+1)^{\beta},$$

and, for all $j = 1, 2, \ldots, 2n+1$,

$$I_{n,\beta}(f)(x_j) = K_{n,\beta}(x_j) \sum_{i=1}^{2n+1} f(x_i) Q_{n,i,\beta}(x_j) = (2n+1)^{-\beta} f(x_j)(2n+1)^{\beta} = f(x_j). \quad \square$$

Fig. 1 shows the ordinary (left) and generalized interpolation (right) of the set of data

$$\{\left(\frac{-9\pi}{11}, 1\right), \left(\frac{-7\pi}{11}, 5\right), \left(\frac{-5\pi}{11}, 8\right), \left(\frac{-3\pi}{11}, 0\right), \left(\frac{-\pi}{11}, -2\right)$$
$$\left(\frac{\pi}{11}, 2\right), \left(\frac{3\pi}{11}, 5\right), \left(\frac{5\pi}{11}, 3\right), \left(\frac{7\pi}{11}, 1\right), \left(\frac{9\pi}{11}, 8\right), (\pi, 4)\},$$

for $n = 5$ and $\beta = 2$ in the second case.

Table 1 depicts the errors in the evaluation of interpolated functions at several points of the interval $[-\pi, \pi]$ for ordinary and generalized method. We have computed the difference between the exact and the approximate values of the functions: $f_1(x) = \sqrt{|\sin(x)|}$ at the points $x = \pi/2, 1$ (first and second rows), $f_2(x) = \sqrt{|\sin(x)|} + \sqrt{|\cos(x)|}$ at the point $x = \pi/3$ (third row), $f_3(x) = x\sin(7x)$ at the point $x = \pi/5$ (fourth row) and $f_4(x) = \min(\sin(x), \cos(x))$ at the point $x = \pi/10$ (fifth row), for different choices of n and exponent β.

We consider now a fractal extension of the function $I_{n,\beta}(f)$. Let us perturb the basis function $Q_{n,i,\beta}(x)$ with proper base function $b_{n,i,\beta}(x)$ and partition of the interval described before and define the generalized fractal trigonometric interpolation

as

$$I^{\alpha}_{n,\beta}(f)(x) = K_{n,\beta}(x) \sum_{i=1}^{2n+1} f(x_i) Q^{\alpha}_{n,i,\beta}(x), \tag{10}$$

where $Q^{\alpha}_{n,i,\beta}$ is the α-fractal function associated to $Q_{n,i,\beta}$ (Definition 1 of Section 2).

The formula proposed in the expression (10) is interpolatory as well since $Q^{\alpha}_{n,i,\beta}(x_j) = Q_{n,i,\beta}(x_j) = (2n+1)^{\beta}\delta_{ij}$, for all $j = 1, 2, \ldots, 2n+1$ (see Remark 2 of Section 2).

Lemma 1. *If $x_{i+1} - x_i = \frac{2\pi}{2n+1}$ for $i = 1, 2, \ldots, 2n$, $m \in \mathbb{N}$ and m is not divisible by $2n+1$, then*

$$\sum_{i=1}^{2n+1} \cos(mx_i) = \sum_{i=1}^{2n+1} \sin(mx_i) = 0.$$

Proof. This is due to the identity:

$$\sum_{i=1}^{2n+1} e^{jmx_i} = 0,$$

being $j^2 = -1$, with the hypotheses prescribed. □

Proposition 2. *If $x_{i+1} - x_i = \frac{2\pi}{2n+1}$ for $i = 1, 2, \ldots, 2n$, then*

$$\sum_{i=1}^{2n+1} \frac{\sin(n+\frac{1}{2})(x_i - x)}{\sin\frac{1}{2}(x_i - x)} = 2n+1.$$

Proof. The Dirichlet kernel defined as

$$\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}, \quad n \in \mathbb{N}, \tag{11}$$

admits the expression [14, p. 295]:

$$\frac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u} = \frac{1}{2} + \cos u + \cos 2u + \cdots + \cos nu. \tag{12}$$

Using (12),

$$\sum_{i=1}^{2n+1} \frac{\sin(n+\frac{1}{2})(x_i - x)}{\sin\frac{1}{2}(x_i - x)} = 2\left(\sum_{i=1}^{2n+1}\left(\frac{1}{2} + \cos(x_i - x) + \cos 2(x_i - x) + \cdots + \cos n(x_i - x)\right)\right).$$

The part of the above summand can be written as

$$\sum_{i=1}^{2n+1} \cos\big(m(x_i - x)\big) = \sum_{i=1}^{2n+1} \cos(mx_i)\cos(mx) + \sin(mx_i)\sin(mx) = 0,$$

for $m = 1, 2, \ldots, n$ by applying Lemma 1. Consequently,

$$\sum_{i=1}^{2n+1} \frac{\sin(n+\frac{1}{2})(x_i - x)}{\sin\frac{1}{2}(x_i - x)} = 2n+1.$$

Another way of arguing is thinking that the expression

$$\frac{1}{2n+1}\sum_{i=1}^{2n+1} \frac{\sin(n+\frac{1}{2})(x_i - x)}{\sin\frac{1}{2}(x_i - x)}$$

is the interpolating trigonometric polynomial of the function $f(x) = 1$ for all $x \in [-\pi, \pi]$. Since both are polynomials of order lower or equal than n, they must agree. □

Remark 3. If $\beta \in \mathbb{N}$, the absolute value can be removed from the kernels $Q_{n,i,\beta}$. The formula proposed in (8) is a generalization of the classical trigonometric interpolation as consequence of Proposition 2.

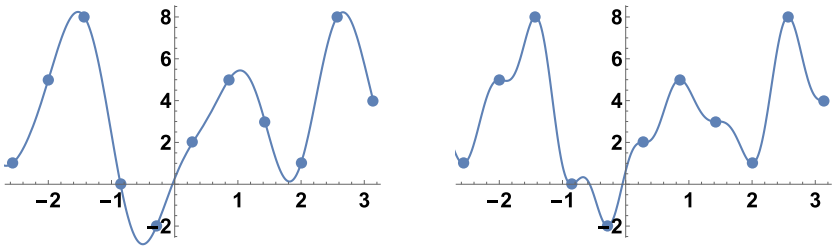

Fig. 1. Ordinary (left) and generalized interpolation function (right) for a set of data, $n = 5$ and $\beta = 2$.

4. Error bounds of generalized fractal trigonometric interpolants

In this section, we face the problem of finding bounds for the error of the interpolations (8) and (10). The following result can be read in Ref. [15].

Lemma 2. *For all $m = 1, 2, \ldots; \beta > 0$, and $v \in \mathbb{R}$:*

$$\left|\frac{\sin(mv)}{m\sin(v)}\right|^{\beta} \leq 1. \tag{13}$$

Lemma 3. *For all n, i, β, x,*

$$0 \leq Q_{n,i,\beta}(x) \leq (2n+1)^{\beta}. \tag{14}$$

Proof. Taking $x - x_i = 2v_i$ in the definition of $Q_{n,i,\beta}$, one has

$$Q_{n,i,\beta}(x) = \left|\frac{\sin((2n+1)v_i)}{\sin(v_i)}\right|^{\beta} \leq (2n+1)^{\beta},$$

according to Lemma 2. □

Lemma 4. *For $v \in [0, \pi/2]$,*

$$\sin(v) \geq \frac{2v}{\pi}. \tag{15}$$

Proof. The function $\sin(v)$ is concave in the interval $[0, \pi/2]$ and thus

$$\sin(v) \geq r(v),$$

where $r(v)$ is the line joining $(0, 0)$ and $(\pi/2, 1)$. But $r(v) = 2v/\pi$, obtaining the result. □

Theorem 4. *Let $f \in \mathcal{C}[-\pi, \pi]$ be Hölder continuous such that for $x, x' \in [-\pi, \pi]$,*

$$|f(x) - f(x')| \leq K|x - x'|^{q}, \quad 0 < q \leq 1.$$

Then for $\beta > q + 1$,

$$\|I^{\alpha}_{n,\beta}(f) - f\|_{\infty} \leq K\left(\frac{\pi}{2n+1}\right)^{q}\left(\frac{\pi}{2}\right)^{\beta}\left(1 + 2^{q} + \frac{1}{\beta-(q+1)} + \frac{1}{\beta-1}\right)$$
$$+ (2n+1)\left(\frac{\pi}{2}\right)^{\beta}\|f\|_{\infty}\frac{|\alpha|_{\infty}}{1-|\alpha|_{\infty}},$$

where α is a suitable scaling vector used to construct the fractal perturbation of $Q_{n,i,\beta}$ over $[-\pi, \pi]$.

Proof. The following inequality is used in our procedure:

$$\|I^{\alpha}_{n,\beta}(f) - f\|_{\infty} \leq \|I^{\alpha}_{n,\beta}(f) - I_{n,\beta}(f)\|_{\infty} + \|I_{n,\beta}(f) - f\|_{\infty}. \tag{16}$$

Let us define the point-wise error for the generalized Jackson interpolant:

$$E_{n,\beta}(f)(x) := I_{n,\beta}(f)(x) - f(x) = K_{n,\beta}(x)\sum_{i=1}^{2n+1}\big(f(x_i) - f(x)\big)Q_{n,i,\beta}(x).$$

With the change of variable $x_i = x + 2u_i$,

$$|E_{n,\beta}(f)(x)| \le K_{n,\beta}(x) \sum_{i=1}^{2n+1} |f(x+2u_i) - f(x)| \left| \frac{\sin((2n+1)u_i)}{\sin(u_i)} \right|^{\beta}, \tag{17}$$

where we assume that $u_i \in [-\pi/2, \pi/2]$ [5]. Let us multiply numerator and denominator by $(2n+1)^{\beta}$.

Let v_0 be the smallest of the numbers $|u_i|$, v_1 the second, and so on. In this way [5, p. 454] using the Hölder condition of f, (17) becomes

$$|E_{n,\beta}(f)(x)| \le K_{n,\beta}(x)(2n+1)^{\beta} \sum_{i=0}^{2n} K 2^q v_i^q \left| \frac{\sin((2n+1)u_i)}{(2n+1)\sin(u_i)} \right|^{\beta}, \tag{18}$$

where

$$\frac{\pi i}{2(2n+1)} \le v_i \le \frac{\pi(i+1)}{2(2n+1)} \le \frac{\pi}{2}, \tag{19}$$

for $i = 0, 1, \ldots, 2n$. Applying Lemma 2 for $i = 0, 1$,

$$\left| \frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)} \right|^{\beta} \le 1. \tag{20}$$

For the rest of the values ($i \ge 2$), we use the left part of the expression (19) to have

$$(2n+1)\sin(v_i) \ge (2n+1)\frac{2v_i}{\pi} \ge i,$$

(the first inequality is due to Lemma 4). As a consequence, for $i \ge 2$,

$$\left| \frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)} \right|^{\beta} \le \left(\frac{1}{(2n+1)\sin(v_i)} \right)^{\beta} \le \left(\frac{1}{i} \right)^{\beta}. \tag{21}$$

Using the estimation of (20) for $i = 0,1$ and (21) for $i \ge 2$, we obtain

$$\begin{aligned} |E_{n,\beta}(f)(x)| &\le K 2^q (2n+1)^{\beta} K_{n,\beta}(x) \left(v_0^q + v_1^q + \sum_{i=2}^{2n} v_i^q (\frac{1}{i})^{\beta} \right) \\ &\le K 2^q (2n+1)^{\beta} K_{n,\beta}(x) \left(\frac{\pi^q}{2^q(2n+1)^q} + \frac{(2\pi)^q}{2^q(2n+1)^q} + \frac{(i+1)^q}{2^q(2n+1)^q i^{\beta}} \right) \\ &\le K(2n+1)^{\beta} K_{n,\beta}(x) \frac{\pi^q}{(2n+1)^q} \left(1 + 2^q + \sum_{i=2}^{2n} \frac{i^q+1}{i^{\beta}} \right), \end{aligned}$$

where we have used (19) in the second step and the inequality $(i+1)^q \le (i^q+1)$ for $0 \le q \le 1$ in the last step. For the estimation of the last two summands in the above expression, we use the lower Riemann sums of the functions $\frac{1}{x^{\beta-q}}$ and $\frac{1}{x^{\beta}}$, in the interval $[1, +\infty)$ with unit step, and thus:

$$\sum_{i=2}^{2n} \frac{1}{i^{\beta-q}} \le \int_1^{\infty} \frac{dx}{x^{\beta-q}} = \frac{1}{\beta-(q+1)},$$

and

$$\sum_{i=2}^{2n} \frac{1}{i^{\beta}} \le \int_1^{\infty} \frac{dx}{x^{\beta}} = \frac{1}{\beta-1}.$$

Using these bounds, we have

$$|E_{n,\beta}(f)(x)| \le K(2n+1)^{\beta} K_{n,\beta}(x) \left(\frac{\pi}{2n+1} \right)^q \left(1 + 2^q + \frac{1}{\beta-(q+1)} + \frac{1}{\beta-1} \right), \tag{22}$$

if $\beta > q+1$. In order to get an upper bound for $(2n+1)^{\beta} K_{n,\beta}(x)$, consider

$$\frac{K_{n,\beta}(x)^{-1}}{(2n+1)^{\beta}} = \sum_{i=0}^{2n} \left| \frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)} \right|^{\beta} > \left| \frac{\sin((2n+1)v_0)}{(2n+1)\sin(v_0)} \right|^{\beta},$$

and

$$\sin(2n+1)v_0 \ge \frac{2(2n+1)v_0}{\pi}.$$

Therefore

$$\frac{K_{n,\beta}(x)^{-1}}{(2n+1)^{\beta}} > \left|\frac{2(2n+1)v_0}{\pi(2n+1)v_0}\right|^{\beta} = \left(\frac{2}{\pi}\right)^{\beta}. \tag{23}$$

Using (23) in (22), we obtain a uniform bound for the difference between f and its generalized approximant as

$$\|f - I_{n,\beta}(f)\|_\infty \le K\left(\frac{\pi}{2n+1}\right)^{q}\left(\frac{\pi}{2}\right)^{\beta}\left(1 + 2^q + \frac{1}{\beta-(q+1)} + \frac{1}{\beta-1}\right). \tag{24}$$

From (23), we have $K_{n,\beta}(x) < \frac{(\frac{\pi}{2})^{\beta}}{(2n+1)^{\beta}}$, and this bound is used for error in the second term of (16) as

$$\begin{aligned}
|I^{\alpha}_{n,\beta}(f)(x) - I_{n,\beta}(f)(x)| &= \left|K_{n,\beta}(x)\sum_{i=1}^{2n+1} f(x_i)\left(Q^{\alpha}_{n,i,\beta}(x) - Q_{n,i,\beta}(x)\right)\right| \\
&\le K_{n,\beta}(x)\sum_{i=1}^{2n+1}|f(x_i)|\left|Q^{\alpha}_{n,i,\beta}(x) - Q_{n,i,\beta}(x)\right| \\
&\le K_{n,\beta}(x)\|f\|_\infty\sum_{i=1}^{2n+1}\left|Q^{\alpha}_{n,i,\beta}(x) - Q_{n,i,\beta}(x)\right| \\
&\le \left(\frac{\pi}{2}\right)^{\beta}\frac{(2n+1)}{(2n+1)^{\beta}}\|f\|_\infty \max_{1\le i\le 2n+1}\|Q^{\alpha}_{n,i,\beta} - Q_{n,i,\beta}\|_\infty \\
&\le \left(\frac{\pi}{2}\right)^{\beta}\frac{1}{(2n+1)^{\beta-1}}\|f\|_\infty\frac{|\alpha|_\infty}{1-|\alpha|_\infty}\max_{1\le i\le 2n+1}\|Q_{n,i,\beta} - b_{n,i,\beta}\|_\infty,
\end{aligned} \tag{25}$$

where the last step follows from (7). The maps $b_{n,i,\beta}$ are the functions used to define the fractal functions $Q^{\alpha}_{n,i,\beta}$, and they can be chosen such that

$$\|Q_{n,i,\beta} - b_{n,i,\beta}\|_\infty \le \|Q_{n,i,\beta}\|_\infty,$$

taking, for instance, the lines joining the extremes of the graph of $Q_{n,i,\beta}$. Thus

$$\max_{1\le i\le 2n+1}\|Q_{n,i,\beta} - b_{n,i,\beta}\|_\infty \le \max_{1\le i\le 2n+1}\|Q_{n,i,\beta}\|_\infty = (2n+1)^{\beta}.$$

Finally using the above bound, (24) and (25) in (16), we get the proposed error estimation. □

Theorem 5. *If $f \in \mathcal{C}[-\pi,\pi]$ and $\beta > 2$,*

$$\begin{aligned}
\|I^{\alpha}_{n,\beta}(f) - f\|_\infty \le{}& \omega\left(\frac{\pi}{2n+1}\right)\left(\frac{\pi}{2}\right)^{\beta}\left(3 + \frac{1}{\beta-2} + \frac{1}{\beta-1}\right) \\
&+ (2n+1)\left(\frac{\pi}{2}\right)^{\beta}\|f\|_\infty\frac{|\alpha|_\infty}{1-|\alpha|_\infty},
\end{aligned}$$

where $\omega(\delta)$ is the modulus of continuity of f and α, $b_{n,i,\beta}$ are suitable scaling vector and functions used to construct the fractal perturbation of $Q_{n,i,\beta}$.

Proof. Following the previous proof until (17), and the same choice of v_i,

$$|E_{n,\beta}(f)(x)| \le K_{n,\beta}(x)\sum_{i=0}^{2n}\omega(2v_i)\left|\frac{\sin((2n+1)v_i)}{\sin(v_i)}\right|^{\beta}.$$

Using the inequality (19) and the properties of ω, we have

$$\omega(2v_i) \le (i+1)\omega\left(\frac{\pi}{2n+1}\right).$$

According to Lemma 2, for $i = 0,1$,

$$\left|\frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)}\right|^{\beta} \le 1,$$

and for the rest of the values ($i \ge 2$), due to (21),

$$\left|\frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)}\right|^{\beta} \le \frac{1}{i^{\beta}}.$$

Thus,

$$|E_{n,\beta}(f)(x)| \le K_{n,\beta}(x)(2n+1)^{\beta} \sum_{i=0}^{2n} \omega(2v_i) \left| \frac{\sin((2n+1)v_i)}{(2n+1)\sin(v_i)} \right|^{\beta}$$

$$\le K_{n,\beta}(x)(2n+1)^{\beta} \omega\left(\frac{\pi}{2n+1}\right) \left(3 + \sum_{i=2}^{+\infty} \left(\frac{1}{i^{\beta-1}} + \frac{1}{i^{\beta}}\right)\right).$$

The rest of the proof follows similar lines of the previous theorem. □

5. Convergence and stability of the interpolatory processes

In this section we face the study of the convergence and stability of the interpolations defined in Section 3. For a scale vector $\alpha = 0$ the operator $I^0_{n,\beta}$ agrees with $I_{n,\beta}$ in (8), since $g^0 = g$ for any function g (Remark 1 of Section 2).

Theorems 4 and 5 provide the main results of convergence.

Theorem 6. *If f is Hölder continuous on the circle with exponent q such that $0 < q \le 1$, then for any $\beta > (q+1)$, the interpolating function $I^0_{n,\beta}(f)$ converges uniformly to f when n tends to infinity. The rate of convergence is $\mathcal{O}(n^{-q})$ and it does not depend on β.*

Theorem 7. *If f is continuous on the circle then for any $\beta > 2$, the interpolating function $I^0_{n,\beta}(f)$ converges uniformly to f when n tends to infinity. The rate of convergence is that of the modulus of continuity of f as $(\omega(n^{-1}))$ and it does not depend on β.*

For the fractal interpolants we have the following results.

Theorem 8. *If f is Hölder continuous on the circle with exponent q such that $0 < q \le 1$, for any $\beta > (q+1)$, and choosing a sequence of scale vectors α^n such that $\alpha^n = \mathcal{O}(n^{-(q+1)})$, the interpolating function $I^{\alpha^n}_{n,\beta}(f)$ converges uniformly to f when n tends to infinity. The rate of convergence is $\mathcal{O}(n^{-q})$ and it does not depend on β.*

Theorem 9. *If f is Hölder continuous on the circle, for any $\beta > 2$, and choosing a sequence of scale vectors α^n such that $\alpha^n = \mathcal{O}(n^{-1}\omega(n^{-1}))$, the interpolating function $I^{\alpha^n}_{n,\beta}(f)$ converges uniformly to f when n tends to infinity. The rate of convergence is that of $\omega(n^{-1})$ and it does not depend on β.*

Regarding the stability, let us remind this concept for an arbitrary continuous interpolation $I_m(f)$ on a partition $\{x_i^m\}_{i=1}^m$ [16].

Definition 2. The interpolation $I_m(f)$ is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\max_{1\le i\le m}|f(x_i^m)| \le \delta$ implies that $\|I_m(f)\|_\infty \le \varepsilon$.

The Lebesgue constant of an interpolation was mentioned in the Introduction (1). The following result can be read in the Ref. [16].

Theorem 10. *A necessary and sufficient condition for the stability of the interpolation is that for the Lebesgue constant sequence $\{\Lambda_m\}$, there exists a positive real number K such that $\Lambda_m \le K$ for any $m \in \mathbb{N}$.*

Applying this result we can conclude that $I^0_{n,\beta}$ is stable since, by the definition of the nodal functions:

$$\Lambda^0_{n,\beta} = \sup_{x\in[-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{2n+1} Q^0_{n,i,\beta}(x) = \sup_{x\in[-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{2n+1} Q_{n,i,\beta}(x) = 1.$$

This fact implies that the interpolation does not increase errors and is stable.

For the fractal case, using arguments similar to those of the last part of Theorem 4 (for $f = 1$):

$$\Lambda^{\alpha}_{n,\beta} = \sup_{x\in[-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{2n+1} |Q^{\alpha}_{n,i,\beta}(x)|$$

$$\le \sup_{x\in[-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{2n+1} |Q^{\alpha}_{n,i,\beta}(x) - Q_{n,i,\beta}(x)| + \Lambda^0_{n,\beta}$$

$$\le 1 + \sup_{x\in[-\pi,\pi]} K_{n,\beta}(x)(2n+1)^{\beta+1} \frac{|\alpha|_\infty}{1-|\alpha|_\infty}$$

$$\le 1 + \left(\frac{\pi}{2}\right)^{\beta} \frac{|\alpha|_\infty (2n+1)}{1-|\alpha|_\infty}.$$

Table 2
Optimal values of β and relative errors corresponding the each exponent for the weekly average temperature for the years 2010 to 2016.

Years	Exponent β	Relative error
2010	1.46393	0.0513211
2011	1.50422	0.0574602
2012	1.12111	0.0379251
2013	1.48593	0.0450024
2014	1.24308	0.0429304
2015	1.30641	0.0417724
2016	1.24439	0.0486658

In conclusion, if we choose a sequence of scale vectors α^n such that $\alpha^n = \mathcal{O}(n^{-1})$, the interpolation $I^{\alpha^n}_{n,\beta}(f)$ is also stable for any $\beta > 2$.

6. Application

We have used the generalized interpolation formula (8) to study the temperature of Chennai in the time period between 2010 to 2016. Our objective was to find the optimal β for which we can fit the data properly. First of all, we have collected the weekly average temperature from [17] for every year from 2010 to 2016. From these data we have taken 27 samples as interpolation nodes, and the rest of the points were considered as targets to find the optimal value of the exponent β in a least square process. We have performed the computations for different values of β and selected the exponent for which the relative error of fitting original data was a minimum. Table 2 depicts the optimal β and the relative error for every year from 2010 to 2016. It can be noted that, in general, the maximum relative error for this process is near 6% and the exponent β always lies between 1 and 2 in all the years studied.

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References

[1] A. Zygmund, Trigonometric Series, second ed., Cambridge University Press, London, New York, 1959.
[2] J. Prestin, Y. Xu, Convergence rate for trigonometric interpolation of non-smooth functions, J. Approx. Theory 77 (1994) 113–122.
[3] J. Marcinkiewicz, Sur la divergence des polynomes d'interpolation, Acta Univ. Szeged 8 (1936-37) 131–135.
[4] R.P. Gosselin, On the convergence behavior of trigonometric polynomials, Pac. J. Math. 5 (6) (1955) 915–922.
[5] D. Jackson, On the accuracy of trigonometric interpolation, Trans. Amer. Math. Soc. 14 (1913) 453–461.
[6] Victor S. Ryaben' kii, Semyon V. Tsynkov, A Theoretical Introduction to Numerical Analysis, Chapman & Hall/CRC, Boca Raton, FL, 2007, p. xiv+537.
[7] M.F. Barnsley, Fractal functions and interpolation, Constr. Approx. 2 (4) (1986) 303–329.
[8] M.F. Barnsley, Fractals Everywhere, Academic Press, Inc., Boston, MA, 1988, p. xii+396.
[9] M.A. Navascués, M., V. Sebastián, Smooth fractal interpolation, J. Inequal. Appl. (2006) 20, Art. ID 78734.
[10] M.A. Navascués, Fractal polynomial interpolation, Z. Anal. Anwend. 24 (2) (2005) 401–418.
[11] D. Jackson, On approximation by trigonometric sums and polynomials, Trans. Amer. Math. Soc. 13 (4) (1912) 491–515.
[12] M.A. Navascués, A.K.B. Chand, Fundamental sets of fractal functions, Acta Appl. Math. 100 (3) (2008) 247–261.
[13] J. Szabados, P. Vértesi, Interpolation of Functions, World Sci. Publ., Singapore, 1990.
[14] P.J. Davis, Interpolation and Approximation, Dover Publications, Inc., New York, 1975, p. xv+393.
[15] M.A. Navascués, S. Jha, A.K.B. Chand, M.V. Sebastián, Fractal approximants on the circle, Chaotic Model. Simul. 3 (2018) 343–353.
[16] H. Hong-ci, On the stability of interpolation, J. Comput. Math. 1 (1) (1983) 34–44.
[17] https://www.timeanddate.com/weather/india/chennai/historic.