

COMPLEX SYMMETRIC WEIGHTED COMPOSITION DIFFERENTIATION OPERATORS

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ABSTRACT. In this note, we completely characterize complex symmetric weighted composition differentiation operator on the Hardy space H^2 with respect to the conjugation operator $C_{\lambda, \alpha}$. Meanwhile, the normal and self-adjoint of the weighted composition differentiation operators on the Hardy space H^2 are also studied. This note could be considered as a continuation of the work initiated by Fatehi and Hammond.

1. INTRODUCTION AND PREPARATION

In this paper, \mathbb{D} denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and \mathbb{T} the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let $H(\mathbb{D})$ be the Hilbert space of all analytic function on \mathbb{D} . The space H^2 , the Hardy space, is the set of functions from $H(\mathbb{D})$ with square summable power series coefficients; that is, $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} < \infty,$$

where $\{a_n\}$ is the sequence of Maclaurin coefficients for f .

Given formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the inner product on H^2 is defined by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n},$$

Let H^∞ denote the space of bounded analytic functions on \mathbb{D} . The norm $\|\cdot\|_\infty$ of $f \in H^\infty$ is defined by

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

For an analytic self-map φ of \mathbb{D} , we define the composition operator C_φ on $H(\mathbb{D})$ by

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z)), \quad z \in \mathbb{D}.$$

This is the first setting in which composition operators were studied. By Littlewood's subordination principle, every composition operator takes H^2 into itself. The differentiation operator is defined by $Df = f'$ for each $f \in H(\mathbb{D})$. For $u \in H(\mathbb{D})$, the weighted composition operator uC_φ is given by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

For $m \in \mathbb{N}$, the weighted differentiation composition operator is denoted by

$$(D_{u, \varphi}^m f)(z) = u(z)f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

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When $m = 0$, the operator $D_{u,\varphi}^m$ becomes the weighted composition operator uC_φ . If $m = 0$ and $u(z) = 1$, we get $D_{u,\varphi}^m = C_\varphi$. If $m = 1$ and $u(z) = 1$, it turns out to be $D_{u,\varphi}^m = C_\varphi D$. If $m = 1$ and $u(z) = \varphi'(z)$, it yields that $D_{u,\varphi}^m = DC_\varphi$. In this article, we study the case $m = 1$. In what follows, we denote $D_{u,\varphi}^{(1)}$ by $D_{u,\varphi}$ for convenience.

Weighted composition operators have arisen in the study of isometries of Hardy spaces. Later, these have been studied by many mathematicians. Recently, many researchers have started investigating weighted composition differentiation operator on various function spaces. For example, Ohno [21] studied the boundedness and compactness of the products of composition and differentiation between Hardy spaces whereas Li and Stević [16,17] investigated the products of composition and differentiation operators between H^∞ and Bloch type spaces. Liang and Zhou studied them on logarithmic Bloch space in [18]. Fatehi and Hammond [2] investigated the adjoint, norm and spectrum of the composition differentiation operator D_φ on Hardy spaces. They only considered the case of $\|\varphi\|_\infty < 1$. In this case, it could guarantee that D_φ is bounded and compact. To ensure that $D_{u,\varphi}$ is bounded and compact on Hardy space, we study the operator $D_{u,\varphi}$ with $u \in H^2$ and $\|\varphi\|_\infty < 1$.

The reproducing kernel of H^2 is

$$K_w(z) = \frac{1}{1 - \bar{w}z}.$$

For $z \in \mathbb{D}$, the normalized reproducing kernels of H^2 are given by

$$k_w(z) = \frac{K_w(z)}{\|K_w(z)\|} = \frac{(1 - |w|^2)^{\frac{1}{2}}}{1 - \bar{w}z}, \quad z \in \mathbb{D}.$$

Then $\langle f(z), K_w(z) \rangle = f(w)$ for all $f \in H^2$ and $z \in \mathbb{D}$. Now, we introduce

$$K_w^{(1)}(z) = \frac{z}{(1 - \bar{w}z)^2}.$$

Then $K_w^{(1)}$ is the reproducing kernel for point-evaluation of the first derivative. In fact, for all $f \in H^2$, we have

$$\langle f(z), K_w^{(1)}(z) \rangle = f'(w)$$

and it is easy to see that

$$\langle f, D_{u,\varphi}^*(K_w) \rangle = \langle D_{u,\varphi}(f), K_w \rangle = \overline{u(w)} f'(\varphi(w)) = \langle f, \overline{u(w)} K_{\varphi(w)}^{(1)} \rangle.$$

Therefore, $D_{u,\varphi}^*(K_w) = \overline{u(w)} K_{\varphi(w)}^{(1)}$.

A linear operator $C : H^2 \rightarrow H^2$ is a *conjugation* if

$$\langle Cf, Cg \rangle = \langle g, f \rangle, \quad \text{for all } f, g \in H^2$$

and $C^2 = I$, where I is identity operator. A bounded operator T on the Hardy space H^2 is said to be *complex symmetric* if there is a conjugation C on H^2 such that $T = CT^*C$.

Complex symmetric operator can be regarded as a generalization of complex symmetric matrices. It is important to the development of operator theory. The study of complex symmetric operators was started by Garcia and Putinar [6, 7], Garcia and Poore [5], and Garcia and Wogen [8, 9]. Recently, the study of complex symmetric weighted composition operator on different spaces has attracted the interest of many researchers. In 2018, Lim and Khoi [19] studied the weighted composition operator on the Hilbert space $\mathcal{H}_\gamma(\mathbb{D})$ of holomorphic functions, which is complex symmetric operator with the conjugation of the form $\mathcal{A}_{u,v}f = u \cdot \overline{f \circ \bar{v}}$. And they also obtained the result about the conjugation $\mathcal{A}_{u,v}$. Hu et al. [13] investigated complex symmetric weighted composition operator on Dirichlet spaces and Hardy spaces. Wang and Yao [23], and Wang and Han [24] studied complex symmetry of weighted composition operators in several variables. Hai

and Khoi [10,11] characterized complex symmetry of weighted composition operators on the Fock space. For further details, we suggest the readers to refer the articles [3, 12, 19, 20, 22].

In this paper, we consider the problem of describing all complex symmetric weighted composition differentiation operator on the Hardy space H^2 with the conjugation $C_{\lambda,\alpha}$. The normal and self adjoint properties of weighted composition differentiation operator is also discussed.

2. COMPLEX SYMMETRIC OPERATOR

For $\lambda, \alpha \in \mathbb{T}$, we define the conjugate linear operator $C_{\lambda,\alpha}$ on the Hardy space H^2 by

$$C_{\lambda,\alpha}f(z) = \lambda \overline{f(\overline{\alpha z})},$$

where $f \in H^2$. It is easy to see that $C_{\lambda,\alpha}$ is a conjugation. For $\alpha = 1$, we denote C_λ by

$$C_\lambda f(z) = \lambda \overline{f(\overline{z})}.$$

In this section, we will characterize complex symmetric weighted composition differentiation operator on the Hardy space H^2 .

Theorem 1. *Suppose that $\varphi(z)$ is an analytic self-map on \mathbb{D} such that $\|\varphi\|_\infty < 1$, and $u \in H^2$ with $u \neq 0$. Then $D_{u,\varphi}$ is a complex symmetric operator on H^2 with the conjugation $C_{\lambda,\alpha}$ if and only if there are complex numbers a, b, c such that*

$$u(z) = \frac{az}{(1-\alpha bz)^2} \quad \text{and} \quad \varphi(z) = b + \frac{cz}{1-\alpha bz}, \quad \text{for all } z \in \mathbb{D}.$$

Proof. Suppose that $D_{u,\varphi}$ is $C_{\lambda,\alpha}$ -symmetric. Then

$$(1) \quad D_{u,\varphi} C_{\lambda,\alpha} K_w(z) = C_{\lambda,\alpha} D_{u,\varphi}^* K_w(z)$$

for all $w, z \in \mathbb{D}$. Thus,

$$D_{u,\varphi} C_{\lambda,\alpha} K_w(z) = D_{u,\varphi} C_{\lambda,\alpha} \left(\frac{1}{1-\overline{w}z} \right) = D_{u,\varphi} \left(\frac{\lambda}{1-\alpha w z} \right) = \frac{\lambda \alpha w u(z)}{(1-\alpha w \varphi(z))^2}$$

and

$$C_{\lambda,\alpha} D_{u,\varphi}^* K_w(z) = C_{\lambda,\alpha} D_{u,\varphi}^* \left(\frac{1}{1-\overline{w}z} \right) = C_{\lambda,\alpha} \left(\frac{\overline{z u(w)}}{(1-z \varphi(w))^2} \right) = \frac{\lambda \alpha z u(w)}{(1-\alpha z \varphi(w))^2}.$$

In view of (1), it follows that

$$(2) \quad \frac{\lambda \alpha w u(z)}{(1-\alpha w \varphi(z))^2} = \frac{\lambda \alpha z u(w)}{(1-\alpha z \varphi(w))^2}$$

for all $w, z \in \mathbb{D}$. Since $\lambda, \alpha \in \mathbb{T}$ and $u \neq 0$, we must have $u(0) = 0$.

Now, we set $u(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_n \in \mathbb{C}$. Substituting $u(z)$ back into the equation (2), we get

$$\left(\sum_{n=1}^{\infty} a_n w z^n \right) (1-\alpha z \varphi(w))^2 = \left(\sum_{n=1}^{\infty} a_n z w^n \right) (1-\alpha w \varphi(z))^2$$

for all $z, w \in \mathbb{D}$. Differentiating the above formula with respect to w , we obtain

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} a_n z^n \right) (1-\alpha z \varphi(w))^2 - 2\alpha z \varphi'(w) \left(\sum_{n=1}^{\infty} a_n w z^n \right) (1-\alpha w \varphi(w)) \\ &= \left(\sum_{n=1}^{\infty} n a_n z w^{n-1} \right) (1-\alpha w \varphi(z))^2 - 2\alpha \varphi'(z) \left(\sum_{n=1}^{\infty} a_n z w^n \right) (1-\alpha w \varphi(z)). \end{aligned}$$

Let $w = 0$ in the above equation. Then, we have

$$u(z) = \frac{az}{(1 - \alpha bz)^2}, \text{ for all } z \in \mathbb{D},$$

where $a = a_1$ and $b = \varphi(0)$. Substituting $u(z) = \frac{az}{(1 - \alpha bz)^2}$ back into the equation (2), we have

$$(1 - \alpha bz)^2(1 - \alpha w\varphi(z))^2 = (1 - \alpha bw)^2(1 - \alpha z\varphi(w))^2$$

for all $z, w \in \mathbb{D}$. Differentiate both sides of the above equation with respect to w , we see that

$$\begin{aligned} & (1 - \alpha bz)^2(1 - \alpha w\varphi(z))(-2\alpha\varphi(z)) \\ &= (1 - \alpha bw)(1 - \alpha z\varphi(w))^2(-2\alpha b) + (1 - \alpha z\varphi(w))(1 - \alpha bw)^2(-2\alpha z\varphi'(w)). \end{aligned}$$

Letting $w = 0$ in the above formula shows that

$$\varphi(z) = b + \frac{cz}{1 - \alpha bz}, \text{ for all } z \in \mathbb{D},$$

where $c = \varphi'(0)$.

Conversely, if $u(z) = \frac{az}{(1 - \alpha bz)^2}$ and $\varphi(z) = b + \frac{cz}{1 - \alpha bz}$, then

$$C_{\lambda, \alpha} K_w(z) = C_{\lambda, \alpha} \left(\frac{1}{1 - \bar{w}z} \right) = \frac{\lambda}{1 - \alpha wz}$$

and thus,

$$(3) \quad D_{u, \varphi} C_{\lambda, \alpha} K_w(z) = \frac{\lambda \alpha w u(z)}{(1 - \alpha w \varphi(z))^2} = \frac{\lambda \alpha a w z}{(1 - \alpha bz - \alpha bw + \alpha^2 b^2 z w - \alpha c z w)^2}.$$

Again, as

$$D_{u, \varphi}^* K_w(z) = D_{u, \varphi}^* \left(\frac{1}{1 - \bar{w}z} \right) = \frac{\overline{z u(w)}}{1 - \overline{\varphi(w)}z},$$

it follows that

$$(4) \quad C_{\lambda, \alpha} D_{u, \varphi}^* K_w(z) = \frac{\lambda \alpha z u(w)}{(1 - \alpha z \varphi(w))^2} = \frac{\lambda \alpha a w z}{(1 - \alpha bw - \alpha bz + \alpha^2 b^2 z w - \alpha c z w)^2}.$$

Comparing (3) and (4) shows that $D_{u, \varphi} C_{\lambda, \alpha} = C_{\lambda, \alpha} D_{u, \varphi}^*$ and hence, $D_{u, \varphi}$ is $C_{\lambda, \alpha}$ -symmetric. \square

The above theorem can be seen as a continuation of the work of [14, Theorem 3.3] and [4, Proposition 2.9].

3. NORMAL AND SELF-ADJOINTS

In the section, we study the adjoint of $D_{u, \varphi}$. For $z \in \mathbb{D}$, let

$$(5) \quad \varphi(z) = \frac{az + b}{cz + d} \text{ and } \sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

If φ is an analytic self-map of \mathbb{D} , then so does σ , see [1]. The following proposition gives the adjoint of the weighted differentiation composition operator.

Lemma 1. *If φ and σ are linear fractional self-maps of \mathbb{D} as in (5), where $\|\varphi\|_\infty < 1$, then $D_{K_{\sigma(0)}, \varphi}^* = D_{K_{\varphi(0)}, \sigma}$.*

Proof. For $z \in \mathbb{D}$, it is easy to obtain that

$$K_{\varphi(0)}^{(1)}(z) = \frac{\bar{d}^2 z}{(\bar{d} - \bar{b}z)^2} \quad \text{and} \quad K_{\sigma(0)}^{(1)}(z) = \frac{d^2 z}{(d + cz)^2}.$$

By simple calculations, we obtain that, for $z, w \in \mathbb{D}$,

$$D_{K_{\varphi(0)}^{(1)}, \sigma} K_w(z) = D_{K_{\varphi(0)}^{(1)}, \sigma} \left(\frac{1}{1 - \bar{w}z} \right) = \frac{\bar{d}^2 z}{(\bar{d} - \bar{b}z)^2} \cdot \frac{\bar{w}}{\left(1 - \bar{w} \left(\frac{\bar{a}z - \bar{c}}{-bz + d} \right)\right)^2}$$

and

$$D_{K_{\sigma(0)}^{(1)}, \varphi}^* K_w(z) = D_{K_{\sigma(0)}^{(1)}, \varphi}^* \left(\frac{1}{1 - \bar{w}z} \right) = \frac{\bar{d}^2 \bar{w}}{(\bar{d} + c\bar{w})^2} \cdot \frac{z}{\left(1 - z \left(\frac{\bar{a}w + \bar{b}}{c\bar{w} + d} \right)\right)^2}$$

which upon simplifications of the right hand side of the last two relations give the desired result. The proof is complete. \square

In the above lemma, we used the weighted composition differentiation operator to express the adjoint formula given by Fatehi and Hammond [2]. A bounded linear operator T in Hardy spaces is called a normal operator if $T^*T = TT^*$. In the following theorem, we discuss when the weighted composition differentiation operator is normal.

Theorem 2. For $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, suppose that φ and σ are given by (5), where $\|\varphi\|_\infty < 1$. Let u be defined as $u = K_{\sigma(0)}^{(1)}$. If $a\bar{b} = -\bar{a}c$ and $\varphi(0) = \sigma(0)$, then $D_{u, \varphi}$ is normal.

Proof. To verify that $D_{u, \varphi}$ is normal, we only need to prove that

$$D_{u, \varphi}^* D_{u, \varphi} K_w(z) = D_{u, \varphi} D_{u, \varphi}^* K_w(z) \quad \text{for all } z \in \mathbb{D}.$$

For the convenience, we let

$$\begin{cases} |a|^2 - |b|^2 = t_1 \\ b\bar{d} - a\bar{c} = t_2 \\ \bar{a}c - \bar{b}d = t_3 \\ |d|^2 - |c|^2 = t_4 \end{cases} \quad \text{and} \quad \begin{cases} |a|^2 - |c|^2 = k_1 \\ \bar{a}b - \bar{c}d = k_2 \\ c\bar{d} - a\bar{b} = k_3 \\ |d|^2 - |b|^2 = k_4. \end{cases}$$

If $\varphi(0) = \sigma(0)$, then we can easily obtain that $\bar{b}d = -c\bar{d}$ and $|c|^2 = |b|^2$. Since $a\bar{b} = -\bar{a}c$, it follows easily that $t_i = k_i$ for $i = 1, 2, 3, 4$. By calculation, we find that

$$\begin{aligned} (\bar{a}z - \bar{c})(\bar{d} - \bar{b}z)(ad - bc) &= [-\bar{a}bz^2 + (\bar{a}d + \bar{b}c)z - \bar{c}d](ad - bc) \\ &= (|b|^2\bar{a}c - |a|^2\bar{b}d)z^2 + (|ad|^2 - \bar{a}d\bar{b}c + \bar{b}c\bar{a}d - |bc|^2)z + |c|^2\bar{b}d - |d|^2\bar{a}c \\ &= (|a|^2\bar{c}d - |c|^2\bar{a}b)z^2 + (|ad|^2 - \bar{a}d\bar{b}c + \bar{a}d\bar{b}c - |bc|^2)z + |d|^2\bar{a}b - |b|^2\bar{c}d \\ &= (\bar{a}d - \bar{b}c)acz^2 + (\bar{a}d - \bar{b}c)(ad + bc)z + (\bar{a}d - \bar{b}c)bd \\ &= (az + b)(cz + d)(\bar{a}d - \bar{b}c) \end{aligned}$$

for all $z \in \mathbb{D}$. Since

$$D_{K_{\sigma(0)}^{(1)}, \varphi}^* = D_{K_{\varphi(0)}^{(1)}, \sigma},$$

we have

$$\begin{aligned}
D_{u,\varphi}^* D_{u,\varphi} K_w(z) &= D_{K_{\varphi(0)}^{(1)},\sigma} D_{K_{\sigma(0)}^{(1)},\varphi} \left(\frac{1}{1-\bar{w}z} \right) \\
&= D_{K_{\varphi(0)}^{(1)},\sigma} \left(\frac{d^2 z \bar{w}}{(d+cz)^2 (1-\bar{w}\varphi(z))^2} \right) \\
&= |d|^4 z \bar{w} \cdot \frac{(d-c\sigma(z))}{(\bar{d}-\bar{b}z)^2 (d+c\sigma(z))^3 (1-\bar{w}\varphi(\sigma(z)))^2} \\
&\quad + |d|^4 z \bar{w} \cdot \frac{2\bar{w}\sigma(z)\varphi'(\sigma(z))}{(\bar{d}-\bar{b}z)^2 (d+c\sigma(z))^2 (1-\bar{w}\varphi(\sigma(z)))^3} \\
&= |d|^4 z \bar{w} \left(\frac{|d|^2 + |c|^2 - (\bar{a}c + \bar{b}d)z}{(t_3z + t_4)[(t_3z + t_4) - (t_1z + t_2)\bar{w}]^2} \right. \\
&\quad \left. + \frac{2(\bar{a}z - \bar{c})(\bar{d} - \bar{b}z)(ad - bc)\bar{w}}{(t_3z + t_4)[(t_3z + t_4) - (t_1z + t_2)\bar{w}]^3} \right) \\
&= |d|^4 z \bar{w} \left(\frac{|d|^2 + |b|^2 + (a\bar{b} + c\bar{d})z}{(k_3z + k_4)[(k_3z + k_4) - (k_1z + k_2)\bar{w}]^2} \right. \\
&\quad \left. + \frac{2(az + b)(cz + d)(\bar{a}\bar{d} - \bar{b}\bar{c})\bar{w}}{(k_3z + k_4)[(k_3z + k_4) - (k_1z + k_2)\bar{w}]^3} \right) \\
&= |d|^4 z \bar{w} \cdot \frac{(\bar{d} + \bar{b}\varphi(z))}{(d+cz)^2 (\bar{d} - \bar{b}\varphi(z))^3 (1 - \bar{w}\sigma(\varphi(z)))^2} \\
&\quad + |d|^4 z \bar{w} \cdot \frac{2\bar{w}\varphi(z)\sigma'(\varphi(z))}{(d+cz)^2 (\bar{d} - \bar{b}\varphi(z))^2 (1 - \bar{w}\sigma(\varphi(z)))^3} \\
&= D_{u,\varphi} D_{u,\varphi}^* K_w(z)
\end{aligned}$$

for all $z, w \in \mathbb{D}$. Hence, $D_{u,\varphi}^* D_{u,\varphi} = D_{u,\varphi} D_{u,\varphi}^*$ which in turn implies that $D_{u,\varphi}$ is normal. \square

In the above theorem, if we choose $a = i, b = 1 + i, c = 1 - i$ and $d = 8i$, then the operator $D_{u,\varphi}$ is a normal operator. However, $D_{u,\varphi}$ is not a self-adjoint operator.

A bounded linear operator T in a Hardy space is called a self-adjoint if $T^* = T$. In the following theorem, we study when the weighted composition differentiation operator is self-adjoint.

Theorem 3. *Let $u(z)$ be a nonzero analytic function in H^2 and $\varphi(z)$ be analytic self-map on \mathbb{D} such that $\|\varphi\|_\infty < 1$. Then $D_{u,\varphi}$ is self-adjoint if and only if*

$$u(z) = \frac{az}{(1-bz)^2} \quad \text{and} \quad \varphi(z) = b + \frac{cz}{1-bz} \quad \text{for all } z \in \mathbb{D},$$

where $a, b, c \in \mathbb{R}$.

Proof. If $u(z) = \frac{az}{(1-bz)^2}$ and $\varphi(z) = b + \frac{cz}{1-bz}$, where $a, b, c \in \mathbb{R}$, for all $z \in \mathbb{D}$, then we have

$$D_{u,\varphi}^* K_w(z) = D_{u,\varphi}^* \left(\frac{1}{1-\bar{w}z} \right) = \frac{\overline{zu(w)}}{(1-z\overline{\varphi(w)})^2} = \frac{\bar{a}z\bar{w}}{(1-\bar{b}\bar{w}-\bar{b}z+z\bar{w}\bar{b}^2-\bar{c}z\bar{w})^2}$$

and

$$D_{u,\varphi} K_w(z) = D_{u,\varphi} \left(\frac{1}{1-\bar{w}z} \right) = \frac{\overline{wu(z)}}{(1-\bar{w}\varphi(z))^2} = \frac{az\bar{w}}{(1-bz-b\bar{w}+z\bar{w}b^2-cz\bar{w})^2},$$

where $z, w \in \mathbb{D}$. Hence, $D_{u,\varphi}^* = D_{u,\varphi}$ which shows that $D_{u,\varphi}$ is self-adjoint.

Conversely, we suppose that $D_{u,\varphi}$ is self-adjoint. Then $D_{u,\varphi}^* = D_{u,\varphi}$. This implies that $D_{u,\varphi}^* K_w(z) = D_{u,\varphi} K_w(z)$ for $w, z \in \mathbb{D}$. Notice that

$$D_{u,\varphi}^* K_w(z) = \frac{\overline{u(w)}z}{(1 - z\varphi(w))^2} \quad \text{and} \quad D_{u,\varphi} K_w(z) = \frac{u(z)\overline{w}}{(1 - \overline{w}\varphi(z))^2}.$$

Hence,

$$(6) \quad \frac{\overline{u(w)}z}{(1 - z\varphi(w))^2} = \frac{u(z)\overline{w}}{(1 - \overline{w}\varphi(z))^2}$$

for $z, w \in \mathbb{D}$. With $w = 0$, we show that $u(0) = 0$.

For $u \in H^2$, we let $u(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_n \in \mathbb{C}$. Substitute $u(z)$ back into (6) to obtain

$$(7) \quad \left(\sum_{n=1}^{\infty} \overline{a_n} z \overline{w}^n \right) (1 - \overline{w}\varphi(z))^2 = \left(\sum_{n=1}^{\infty} a_n \overline{w} z^n \right) (1 - z\varphi(w))^2$$

for all $z, w \in \mathbb{D}$.

Now, differentiating the equation (7) with respect to \overline{w} , we obtain

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} n \overline{a_n} z \overline{w}^{n-1} \right) (1 - \overline{w}\varphi(z))^2 - 2\varphi(z) \left(\sum_{n=1}^{\infty} \overline{a_n} z \overline{w}^n \right) (1 - \overline{w}\varphi(z)) \\ & = \left(\sum_{n=1}^{\infty} a_n z^n \right) (1 - z\varphi(w))^2 - 2z\varphi'(w) \left(\sum_{n=1}^{\infty} a_n \overline{w} z^n \right) (1 - z\varphi(w)). \end{aligned}$$

Set $w = 0$. Then $\overline{a_1}z = (1 - \overline{\varphi(0)}z)^2 u(z)$ so that

$$(8) \quad u(z) = \frac{\overline{a_1}z}{(1 - \overline{\varphi(0)}z)^2} = \frac{\overline{a}z}{(1 - \overline{b}z)^2},$$

with $a = a_1$ and $b = \varphi(0)$.

Substituting (8) back into (6) yields that

$$(9) \quad a(1 - \overline{b}z)^2 (1 - \overline{w}\varphi(z))^2 = \overline{a}(1 - b\overline{w})^2 (1 - z\overline{\varphi(w)})^2,$$

where $z \in \mathbb{D}$. Setting $w = 0$ shows that $\overline{a} = a$, i.e., $a \in \mathbb{R}$.

Differentiating the equation (9) with respect to \overline{w} gives

$$\begin{aligned} & (1 - \overline{b}z)^2 (1 - \overline{w}\varphi(z)) (-2a\varphi(z)) \\ & = (1 - z\overline{\varphi(w)})^2 (-2ab)(1 - b\overline{w}) + (1 - b\overline{w})^2 (1 - z\overline{\varphi(w)}) (-2az\overline{\varphi'(w)}) \end{aligned}$$

for all $z \in \mathbb{D}$. If $w = 0$, then we have

$$\varphi(z) = b + \frac{cz}{1 - bz}, \quad z \in \mathbb{D},$$

where $b = \varphi(0)$ and $c = \overline{\varphi'(0)}$.

Now the fact that $D_{u,\varphi}$ is self-adjoint implies that

$$D_{u,\varphi}^* K_w(z) = D_{u,\varphi} K_w(z).$$

Similar to the sufficiency part of the proof of the theorem, we get $b, c \in \mathbb{R}$. □

Recall that a bounded linear operator T on H^2 is unitary if and only if

$$TT^* = T^*T = I.$$

In the above theorem, we considered conditions such that the operator $D_{u,\varphi}$ is self-adjoint. Clearly, $D_{u,\varphi}$ is also normal. But it is not unitary. We have $D_{u,\varphi}^* D_{u,\varphi} \neq I$ by a simple calculation. Let

$u = az$ and $\varphi = cz$, where $a, c \in \mathbb{R}$. If $|a| < \infty$ and $0 < c < 1$, then $D_{u,\varphi}$ is self-adjoint. In this case, if we can find the spectrum of the operator $D_{u,\varphi}$, we will obtain the norm of $D_{u,\varphi}$.

Theorem 4. *Assume that $u(z) = az$ and $\varphi(z) = cz$, where $a \in \mathbb{R}$ and $c \in (0, 1)$. Then*

$$\|D_{u,\varphi}\| = akc^k,$$

where $k = \lfloor \frac{1}{1-c} \rfloor$, and $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Proof. For $u(z) = az$ and $\varphi(z) = cz$, we have

$$D_{u,\varphi}(z^{n-1}) = acz(n-1)(cz)^{n-2} = (n-1)ac^{n-1}z^{n-1}$$

for all $n \in \mathbb{N} = \{1, 2, \dots\}$. We see that $\{(n-1)ac^{n-1} : n \in \mathbb{N}\}$ belongs to the spectrum of $D_{u,\varphi}$. Next we let λ be an arbitrary eigenvalue of $D_{u,\varphi}$ with the corresponding eigenvector f . Then

$$(10) \quad \lambda f(z) = acz f'(cz).$$

If $f(0) \neq 0$, then $\lambda = 0$. If $f(0) = 0$, then differentiate the equation (10) with respect to z to obtain

$$(11) \quad \lambda f'(z) = acf'(cz) + ac^2 z f''(cz).$$

If $f'(0) \neq 0$, then $\lambda = ac$. If $f''(0) = 0$, then we differentiate both sides of the equation (11) to get

$$(12) \quad \lambda f''(z) = 2ac^2 f''(cz) + ac^3 z f'''(cz).$$

If $f''(0) \neq 0$, then $\lambda = 2ac^2$. Also, $(n-1)$ -times differentiation of the equation (10) gives

$$\lambda f^{(n-1)}(z) = (n-1)ac^{n-1} f^{(n-1)}(cz) + ac^n z f^{(n)}(cz).$$

If $f^{(n-1)}(0) \neq 0$, then $\lambda = (n-1)ac^{(n-1)}$. Therefore, any eigenvalue can be represented in this form $(n-1)ac^{(n-1)}$ with $n \in \mathbb{N}$. Since $D_{u,\varphi}$ is compact, the spectrum of $D_{u,\varphi}$ is precisely $\{0\} \cup \{(n-1)ac^{(n-1)} : n \in \mathbb{N}\}$. Hence

$$\|D_{u,\varphi}\| = \max\{(n-1)ac^{(n-1)} : n \in \mathbb{N}\}.$$

Let $g(x) = xc^x$. It can be found that $g(x)$ has maximum in $[0, \infty)$. To obtain the maximum, we need to find that the greatest natural number n such that

$$(n-2)c^{(n-2)} \leq (n-1)c^{n-1}, \quad \text{i.e., } n \leq \frac{1}{1-c} + 1.$$

This gives $\max\{(n-1)ac^{(n-1)}\} = akc^k$, where $k = \lfloor \frac{1}{1-c} \rfloor$. □

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