Topology and its Applications 194 (2015) 386–399

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Complex cobordism of quasitoric orbifolds

Soumen Sarkar

Department of Mathematics and Statistics, University of Regina, Regina S4S 6A7, Canada

ARTICLE INFO

Article history:
Received 16 October 2014
Received in revised form 25 August 2015
Accepted 25 August 2015
Available online 11 September 2015

MSC:
55N22
57R90

Keywords:
Quasitoric orbifolds
Stable and almost complex structure
Complex cobordism

ABSTRACT

We construct orbifolds with quasitoric boundary and show that they have stable almost complex structure. We show that a quasitoric orbifold is complex cobordant to finite disjoint copies of complex orbifold projective spaces. Finally some computations in the complex cobordism ring for manifolds are given.

1. Introduction

Cobordism was explicitly introduced by Lev Pontryagin in his seminal paper [15]. In [20] Thom showed that the cobordism groups could be computed by results of homotopy theory using the Thom complex construction. Later, Atiyah [2] showed that complex cobordism is a generalized cohomology theory. In Section 1 of [17], Quillen discussed geometric interpretation of complex cobordism rings. Following his definition, we define the complex orbifold (co)bordism groups and rings for the category of stable almost complex orbifolds. It seems that complex cobordism for orbifolds did not appear in the literature until now. However oriented cobordism of orbifolds is studied in [8] and [9].

In the pioneering paper [7], Davis and Januskiewicz introduced the topological counterpart of nonsingular projective toric varieties. They called this class of manifolds toric manifolds. Since "toric manifold" is used in algebraic geometry for "nonsingular toric variety", Buchstaber and Panov [3] introduced the term "quasitoric manifold" instead. Quasitoric orbifolds are generalization of quasitoric manifolds and they are studied in [16]. An orbifold with quasitoric boundary is an orbifold with boundary where the boundary is a disjoint union of some quasitoric orbifolds.

E-mail address: soumensarkar20@gmail.com.

http://dx.doi.org/10.1016/j.topol.2015.08.020

In this article we study the complex cobordism of quasitoric orbifolds. The article is organized as follows. In Section 2, we recall the definition of stable complex structure on an orbifold. In Section 3, we recall the definition of quasitoric orbifold, omniorientation on a quasitoric orbifold and equivariant classification of quasitoric orbifolds. Also we show that a quasitoric orbifold over a simplex is equivariantly homeomorphic to a complex orbifold projective space, see Lemma 3.9. In Section 4, we construct oriented orbifolds with quasitoric boundary. In Section 5, first we show that the orbifolds with quasitoric boundary which are constructed in Section 4 have stable complex structure, see Theorem 5.5. Then we show that a quasitoric orbifold is complex cobordant to finite disjoint copies of complex orbifold projective spaces, see Theorem 5.6. This process produces explicit complex cobordism relations among quasitoric orbifolds. We show that the set of all complex orbifold cobordism classes of complex orbifold projective spaces is not linearly independent, see Observation 5.8. Note that this is in contrast to manifold case. As a particular case when the orbifold singularity is trivial, we get explicit complex cobordism relations among quasitoric manifolds. At the end of Section 5, we give some sufficient conditions to the famous problem of Hirzebruch which asks when a complex cobordism class in the complex cobordism ring Ω^U for manifolds may contain a connected nonsingular algebraic variety. In [21], Andrew Wilfong gives some necessary condition of this problem up to dimension 8. We also compute the Chern numbers of a quasitoric manifold over a simplex, see Example 5.10.

2. Orbifolds

Orbifolds were introduced by Satake [19], who called them V-manifolds. An orbifold is a singular space that is locally look like a quotient of an open subset of Euclidean space by an action of a finite group. The readers are referred to the Section 1.1 in [1] for the definition and basic facts concerning effective orbifolds. Also they may see [12] for an excellent exposition of the foundation of the theory of the reduced differentiable orbifolds.

Similarly as the definition of manifold with boundary, we can talk about orbifold with boundary, see Definition 1.3 in [8]. In this article Druschel studies the orientation on orbifolds in Section 1.

Many concepts in orbifold theory are defined in the context of groupoid, see [1] to enjoy this approach. For example, Section 2.3 in [1] talks about orbifold vector bundle in the language of groupoid. Most relevant example of the orbibundle of an orbifold is its tangent bundle. An explicit description of the tangent bundle of an effective orbifold is given in Section 1.3 of [1].

Definition 2.1. Let Y be a smooth orbifold with the tangent bundle (TY, p_Y, Y) where $p_Y : TY \to Y$ is the projection map.

(1) An almost complex structure on Y is an endomorphism $J : TY \to TY$ such that $J^2 = -Id$.
(2) A stable almost complex (or stable complex) structure on Y is an endomorphism

$$J : TY \oplus (Y \times \mathbb{R}^k) \to TY \oplus (Y \times \mathbb{R}^k) \tag{2.1}$$

such that $J^2 = -Id$ for some positive integer k.

3. Quasitoric orbifolds

In this section we review the definition of quasitoric orbifold following [16]. We also discuss several results on quasitoric orbifolds. An n-dimensional *simple polytope* in $\mathbb{R}^n$ is a convex polytope where exactly n bounding hyperplanes meet at each vertex. A *facet* is a codimension one face of a convex polytope. If P is a convex polytope then we denote the set of all facets of P by $\mathcal{F}(P)$. Let $\mathbb{T}^n = (\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Z}^n$ and $\mathbb{T}_M = (M \otimes_{\mathbb{Z}} \mathbb{R})/M$ for a free $\mathbb{Z}$-module M.

Definition 3.1. A $2n$-dimensional quasitoric orbifold X is a smooth orbifold with a $\mathbb{T}^n$-action, such that the orbit space is (diffeomorphic as manifold with corners to) an n-dimensional simple polytope P. Denote the projection map from X to P by $\pi: X \to P$. Furthermore every point $x \in X$ has

A1) a $\mathbb{T}^n$-invariant neighborhood V,
A2) an associated free $\mathbb{Z}$-module M of rank n with an isomorphism $\theta: \mathbb{T}_M \to U(1)^n$ and an injective module homomorphism $\iota: M \to \mathbb{Z}^n$ which induces a surjective covering homomorphism $\iota_M: \mathbb{T}_M \to \mathbb{T}^n$,
A3) an orbifold chart (W, G, η) over V where W is θ-equivariantly diffeomorphic to an open set in $\mathbb{C}^n$, $G = \ker\iota_M$ and $\eta: W \to V$ is an equivariant map i.e. $\eta(t \cdot y) = \iota_M(t) \cdot \eta(y)$ inducing a homeomorphism between W/G and V.

Note that this definition is a generalization of the axiomatic definition of a quasitoric manifold, see Section 1 in [7]. Let P be an n-dimensional simple polytope and $\mathcal{F}(P) = \{P_1, \ldots, P_s\}$.

Definition 3.2. A function $\xi: \mathcal{F}(P) \to \mathbb{Z}^n$ is called a di-characteristic function if the vectors $\xi(P_{j_1}), \ldots, \xi(P_{j_l})$ are linearly independent over $\mathbb{Z}$ whenever the intersection of the facets $P_{j_1}, \ldots, P_{j_l}$ is nonempty.

The vector $\xi_j = \xi(P_j)$ is called the di-characteristic vector corresponding to the facet P_j and the pair (P, ξ) is called a characteristic model on P.

Remark 3.3. If the set $\{\xi(P_{j_1}), \ldots, \xi(P_{j_l})\}$ is a part of a basis of $\mathbb{Z}^n$ over $\mathbb{Z}$ whenever the intersection of the facets $P_{j_1}, \ldots, P_{j_l}$ is nonempty, then the map ξ is called characteristic function on P, see Section 1 of [7].

In Subsection 2.1 of [16], the authors construct a quasitoric orbifold from the characteristic model (P, ξ). Also given a quasitoric orbifold we can associate a characteristic model to it up to choice of signs of di-characteristic vectors.

Example 3.4. Let $S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}: |z_0|^2 + \cdots + |z_n|^2 = 1\}$ and $a_0, \ldots, a_n$ be coprime positive integers. Then a weighted action of the circle S^1 on S^{2n+1} is given by

$$\alpha \cdot (z_0, \ldots, z_n) = (\alpha^{a_0} z_0, \ldots, \alpha^{n} z_n) \quad \text{for } \alpha \in S^1.$$

The orbit space $\mathbb{WP}(a_0, \ldots, a_n) = S^{2n+1}/S^1$ is called a weighted projective space. Since $a_0, \ldots, a_n$ are coprime integers, the vector $\mathfrak{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ determines a circle subgroup $S^1_{\mathfrak{a}}$ of $\mathbb{T}^{n+1}$. Then the natural $\mathbb{T}^{n+1}$-action on S^{2n+1} induces an action of $\mathbb{T}^n \cong \mathbb{T}^{n+1}/S^1_{\mathfrak{a}}$ on $\mathbb{WP}(a_0, \ldots, a_n)$. With respect to this $\mathbb{T}^n$-action, $\mathbb{WP}(a_0, \ldots, a_n)$ is a quasitoric orbifold over an n-simplex. For an integer $a > 1$, the space $\mathbb{WP}(1, a)$ is called the teardrop. The characteristic model for the teardrop is given by $([0, 1], \xi)$ where $\xi(\{0\}) = -1$ and $\xi(\{1\}) = a$ (possibly up to sign). $\square$

Definition 3.5. Let $\delta: \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism. Two quasitoric orbifolds X_1 and X_2 over the same polytope P are called δ-equivariantly homeomorphic if there is a homeomorphism $f: X_1 \to X_2$ such that $f(t \cdot x) = \delta(t) \cdot f(x)$ for all $(t, x) \in \mathbb{T}^n \times X_1$.

The automorphism δ induces an automorphism δ_* of $\mathbb{Z}^n$. For the automorphism δ, two characteristic models (P, ξ) and (P, η) are called *δ-equivalent* if there is a diffeomorphism $\psi : P \to P$ (as manifold with corners) such that $\eta(\psi(F)) = \pm\delta_*(\xi(F))$ for all $F \in \mathcal{F}(P)$. If δ is identity, then (P, ξ) and (P, η) are called *equivalent*. The following proposition is a classification result which can be found in [3] for quasitoric manifolds (Proposition 5.14) and in [16] for quasitoric orbifolds (Lemma 2.2).

Proposition 3.6. *For an automorphism δ of the torus $\mathbb{T}^n$, there is a bijection between δ-equivariant homeomorphism classes of quasitoric orbifolds over P and δ-equivalent classes of characteristic models on P.*

Suppose δ is the identity automorphism of $\mathbb{T}^n$. Proposition 3.6 implies that two quasitoric orbifolds over P are equivariantly homeomorphic if and only if their di-characteristic models are equivalent.

Let X be a quasitoric orbifold with the orbit map $\pi: X \to P$. There are important closed $\mathbb{T}^n$-invariant suborbifolds of X which are corresponding to the faces of P. If F is a codimension k face of P, then define $X(F) = \pi^{-1}(F)$. The space $X(F)$ with subspace topology is a quasitoric orbifold of dimension $2n - 2k$. If F is a facet of P then $X(F)$ is called a *characteristic suborbifold* of X. Note that a choice of orientation on $\mathbb{T}^n$ and P give an orientation on the orbifold X.

Definition 3.7. An *omniorientation* of a quasitoric orbifold X is a choice of orientation for X as well as an orientation for each characteristic suborbifold of X.

Clearly the isotropy group of a characteristic suborbifold is a circle subgroup of $\mathbb{T}^n$. So there is a natural S^1-action on the normal bundle (possibly an orbibundle) of that characteristic suborbifold. Thus the normal bundle has a complex structure and consequently an orientation. Whenever the sign of the characteristic vector of a facet is reverse, we get the opposite orientation on the normal bundle. An orientation on the normal bundle together with an orientation on X induces an orientation on the characteristic suborbifold. So a di-characteristic function determines a natural omniorientation. We call this omniorientation the *characteristic omniorientation*.

A toric variety X_Σ associated to the simplicial fan Σ is called a toric orbifold. The space X_Σ is compact if and only if Σ is complete. More about toric varieties can be found in [6].

Definition 3.8. Let Σ be a complete simplicial fan in $\mathbb{R}^n$ with $n+1$ many one-dimensional cones. The associated toric orbifold X_Σ is called a $2n$-dimensional complex orbifold projective space.

Lemma 3.9. *Let X be a quasitoric orbifold over $\triangle^n$. Then X is equivariantly homeomorphic to a $2n$-dimensional complex orbifold projective space.*

Proof. Let X be a quasitoric orbifold over $\triangle^n$ and $\mathcal{F}(\triangle^n) = \{F_0, \ldots, F_n\}$. Let

$$\xi : \mathcal{F}(\triangle^n) \to \mathbb{Z}^n$$

be the associated di-characteristic function. Suppose $\xi_i = \xi(F_i)$ for $i = 0, \ldots, n$. So $\{\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_n\}$ is a linearly independent set in $\mathbb{Z}^n$ for $i = 0, \ldots, n$ where $\hat{}$ represents the omission of the corresponding entry. Let

$$\xi_0 = a_1\xi_1 + \cdots + a_n\xi_n$$

for some $a_1, \ldots, a_n \in \mathbb{Q}$. Then $a_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Suppose $a_{i_1}, \ldots, a_{i_l} \in \mathbb{Q}_{>0}$. Define $\eta: : \mathcal{F}(\triangle^n) \to \mathbb{Z}^n$ by

$$\eta(F_j) = \begin{cases} -\xi_j & \text{if } j \in \{i_1, \ldots, i_l\} \\ \xi_j & \text{if } j \in \{0, \ldots, n\} - \{i_1, \ldots, i_l\}. \end{cases} \tag{3.1}$$

Let $\eta_j = \eta(F_j)$ for $j = 0, \ldots, n$ and

$$b_j = \begin{cases} -a_j & \text{if } j \in \{i_1, \ldots, i_l\} \\ a_j & \text{if } j \in \{0, \ldots, n\} - \{i_1, \ldots, i_l\}. \end{cases}$$

So $b_j < 0$ for $j = 1, \ldots, n$ and $\eta_0 = b_1\eta_1 + \cdots + b_n\eta_n$. Therefore $\eta_0, \ldots, \eta_n$ are the one-dimensional cones of a complete simplicial fan Σ in $\mathbb{R}^n$. Let X_Σ be the associated toric orbifold. So X_Σ is a complex orbifold projective space. With respect to the compact n-torus action on X_Σ, it is a quasitoric orbifold where the corresponding characteristic model is $(\triangle^n, \eta)$. Therefore by Proposition 3.6, X and X_Σ are equivariantly homeomorphic. □

Fake weighted projective space is a holomorphic generalization of weighted projective space, see [10]. A $2n$-dimensional fake weighted projective space is defined by a complete simplicial fan generated by $n+1$ many primitive vectors in $\mathbb{Z}^n$. So a fake weighted projective space is a complex orbifold projective space. Since the primitive vectors in $\mathbb{Z}$ are -1 and 1, the teardrop $\mathbb{WP}(1,a)$ is not a fake weighted projective space if $a > 1$. But $\mathbb{WP}(1,a)$ is a complex orbifold projective space.

4. Construction of orbifolds with quasitoric boundary

In this section we construct some oriented orbifolds with quasitoric boundary. This construction is a generalization of the construction of manifolds with quasitoric boundary of Section 4 in [18].

Definition 4.1. An $(n+1)$-dimensional simple polytope Q in $\mathbb{R}^{n+1}$ is said to be polytope with exceptional facets $\{Q_1, \dots, Q_k\}$ if $Q_i \cap Q_j$ is empty for $i \neq j$ with $1 \leq i, j \leq k$ and $V(Q) = \cup_{i=1}^k V(Q_i)$. We denote $\{Q \backslash Q_1, \dots, Q_k\}$ for a simple polytope with exceptional facets.

Let $\mathcal{F}(Q) = \{F_1, \dots, F_m\} \cup \{Q_1, \dots, Q_k\}$ be the facets of Q where $\{Q_1, \dots, Q_k\}$ are the exceptional facets.

Definition 4.2. A function $\lambda\colon \{F_1, \dots, F_m\} \to \mathbb{Z}^n$ is called an isotropy function on $\{Q \backslash Q_1, \dots, Q_k\}$ if the vectors $\lambda(F_{i_1}), \dots, \lambda(F_{i_q})$ are linearly independent in $\mathbb{Z}^n$ whenever the intersection of the facets $F_{i_1}, \dots, F_{i_q}$ is nonempty. The vector $\lambda_i = \lambda(F_i)$ is called an isotropy vector assigned to the facet F_i for $i = 1, \dots, m$.

We define the isotropy function on some polytopes with exceptional facets in Example 4.6.

Remark 4.3. Since $\mathbb{Z}^n$ is not the union of finitely many proper submodules over $\mathbb{Z}$, we can define the isotropy function on any polytope with exceptional facets.

We adhere the notations of Definition 4.2. Let F be a codimension l face of Q with $0 < l \leq n+1$. If F is a face of Q_i for some $i \in \{1, \dots, k\}$, then F is the intersection of a unique collection of l facets $F_{i_1}, \dots, F_{i_{l-1}}, Q_i$ of Q. Otherwise, F is the intersection of a unique collection of l facets $F_{i_1}, \dots, F_{i_l} \in \{F_1, \dots, F_m\}$ of Q. Let

$$M(F) = \begin{cases} \langle \{\lambda_{i_j} \colon j = 1, \dots, l-1\} \rangle \subseteq \mathbb{Z}^n & \text{if } F = F_{i_1} \cap \dots \cap F_{i_{l-1}} \cap Q_i \\ \langle \{\lambda_{i_j} \colon j = 1, \dots, l\} \rangle \subseteq \mathbb{Z}^n & \text{if } F = F_{i_1} \cap \dots \cap F_{i_l} \end{cases} \tag{4.1}$$

where $\langle \{\alpha_i : i = 1, \dots, s\} \rangle$ denotes the submodule generated by the vectors $\{\alpha_i \colon i = 1, \dots, s\}$ of $\mathbb{Z}^n$. So

$$\mathbb{T}_{M(F)} = (M(F) \otimes_{\mathbb{Z}} \mathbb{R}) / M(F) \tag{4.2}$$

is a compact torus of dimension $l-1$ or l depending on the situation of the face F. Adopt the convention that $\mathbb{T}_{M(Q)} = 1 = \mathbb{T}_{M(Q_i)}$ for $i = 1, \dots, k$. The inclusion $M(F) \hookrightarrow \mathbb{Z}^n$ induces a natural homomorphism

$$f_F \colon \mathbb{T}_{M(F)} \to \mathbb{T}^n$$

for any face F of Q. Denote the image of f_F by $Im(f_F)$. Define an equivalence relation $\sim_b$ on the product $\mathbb{T}^n \times Q$ as follows,

$$(t, x) \sim_b (u, y) \quad \text{if and only if} \quad x = y \ \text{ and } tu^{-1} \in Im(f_F) \tag{4.3}$$

where F is the unique face of Q containing y in its relative interior. We denote the quotient space $(\mathbb{T}^n \times Q)/\sim_b$ by $W(Q, \lambda)$ and the equivalence class of (t, x) by $[t, x]^{\sim_b}$. The space $W(Q, \lambda)$ is a $\mathbb{T}^n$-space where the action is induced by the group operation in $\mathbb{T}^n$. Let

$$\mathfrak{q} \colon W(Q, \lambda) \to Q$$

be the projection map defined by $\mathfrak{q}([t, x]^{\sim_b}) = x$. We consider the standard orientation of $\mathbb{T}^n$ and the orientation of Q induced from the ambient space $\mathbb{R}^{n+1}$.

Lemma 4.4. *The space $W(Q, \lambda)$ is a $(2n+1)$-dimensional oriented orbifold with boundary and the boundary is a disjoint union of $2n$-dimensional quasitoric orbifolds.*

Proof. Let $C_j = \{F : F \text{ is a face of } Q \text{ and } F \cap Q_j \text{ is empty}\}$ and

$$U_j = Q - \cup_{F \in C_j} F$$

for $j = 1, \ldots, k$. Since Q is a simple polytope, U_j is diffeomorphic as manifold with corners to $Q_j \times [0, 1)$ and $Q = \cup_{j=1}^k U_j$. Let

$$f_j \colon U_j \to Q_j \times [0, 1)$$

be a diffeomorphism. Note that the facets of Q_j are $\{Q_j \cap F_{j_1}, \ldots, Q_j \cap F_{j_l}\}$ for some facets $F_{j_1}, \ldots, F_{j_l} \in \{F_1, \ldots, F_m\}$. The restriction of the isotropy function λ on the facets of Q_j is given by

$$\xi^j(Q_j \cap F_{j_i}) = \lambda_{j_i} \quad \text{for} \quad i = 1, \ldots, l.$$

By definition of λ, we can see that ξ^j is a di-characteristic function on Q_j. So by Subsection 2.1 of [16] the space $X(Q_j, \xi^j) = (\mathbb{T}^n \times Q_j)/\sim_b$ is a $2n$-dimensional quasitoric orbifold for $j = 1, \ldots, k$. From the equivalence relation $\sim_b$ in (4.3), we have the following commutative diagram where lower horizontal maps are homeomorphisms.

$$\begin{array}{ccccc}
\mathbb{T}^n \times U_j & \xrightarrow{Id \times f_i} & \mathbb{T}^n \times Q_j \times [0,1) & & \\
\downarrow & & \downarrow & & \\
(\mathbb{T}^n \times U_j)/\sim_b & \xrightarrow{h_i} & ((\mathbb{T}^n \times Q_j)/\sim_b) \times [0,1) & \xrightarrow{\cong} & X(Q_j, \xi^j) \times [0,1).
\end{array}$$

So

$$W(Q, \lambda) = \bigcup_{j=1}^{k} (T^n \times U_j)/\sim_b \;\cong\; \bigcup_{j=1}^{k} (X(Q_j, \xi^j) \times [0, 1)).$$

Hence $W(Q, \lambda)$ is an orbifold with boundary where the boundary is the disjoint union of quasitoric orbifolds $\{X(Q_j, \xi^j) \colon j = 1, \ldots, k\}$. Clearly orientations of $\mathbb{T}^n$ and Q induce an orientation of $W(Q, \lambda)$. $\square$

Suppose λ satisfies the following condition: the set of vectors $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_l})\}$ is a part of a basis of $\mathbb{Z}^n$ over $\mathbb{Z}$ whenever the intersection of the facets $\{F_{i_1}, \ldots, F_{i_l}\}$ is nonempty. Then all the quasitoric orbifolds in the proof of Lemma 4.4 are quasitoric manifolds. So, in this case we have the following corollary.

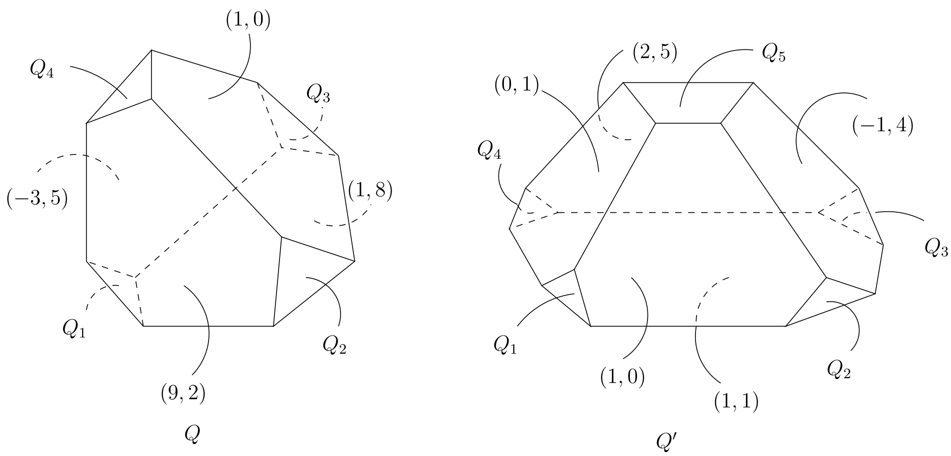

Fig. 1. Isotropy functions of some polytopes with exceptional facets.

Corollary 4.5. *With the assumption in the above paragraph, the space $W(Q, \lambda)$ is a $(2n + 1)$-dimensional oriented manifold with boundary. The boundary is a disjoint union of quasitoric manifolds.*

Example 4.6. Some isotropy functions on the polytopes Q and Q' with exceptional facets are given in the Fig. 1. In the left picture of Fig. 1, Q_1, Q_2, Q_3, Q_4 are exceptional facets which are triangles. The restriction of the isotropy function on Q_i gives that the space $(\mathbb{T}^2 \times Q_i)/\sim_b$ is a complex orbifold projective space for $i \in \{1, 2, 3, 4\}$. So $W(Q, \lambda)$ is an oriented orbifold with boundary where the boundary is the disjoint union of distinct 4-dimensional complex orbifold projective spaces.

In the right picture of Fig. 1, Q_1, Q_2, Q_3, Q_4 and Q_5 are exceptional facets where $Q_1, \ldots, Q_4$ are triangles and Q_5 is a rectangle. The restriction of the isotropy function on Q_i gives that the space $M_i = (\mathbb{T}^2 \times Q_i)/\sim_b$ is a complex orbifold projective space for $i \in \{1, 2, 3, 4\}$ and $X(Q_5, \lambda^5) = (\mathbb{T}^2 \times Q_5)/\sim_b$ is a quasitoric orbifold. Hence the space $\sqcup_{i=1}^4 M_i \sqcup X(Q_5, \lambda^5)$ is the boundary of the oriented orbifold $W(Q', \lambda)$. □

5. Stable complex structure and complex cobordism

Stable complex structure of quasitoric manifolds and quasitoric orbifolds are studied in [7] and [16] respectively. In this section first we show the existence of stable complex structure on orbifolds with quasitoric boundary $W(Q, \lambda)$. Then we compute the complex orbifold cobordism class of a quasitoric orbifold explicitly. At the end of this section we give some computation in Ω^U. Similarly as the manifolds case, we may define complex cobordism of orbifolds.

Definition 5.1. Let Y be a topological space and X_1, X_2 be two n-dimensional stable complex orbifolds. Let $h_i \colon X_i \to Y$ be a continuous map for $i = 1, 2$. Then h_1 and h_2 are bordant if there exists a stable complex orbifold Z of dimension $n+1$ with $\partial Z = X_1 \sqcup X_2$ and a continuous map $H \colon Z \to Y$ such that $H|_{\partial Z} = h_1 \sqcup h_2$.

So the Definition 5.1 induces an equivalence relation on the collection

$$\{(X, h) \colon X \text{ is a stable complex orbifold and } h \colon X \to Y \text{ is a continuous map}\}.$$

We denote the equivalence class of (X, h) by $[X, h]$ or $[X]$ if the map h and the stable complex structure on X are clear. Let $OB_n^U(Y) = \{[X, h] \colon \dim X = n\}$. The disjoint union induces an abelian group structure on $OB_n^U(Y)$. The group $OB_n^U(Y)$ is called the n-th complex orbifold bordism group of Y. Let $OB_*^U(Y) = \cup_n OB_n^U(Y)$. Then the Cartesian product endows the structure of a graded ring on $OB_*^U(Y)$, called the complex orbifold bordism ring of Y.

Definition 5.2. The complex orbifold bordism groups and ring of a point are called complex orbifold cobordism groups and ring respectively.

At this moment we do not know about the generators of the group $OB^U_n(Y)$ as well as many other questions which may arise from the theory of complex cobordism for manifolds. However the complex cobordism ring Ω^U is a subring of $OB^U_*(pt)$. We adhere the notations of Section 4.

Lemma 5.3. *The orbifold with boundary $W(Q, \lambda)$ is an orbit space of a circle action on a quasitoric orbifold.*

Proof. Let $\{Q\backslash Q_1, \dots, Q_k\}$ be a simple $(n+1)$-polytope with exceptional facets and

$$\mathcal{F}(Q) = \{F_i : i = 1, \dots, m\} \cup \{Q_j : j = 1, \dots, k\}.$$

Let λ be an isotropy function on $\{Q\backslash Q_1, \dots, Q_k\}$. We define a function $\eta: \mathcal{F}(Q) \to \mathbb{Z}^{n+1}$ as follows,

$$\eta(F) = \begin{cases} (0, \dots, 0, 1) \in \mathbb{Z}^{n+1} & \text{if } F = Q_j \text{ and } j \in \{1, \dots, k\} \\ (\lambda_i, 0) \in \mathbb{Z}^n \times \{0\} \subset \mathbb{Z}^{n+1} & \text{if } F = F_i \text{ and } i \in \{1, \dots, m\}. \end{cases} \tag{5.1}$$

So the function η is a di-characteristic function on Q. Let $X(Q, \eta)$ be the quasitoric orbifold constructed from the characteristic model (Q, η). There is a natural $\mathbb{T}^{n+1}$-action on $X(Q, \eta)$, see Subsection 2.1 of [16]. Let

$$\pi_Q: X(Q, \eta) \to Q \tag{5.2}$$

be the orbit map of this action and $\mathbb{T}_Q$ be the circle subgroup of $\mathbb{T}^{n+1}$ determined by the submodule $\{0\} \times \ldots \times \{0\} \times \mathbb{Z}$ of $\mathbb{Z}^{n+1}$. From the definition of η it is clear that $W(Q, \lambda)$ is the orbit space of the circle $\mathbb{T}_Q$ action on $X(Q, \eta)$. $\square$

Remark 5.4. The quotient map $\phi_Q: X(Q, \eta) \to W(Q, \lambda)$ is not a fiber bundle map.

Theorem 5.5. *Let $\{Q\backslash Q_1, \dots, Q_k\}$ be a simple $(n+1)$-polytope with exceptional facets in $\mathbb{R}^{n+1}$ and λ be an isotropy function on $\{Q\backslash Q_1, \dots, Q_k\}$. Then there is a stable complex structure on $W(Q, \lambda)$. Moreover $[X(Q_1, \xi^1)] + \cdots + [X(Q_k, \xi^k)] = 0$ in $OB^U_{2n}(pt)$ where ξ^i is the restriction of the isotropy function λ on the facets of Q_j for $j = 1, \dots, k$.*

Proof. We construct a di-characteristic model (Q, η) from the pair (Q, λ), see Equation (5.1). Let $X(Q, \eta)$ be the quasitoric orbifold constructed from the characteristic model (Q, η), see Subsection 2.1 of [16]. Let $X_1, \dots, X_m$ be the characteristic suborbifolds of $X(Q, \eta)$ and the omniorientation of $X(Q, \eta)$ be the characteristic omniorientation. By Section 6 of [16], this omniorientation determines a stably complex structure on $X(Q, \eta)$ by means of the following isomorphism of orbifold real $2m$-bundles

$$T(X(Q, \eta)) \oplus \mathbb{R}^{2(m-n-1)} \cong \rho_1 \oplus \cdots \oplus \rho_m \tag{5.3}$$

where the orbifold complex line bundles ρ_i's can be interpreted in the following way. The orientation of the orbifold normal bundle μ_i over the characteristic suborbifold X_i defines a rational Thom class in the cohomology group $H^2(\mathbf{T}(\mu_i), \mathbb{Q})$, represented by a complex line bundle over the Thom complex $\mathbf{T}(\mu_i)$. We pull this back along the Pontryagin–Thom collapse $X(Q, \eta) \to \mathbf{T}(\mu_i)$, and denote the resulting orbibundle by ρ_i.

Cut off a neighborhood of each facets Q_j of Q by an affine hyperplane H_j in $\mathbb{R}^{n+1}$ for $j = 1, \dots, k$ such that $H_i \cap H_j \cap Q$ is empty for $i \neq j$, $1 \leq i, j \leq k$. Then the remaining subset of Q, denoted by Q_P, is an $(n+1)$-dimensional simple polytope which is naturally diffeomorphic as manifold with corners to Q. Suppose

$$Q_{H_j} = Q \cap H_j = H_j \cap Q_P$$

for $j = 1, \dots k$. Then Q_{H_j} is a facet of Q_P for each $j \in \{1, \dots, k\}$. Note that Q_{H_j} is diffeomorphic as manifold with corners to Q_j for $j = 1, \dots, k$. Clearly, $W(Q_P, \lambda) = (\mathbb{T}^n \times Q_P)/\sim_b$ is equivariantly diffeomorphic to $W(Q, \lambda)$. Let $\overline{W}$ be the pullback of the following diagram:

$$\begin{array}{ccc} \overline{W} & \longrightarrow & X(Q,\eta) \\ {\scriptstyle \pi_v}\downarrow & & {\scriptstyle \pi_Q}\downarrow \\ Q_P & \xrightarrow{\iota} & Q. \end{array} \tag{5.4}$$

Then $\overline{W} = W(Q_P, \lambda) \times \mathbb{T}_Q$ where $\mathbb{T}_Q$ is the circle subgroup of $\mathbb{T}^{n+1}$ determined by the vector $(0, \dots, 0, 1)$ in $\mathbb{Z}^{n+1}$ and π_Q is given in Equation (5.2). We have the following commutative diagrams of complex orbibundles.

$$\begin{array}{ccccccc} E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & T(X(Q,\eta)) \oplus \mathbb{R}^{2(m-n-1)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X(Q_i,\xi^i) & \longrightarrow & W(Q_P,\lambda) & \longrightarrow & \overline{W} & \longrightarrow & X(Q,\eta) \end{array} \tag{5.5}$$

where E_1, E_2 and E_3 are the pullback bundles. Since $\overline{W} = W(Q_P, \lambda) \times T_Q$, we have the following isomorphism of bundles:

$$E_2 \cong T(W(Q_P, \lambda)) \oplus \mathbb{R}^{2(m-n)-1}.$$

Hence for a choice of sign of isotropy vectors of $\{Q \backslash Q_1, \dots, Q_k\}$ the orbifold with boundary $W(Q, \lambda)$ has a stable complex structure.

Observe that the bundle E_3 is isomorphic to

$$T(X(Q_i, \xi^i)) \oplus \mathbb{R}^{2(m-n)} \cong \rho_{i_1} \oplus \cdots \oplus \rho_{i_l} \oplus \mathbb{R}^{2(m-l)} \tag{5.6}$$

where ρ_{i_j} is the complex line bundle corresponding to the i_j-th characteristic suborbifold of $X(Q_i, \xi^i)$. Hence $[X(Q_1, \xi^1)] + \cdots + [X(Q_k, \xi^k)] = 0$ in the orbifold complex cobordism group $OB^U_{2n}(pt)$. $\square$

Theorem 5.6. *Let X be a $2n$-dimensional omnioriented quasitoric orbifold over the simple polytope P. Then $[X] = [M_1] + \cdots + [M_k]$ in $OB^U_{2n}(pt)$, where $M_1, \dots, M_k$ are complex orbifold projective spaces.*

Proof. Let $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ be the facets and $\{v_1, \dots, v_k\}$ be the vertices of $P \subset \mathbb{R}^n$. Let $Q = P \times [0,1]$. So Q is an $(n+1)$-dimensional simple polytope in $\mathbb{R}^{n+1}$. Cut off a neighborhood of each vertex $v_j \times \{0\}$ of Q by an affine hyperplane H_j for $j = 1, \dots, k$ in $\mathbb{R}^{n+1}$ such that

$$H_i \cap H_j \cap Q \text{ is empty for } i \neq j, \text{ and } H_j \cap P \text{ is empty for } i, j \in \{1, \dots, k\}.$$

Then the remaining subset of Q, denoted by Q_P, is an $(n+1)$-dimensional simple polytope. Observe that $\triangle^n_i = Q \cap H_i = H_i \cap Q_P$ is a facet of Q_P. Also $\triangle^n_i$ is an n-dimensional simplex in $\mathbb{R}^{n+1}$ for $i = 1, \dots, k$. So $\{Q_P \backslash P \times \{1\}, \triangle^n_1, \dots, \triangle^n_k\}$ is an $(n+1)$-dimensional simple polytope with exceptional facets. Let

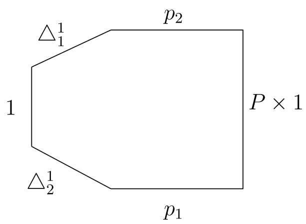

Fig. 2. An isotropy function on a 2-polytope with exceptional facets.

$$\overline{F}_0 = Q_P \cap (P \times \{0\}), \text{ and } \overline{F}_i = Q_P \cap (F_i \times [0,1]) \text{ for } i = 1, \ldots, m.$$

So $\mathcal{F}(Q_P) = \{\overline{F}_i : i = 0, \ldots, m\} \cup \{P \times \{1\}, \triangle_1^n, \ldots, \triangle_k^n\}$.

Let $\xi: \mathcal{F}(P) \to \mathbb{Z}^n$ be the di-characteristic function associated to the omniorientation of X. Let $E(P)$ be the set of all edges of P and $e \in E(P)$. Then $e = F_{i_1} \cap \cdots \cap F_{i_{n-1}}$ for a unique collection of facets $F_{i_1}, \ldots, F_{i_{n-1}}$ of P. Let $\mathbb{Z}(e)$ be the submodule of $\mathbb{Z}^n$ generated by $\{\xi(F_{i_1}), \ldots, \xi(F_{i_{n-1}})\}$. So $\mathbb{Z}(e)$ is a free $\mathbb{Z}$-module of rank $n-1$ for any $e \in E(P)$. From Remark 4.3, there exists $\lambda_0 \in \mathbb{Z}^n - \cup_{e \in E(P)} \mathbb{Z}(e)$. So λ_0 is nonzero and $\{\lambda_0, \xi(F_{i_1}), \ldots, \xi(F_{i_{n-1}})\}$ is a linearly independent set in $\mathbb{Z}^n$ for the edge $e = F_{i_1} \cap \cdots \cap F_{i_{n-1}}$. Define $\lambda: \{\overline{F}_i : i = 0, \ldots, m\} \to \mathbb{Z}^n$ by

$$\lambda(F) = \begin{cases} \xi(F_i) & \text{if } F = \overline{F}_i \\ \lambda_0 & \text{if } F = \overline{F}_0. \end{cases} \tag{5.7}$$

So the function λ is an isotropy function on $\{Q_P \backslash P \times \{1\}, \triangle_1^n, \ldots, \triangle_k^n\}$. Let

$$\xi^j : \mathcal{F}(\triangle_j^n) \to \mathbb{Z}^n$$

be the restriction of λ on the facets of $\triangle_j^n$ for $j = 1, \ldots, k$. Then $(\triangle_j^n, \xi^j)$ is a characteristic model for $j = 1, \ldots, k$. Let M_j be the complex orbifold projective space constructed from the characteristic model $(\triangle_j^n, \xi^j)$ for $j = 1, \ldots, k$. So by Lemma 4.4, the boundary of $(2n+1)$-dimensional oriented orbifold $W(Q_P, \lambda)$ is $X \sqcup M_1 \sqcup \cdots \sqcup M_k$. Hence by Theorem 5.5 we get the orbifold complex cobordism relation $[X] = [M_1] + \cdots + [M_k]$ in $OB_{2n}^U(pt)$. □

Example 5.7. Let X be a 2-dimensional quasitoric orbifold over P. Then P is a closed interval say $[0,1]$ and the corresponding di-characteristic function on P is given by $\xi(\{0\}) = p_1$, $\xi(\{1\}) = p_2$ (possibly up to sign) for some nonzero integers p_1 and p_2. So the Fig. 2 gives an isotropy function λ on $\{Q_P \backslash P \times 1, \triangle_1^1, \triangle_2^1\}$. Then $W(Q_P, \lambda)$ is a 3-dimensional stably complex orbifold with boundary $X \sqcup M_1 \sqcup M_2$ where M_i is the quasitoric orbifold corresponding to the facet $\triangle_i^1$ for $i = 1, 2$. By Example 3.4 and Proposition 3.6, M_i is equivariantly homeomorphic to the teardrop $\mathbb{WP}(1, p_i)$ for $i = 1, 2$. Therefore X is complex orbifold cobordant to two copies of teardrop. □

Observation 5.8. Let

$$A = \{[M] : M \text{ is a } 2n\text{-dimensional complex orbifold projective space}\}.$$

We show that A is not a linearly independent set in $OB_{2n}^U(pt)$. Let Q be an $(n+1)$-dimensional simple polytope in $\mathbb{R}^{n+1}$ with vertices $\{v_1, \ldots, v_k\}$ and facets $\{F_1, \ldots, F_m\}$. We delete a neighborhood of each vertex v_i by cutting with a hyperplane H_i in $\mathbb{R}^{n+1}$ for $i = 1, \ldots, k$ such that $H_i \cap H_j \cap Q$ is empty for $i \neq j$, $1 \leq i, j \leq k$. Let Q_V be the remaining subset of Q and

$$\triangle_i^n = Q \cap H_i = Q_V \cap H_i \quad \text{for} \quad i = 1, \ldots, k.$$

Since Q is an $(n+1)$-dimensional simple polytope, $\triangle_i^n$ is an n-simplex for $i = 1, \ldots, k$. Let $\overline{F}_i = F_i \cap Q_V$ for $i = 1, \ldots, m$. So $\{Q_V \backslash \triangle_1^n, \ldots, \triangle_k^n\}$ is an $(n+1)$-dimensional simple polytope with exceptional facets and

$$\mathcal{F}(Q_V) = \{\overline{F}_1, \ldots, \overline{F}_m\} \cup \{\triangle_1^n, \ldots, \triangle_k^n\}.$$

Since $\mathbb{Z}^n$ is not a union of finitely many proper submodules over $\mathbb{Z}$, we can define an isotropy function

$$\lambda: \{\overline{F}_1, \ldots, \overline{F}_m\} \to \mathbb{Z}^n$$

on $\{Q_V \backslash \triangle_1^n, \ldots, \triangle_k^n\}$. Since Q is an $(n+1)$-dimension simple polytope, each vertex v_i of Q is the intersection of unique collection of facets $\{F_{i_1}, \ldots, F_{i_{n+1}}\}$ for $i = 1, \ldots, k$. So $\mathcal{F}(\triangle_i^n) = \{F_{i_j} \cap \triangle_i^n : j = 1, \ldots, n+1\}$. We define a map $\xi^i: \mathcal{F}(\triangle_i^n) \to \mathbb{Z}^n$ by

$$\xi^i(F_{i_j} \cap \triangle_i^n) = \lambda(\overline{F}_{i_j}) \quad \text{for} \quad j = 1, \ldots, n+1. \tag{5.8}$$

Then ξ^i is a di-characteristic function on $\triangle_i^n$. Let $X(\triangle_i^n, \xi^i)$ be the complex orbifold projective space for the characteristic model $(\triangle_i^n, \xi^i)$ for $i = 1, \ldots, k$. So by Lemma 4.4 the space $W(Q_V, \lambda)$ is an oriented orbifold with boundary where the boundary $\partial W(Q_V, \lambda)$ is a disjoint union of $\{X(\triangle_i^n, \xi^i): i = 1, \ldots, k\}$. Thus by Theorem 5.5, we have $[X(\triangle_i^n, \xi^1)] + \cdots + [X(\triangle_i^n, \xi^k)] = 0$ in $OB_{2n}^U(pt)$. Hence the set of vectors in A is not linearly independent. This also follows from Theorem 5.6 if X is a complex orbifold projective space. In that case k is $n+1$. But if Q is any $(n+1)$-dimensional simple polytope, k can be as large as possible. So we get more relations among complex orbifold projective spaces. Now we may ask the following question.

Question 5.9. *Are there any other types of relation among complex orbifold projective spaces, and if so how do they arise?*

Now we give some computations in the complex cobordism ring Ω^U. Milnor and Novikov independently showed that the ring Ω^U is isomorphic to the polynomial ring $\mathbb{Z}[a_1, a_2, \ldots]$ where $\deg a_i = 2i$, see [14]. The complex projective spaces $\mathbb{CP}^n$ for $n \geq 0$ and Milnor hypersurfaces can be chosen as the standard set of multiplicative generators for Ω^U, see example 5.39 in [3]. In [4], Buchstaber and Ray introduced a new set of multiplicative generators for Ω^U which are quasitoric manifolds. In [5], they proved that any complex cobordism class contains a quasitoric manifold in dimension > 2 by showing that a disjoint union of products of this new generators is complex cobordant to a quasitoric manifold. See example 5.28 in [3] for the case of dimension 2.

Example 5.10. Let $\triangle^n$ be the n-simplex with $\mathcal{F}(\triangle^n) = \{F_0, \ldots, F_n\}$ and

$$\xi: \mathcal{F}(\triangle^n) \to \mathbb{Z}^n$$

be the characteristic function such that the set $\{\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_n\}$ is a basis of $\mathbb{Z}^n$ where $\widehat{\ }$ represents the omission of the corresponding entry. Let $\xi_i = \xi(F_i)$ for $i = 0, \ldots, n$. Suppose $\xi_i = (a_{i1}, \ldots, a_{in})$ for $i = 0, \ldots, n$. Let $X(\triangle^n, \xi)$ be the quasitoric manifold constructed from the characteristic model $(\triangle^n, \xi)$. Then $X(\triangle^n, \xi)$ is δ-equivariantly homeomorphic to $\mathbb{CP}^n$ for some automorphism δ of $\mathbb{T}^n$, see Proposition 5.63 in [3]. The map δ induces an automorphism δ_* of $\mathbb{Z}^n$. We may assume that $\delta_*((\xi_i)^t) = e_i^t$ where e_i is the i-th standard vector of $\mathbb{Z}^n$ and e_i^t is the transpose of e_i for $i = 1, \ldots, n$. By condition on the characteristic function ξ one can show that the set $\{\delta_*((\xi_0)^t), \ldots, \widehat{\delta_*((\xi_i)^t)}, \ldots, \delta_*((\xi_n)^t)\}$ is a basis of $\mathbb{Z}^n$ for $i = 0, \ldots, n$. Hence $\delta_*((\xi_0)^t) = (a_1, \ldots, a_n)^t$ where $a_i = \pm 1$.

By Theorem 5.18 in [3] the cohomology ring of $X(\triangle^n, \xi)$ with integer coefficients is $\mathbb{Z}[x_0, \dots, x_n]/I + J$ where $I = \langle x_0 \dots x_n \rangle$ and

$$J = \langle \{a_{1i}x_1 + \cdots + a_{ni}x_n + a_{0i}x_0 : i = 1, \dots, n\} \rangle$$

and x_i is the Poincare dual of the characteristic submanifold corresponding to the facet F_i for $i = 1, \dots, n$. So in the cohomology ring of $X(\triangle^n, \xi)$, we get a system of homogeneous equation $A\mathbf{x} = \mathbf{b}x_0$ where i-th column of A is $(\xi_i)^t$, $\mathbf{x} = (x_1, \dots, x_n)^t$ and $\mathbf{b} = -(\xi_0)^t$. Then

$$\mathbf{x} = \delta^* A\mathbf{x} = \delta^* \mathbf{b} x_0 = -(a_1, \dots, a_n)^t x_0.$$

Suppose $a_j = 1$ for $j \in \{i_1, \dots, i_l\} \subseteq \{1, \dots, n\}$ and $a_j = -1$ for $j \notin \{i_1, \dots, i_l\}$. So in the cohomology ring of $X(\triangle^n, \xi)$, $x_0^{n+1} = 0$, $x_{i_j} = -x_0$ for $i_j \in \{i_1, \dots, i_l\}$ and $x_j = x_0$ for $j \notin \{i_1, \dots, i_l\}$. Assume the omniorientation of $X(\triangle^n, \xi)$ is the characteristic omniorientation. Then by Theorem 5.34 in [3], the total Chern class of the corresponding stable complex bundle on $X(\triangle^n, \xi)$ is given by

$$\mathcal{C}_\xi = (1 + x_0) \cdots (1 + x_n) = (1 - x_0)^l (1 + x_0)^{n+1-l}.$$

Using the binomial theorem and Cauchy product formula one can compute the coefficient of x_0^i in this expression. Let $c_{\xi,i}$ be the coefficient of x_0^i and $K = (k_1, \dots, k_s)$ be a partition of n. Then K-th Chern number of $X(\triangle^n, \xi)$ is $C_{\xi,K} = c_{\xi,k_1} \cdots c_{\xi,k_s}$. $\square$

Now we discuss some computations in the complex cobordism ring Ω^U. Let M be a $2n$-dimensional omnioriented quasitoric manifold over a simple polytope P. Let $\xi : \mathcal{F}(P) \to \mathbb{Z}^n$ be the corresponding characteristic function on P. We introduce some combinatorial data in the following. Let $\{Q \backslash P, \triangle_1^n, \dots, \triangle_k^n\}$ be an $(n+1)$-dimensional simple polytope with exceptional facets, where $\triangle_1^n, \dots, \triangle_k^n$ are n-dimensional simplices. Let

$$\mathcal{F}(Q) = \{F_1, \dots, F_m\} \cup \{P, \triangle_1^n, \dots, \triangle_k^n\}$$

and

$$\lambda : \{F_1, \dots, F_m\} \to \mathbb{Z}^n$$

be an isotropy function on $\{Q \backslash P, \triangle_1^n, \dots, \triangle_k^n\}$ such that the following holds:

(1) $\lambda(F_i) = \xi(F_i \cap P)$ if $F_i \cap P$ is nonempty.
(2) If e is an edge of Q not contained in $\cup_{i=1}^k \triangle_i^n \cup P$ then $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ is a basis of $\mathbb{Z}^n$ where $e = F_{i_1} \cap \cdots \cap F_{i_n}$.

Note that $F_i \cap P$ is nonempty if and only if $F_i \cap P$ is a facet of P. So the restriction of λ on the facets of P is the map ξ. Thus we may assume λ is an extension of ξ.

Let $\xi^i : \mathcal{F}(\triangle_i^n) \to \mathbb{Z}^n$ be the map defined by

$$\xi^i(F) = \lambda(F_j) \text{ if } F = F_j \cap \triangle_i^n.$$

So ξ^i is a characteristic function on $\triangle_i^n$ for $i = 1, \dots, k$. Let $X(\triangle_i^n, \xi^i)$ be the quasitoric manifold constructed from the characteristic model $(\triangle_i^n, \xi^i)$ for $i = 1, \dots, k$. Recall that $X(\triangle_i^n, \xi^i)$ is δ-equivariantly diffeomorphic to $\mathbb{CP}^n$ for some $\delta \in \mathrm{Aut}(\mathbb{T}^n)$. Then by Corollary 4.5, the space $W(Q, \lambda)$ is a manifold with quasitoric boundary where the boundary is $M \sqcup X(\triangle_1^n, \xi^1) \sqcup \ldots \sqcup X(\triangle_k^n, \xi^k)$. Therefore by Theorem 5.6 we get

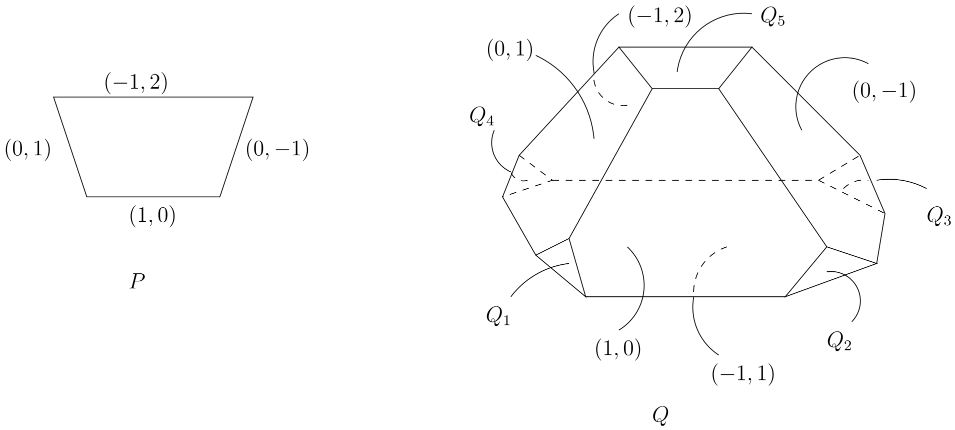

Fig. 3. A characteristic and isotropy function on a 2- and 3-polytope.

$$[M] = k[\mathbb{CP}^n]$$

in Ω^U, where the stable complex structure on each $\mathbb{CP}^n$ is determined by the corresponding characteristic function.

Let K be a partition of n and $C_{i,K}$ be the K-th Chern number of $X(\triangle_i^n, \xi^i)$ for $i = 1, \ldots, k$. Since two stably complex manifolds are cobordant if and only if their Chern numbers are identical (by Milnor [11] and Novikov [13]), we get the following formula for the K-th Chern number C_K of M,

$$C_K(M) = C_{1,K} + \cdots + C_{k,K}.$$

Recall that every quasitoric manifold has a stable complex structure which depends on the omniorientation on it. Not all quasitoric manifold admit an almost complex structure. For example, $\mathbb{CP}^2 \# \mathbb{CP}^2$ is a quasitoric manifold, but not an almost complex manifold.

Theorem 5.11. *In the above discussion, if M is an almost complex quasitoric manifold then the complex cobordism class of $\sqcup_{i=1}^k X(\triangle_i^n, \xi^i)$ contains the almost complex quasitoric manifold.*

Remark 5.12. In the Theorem 5.11, if M is a smooth projective space then we get a representation of $[M]$ in term of some generators of complex cobordism ring Ω^U. Therefore the above combinatorial process gives some sufficient conditions for the Hirzebruch problem which is mentioned in the introduction.

Example 5.13. The quasitoric manifold corresponding to the characteristic function on P in Fig. 3 is the Hirzebruch surface M_2^4, see Example 1.19 in [7]. The function on $\{Q \backslash Q_1, \ldots, Q_5\}$ is an isotropy function which extends the characteristic function on $P \cong Q_5$. Observe that the restriction of the isotropy function on the facets of $\triangle_i^2$ is a characteristic function $\xi^i : \mathcal{F}(\triangle_i^2) \to \mathbb{Z}^2$ for $i = 1, \ldots, 4$. For each $i \in \{1, \ldots, 4\}$, the corresponding quasitoric manifold $X(\triangle_i^2, \xi^i)$ is δ-equivariantly homeomorphic to $\mathbb{CP}^2$. Then $[M_2^2] = 4[\mathbb{CP}^2]$ where the stable complex structure on $\mathbb{CP}^2$ is determined by the corresponding characteristic function. Hence the complex cobordism class of $4[\mathbb{CP}^2]$ contains a connected algebraic variety. $\square$

Acknowledgements

The author would like to thank Professor Mainak Poddar, Marzieh Bayeh and Professor Goutam Mukherjee for helpful suggestions and stimulating discussions. He also thanks anonymous referee for many helpful suggestions. He thanks University of Regina and Pacific Institute for the Mathematical Sciences for support.

References

[1] A. Adem, J. Leida, Y. Ruan, Orbifolds and Stringy Topology, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007.
[2] M.F. Atiyah, Bordism and cobordism, Proc. Camb. Philos. Soc. 57 (1961) 200–208.
[3] V.M. Buchstaber, T.E. Panov, Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002.
[4] V.M. Bukhshtaber, N. Ray, Toric manifolds and complex cobordisms, Usp. Mat. Nauk 53 (2) (1998) 139–140.
[5] V.M. Buchstaber, N. Ray, Tangential structures on toric manifolds, and connected sums of polytopes, Int. Math. Res. Not. (4) (2001) 193–219.
[6] D.A. Cox, S. Katz, Mirror Symmetry and Algebraic Geometry, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999.
[7] M.W. Davis, T. Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (2) (1991) 417–451.
[8] K.S. Druschel, Oriented orbifold cobordism, Pac. J. Math. 164 (2) (1994) 299–319.
[9] K.S. Druschel, The cobordism of oriented three dimensional orbifolds, Pac. J. Math. 193 (1) (2000) 45–55.
[10] A.M. Kasprzyk, Bounds on fake weighted projective space, Kodai Math. J. 32 (2) (2009) 197–208.
[11] J. Milnor, On the cobordism ring Ω^* and a complex analogue. I, Am. J. Math. 82 (1960) 505–521.
[12] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids, Cambridge Studies in Advanced Mathematics, vol. 91, Cambridge University Press, Cambridge, 2003.
[13] S.P. Novikov, Some problems in the topology of manifolds connected with the theory of Thom spaces, Sov. Math. Dokl. 1 (1960) 717–720.
[14] S.P. Novikov, Topology i, Encyclopaedia Math. Sci., vol. 12, Springer, Berlin, 1996, pp. 1–319.
[15] L.S. Pontryagin, Smooth Manifolds and Their Applications in Homotopy Theory, American Mathematical Society Translations, Ser. 2, vol. 11, American Mathematical Society, Providence, R.I., 1959, pp. 1–114.
[16] M. Poddar, S. Sarkar, On quasitoric orbifolds, Osaka J. Math. 47 (4) (2010) 1055–1076.
[17] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, Adv. Math. 7 (1971) 29–56.
[18] S. Sarkar, $\mathbb{T}^2$-cobordism of quasitoric 4-manifolds, Algebr. Geom. Topol. 12 (4) (2012) 2003–2025.
[19] I. Satake, The Gauss–Bonnet theorem for V-manifolds, J. Math. Soc. Jpn. 9 (1957) 464–492.
[20] R. Thom, Quelques propriétés globales des variétés différentiables, Comment. Math. Helv. 28 (1954) 17–86.
[21] A. Wilfong, Smooth projective toric variety representatives in complex cobordism, ArXiv e-prints, December 2013.