BULLETIN DES SCIENCES MATHÉMATIQUES

Bull. Sci. math. 134 (2010) 54–63

www.elsevier.com/locate/bulsci

Chen–Ruan cohomology of some moduli spaces of parabolic vector bundles

Indranil Biswas *, Arijit Dey

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Received 19 May 2009

Available online 26 June 2009

Abstract

Let (X, D) be an ℓ-pointed compact Riemann surface of genus at least two. For each point $x \in D$, fix parabolic weights $(\alpha_1^{(x)}, \alpha_2^{(x)})$ such that $\sum_{x\in D}(\alpha_1^{(x)} - \alpha_2^{(x)}) < 1/2$. Fix a holomorphic line bundle ξ over X of degree one. Let PM_ξ denote the moduli space of stable parabolic vector bundles, of rank two and determinant ξ, with parabolic structure over D and parabolic weights $\{(\alpha_1^{(x)}, \alpha_2^{(x)})\}_{x\in D}$. The group of order two line bundles over X acts on PM_ξ by the rule $E_* \otimes L \longmapsto E_* \otimes L$. We compute the Chen–Ruan cohomology ring of the corresponding orbifold.

MSC: 14D20; 14F05

Keywords: Chen–Ruan cohomology; Parabolic bundle; Moduli space

1. Introduction

Chen and Ruan introduced a new cohomology theory of smooth orbifolds [5], which has turned out to be very useful. Moduli spaces associated to a Riemann surface provide a class of orbifolds. In [3], the Chen and Ruan cohomology of an orbifold associated to the smooth moduli space of rank two vector bundles was computed. Here we consider the more general case of parabolic vector bundles.

* Corresponding author.

E-mail addresses: indranil@math.tifr.res.in (I. Biswas), arijit@math.tifr.res.in (A. Dey).

0007-4497/$ – see front matter

doi:10.1016/j.bulsci.2009.06.001

Let X be a compact Riemann surface of genus g, with $g \geqslant 2$, and let

$$D = \{x_1, \dots, x_\ell\} \subset X \tag{1.1}$$

be a nonempty finite collection of distinct points. For each point $x_i \in D$, fix real numbers

$$0 \leqslant \alpha_2^{(i)} < \alpha_1^{(i)} < 1.$$

We will assume that

$$\sum_{i=1}^{\ell} \bigl(\alpha_1^{(i)} - \alpha_2^{(i)}\bigr) < \frac{1}{2}. \tag{1.2}$$

Fix a holomorphic line bundle $\xi \longrightarrow X$ of degree one.

A parabolic vector bundle of rank two over X with parabolic structure over D is a holomorphic vector bundle $V \longrightarrow X$ of rank two together with a line $L_i \subset V_{x_i}$ for each $x_i \in D$ and a pair of real numbers $0 \leqslant b_1^{(i)} < b_2^{(i)} < 1$. We will consider parabolic vector bundles E_* of rank two on X with parabolic structure over D such that the parabolic weights for each $x_i \in D$ are $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ (as in (1.2)). The holomorphic vector bundle underlying E_* will be denoted by E. (See [6,2] for more on parabolic vector bundles.)

Let PM_ξ denote the moduli space of semistable parabolic vector bundles E_* over X of rank two with $\bigwedge^2 E = \xi$. The condition in (1.2) implies that any $E_* \in PM_\xi$ is parabolic stable; this condition also implies that the underlying vector bundle E is stable. Therefore, PM_ξ is a smooth complex projective variety.

Let $\Gamma \subset \mathrm{Pic}^0(X)$ be the group of order two line bundles on X. This group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}$. We will describe an action of Γ on PM_ξ.

For any $E_* \in PM_\xi$, and any $L \in \Gamma$, consider the vector bundle $E \otimes L$. We have

$$\bigwedge^2 (E \otimes L) = \bigwedge^2 E \otimes L^{\otimes 2} = \xi. \tag{1.3}$$

For each $x_i \in D$, let

$$\zeta_i' := \zeta_i \otimes L_{x_i} \subset (E \otimes L)_{x_i}$$

be the line, where $\zeta_i \subset E_{x_i}$ is the line defining the parabolic vector bundle E_*. The resulting parabolic vector bundle with $E \otimes L$ as the underlying vector bundle will be denoted by $E_* \otimes L$. We note that $E_* \otimes L$ is the parabolic tensor product of E_* with the line bundle L equipped with the trivial parabolic structure (see [2] for the parabolic tensor product). Let

$$\widetilde{\phi}_L : PM_\xi \longrightarrow PM_\xi \tag{1.4}$$

be the automorphism defined by $E_* \longmapsto E_* \otimes L$. The map

$$\widetilde{\phi} : \Gamma \longrightarrow \mathrm{Aut}(PM_\xi) \tag{1.5}$$

defined by $L \longmapsto \widetilde{\phi}_L$ is a homomorphism of groups. The group Γ acts on PM_ξ through the homomorphism $\widetilde{\phi}$.

Let PM_ξ/Γ be the corresponding smooth orbifold. Our aim here is to compute the Chen–Ruan cohomology of this orbifold PM_ξ/Γ.

2. Fixed point sets

We continue with the notation of the introduction.

Take any $L \in \Gamma$. Let

$$PS(L) := (PM_\xi)^{\widetilde{\phi}_L} \subset PM_\xi \tag{2.1}$$

be the fixed point set of the automorphism $\widetilde{\phi}_L$ in (1.4). This $PS(L)$ is a smooth compact complex manifold, but it is not necessarily connected. Our aim in this section is to describe $PS(L)$.

Let M_ξ denote the moduli space of stable vector bundles $E \longrightarrow X$ of rank two with $\bigwedge^2 E = \xi$. We have a projection

$$\gamma : PM_\xi \longrightarrow M_\xi \tag{2.2}$$

that sends any parabolic vector bundle E_* to the underlying vector bundle E; as noted earlier, the condition in (1.2) ensures that $E \in M_\xi$ (see (1.3)). Given any $E \in M_\xi$, the condition in (1.2) also ensures that for any choice of lines in E_{x_i}, $i \in [1, \ell]$, the corresponding parabolic vector bundle is stable. For any $L \in \Gamma$, let

$$\phi_L : M_\xi \longrightarrow M_\xi \tag{2.3}$$

be the automorphism defined by $V \longmapsto V \otimes L$ (see (1.3)). Let

$$\phi : \mathrm{Aut}(M_\xi) \longrightarrow \mathrm{Aut}(M_\xi)$$

be the homomorphism defined by $L \longrightarrow \phi_L$. The group Γ acts on M_ξ through the homomorphism ϕ. It is easy to see that the projection γ in (2.2) is equivariant for the actions of Γ on PM_ξ and M_ξ.

Let

$$S(L) := (M_\xi)^{\phi_L} \subset M_\xi \tag{2.4}$$

be the fixed point set, of the automorphism ϕ_L in (2.3), which is a smooth compact complex manifold. Since γ in (2.2) is equivariant for the actions of Γ, we have

$$\gamma\big(PS(L)\big) \subset S(L), \tag{2.5}$$

where $PS(L)$ is defined in (2.1).

The complex manifold $S(L)$ is described in [3]. If the line bundle L is nontrivial, then $S(L)$ is the quotient by $\mathbb{Z}/2\mathbb{Z}$ of a complex torus of dimension $g-1$ [3, Proposition 2.2]. In particular, $S(L)$ is connected. Let

$$f_L : PS(L) \longrightarrow S(L) \tag{2.6}$$

be the restriction of the map γ constructed in (2.2).

Lemma 2.1. *Let $L \in \Gamma$ be a nontrivial line bundle on X.*

(1) *The map f_L in* (2.6) *is surjective.*
(2) *The map f_L is an isomorphism on each connected component of $PS(L)$. The number of connected components of $PS(L)$ is 2^ℓ.*

Proof. The line bundle L defines a nontrivial unramified double cover

$$\gamma_L : Y_L \longrightarrow X \tag{2.7}$$

(see [3, (2.2)]). Let

$$\mathrm{Prym}_\xi \subset \mathrm{Pic}^1(Y_L) \tag{2.8}$$

be the sub-torus consisting of all line bundles $\eta \longrightarrow Y_L$ such that $\bigwedge^2 \gamma_{L*}\eta = \xi$. The Galois group $\mathrm{Gal}(\gamma_L) = \mathbb{Z}/2\mathbb{Z}$ acts on Prym_ξ by sending any η to $\sigma^*\eta$, where

$$\sigma : Y_L \longrightarrow Y_L \tag{2.9}$$

is the nontrivial deck transformation of the covering in (2.7). We have

$$S(L) = \mathrm{Prym}_\xi \,/\, \mathrm{Gal}(\gamma_L) \tag{2.10}$$

(see [3, Proposition 2.2]).

Take any $\eta \in \mathrm{Prym}_\xi$. For each point $x_i \in D$, fix a point

$$y_i \in \gamma_L^{-1}(x_i)$$

in the inverse image. The fiber of $\gamma_{L*}\eta$ over x_i has the following decomposition:

$$(\gamma_{L*}\eta)_{x_i} = \bigoplus_{z \in \gamma_L^{-1}(x_i)} \eta_z = \eta_{y_i} \oplus \eta_{\sigma(y_i)}, \tag{2.11}$$

where σ is the automorphism in (2.9). Therefore, the line

$$\eta_{y_i} \subset (\gamma_{L*}\eta)_{x_i}$$

defines a parabolic structure on the vector bundle $\gamma_{L*}\eta \longrightarrow X$ over the point x_i.

Let

$$E_* \longrightarrow X \tag{2.12}$$

be the parabolic vector bundle obtained this way from η and $\{y_i\}_{i=1}^{\ell}$.

The vector bundle $\gamma_{L*}\eta$ is canonically isomorphic to $(\gamma_{L*}\eta) \otimes L$ [3]. We recall from [3] the construction of the isomorphism of $\gamma_{L*}\eta$ with $(\gamma_{L*}\eta) \otimes L$. The Riemann surface Y_L lies in the complement $L \setminus \{0_X\}$, where $0_X \subset L$ is the image of the zero section of L. Hence $\gamma_L^* L$ is canonically trivialized. Let

$$s : Y_L \longrightarrow \gamma_L^* L$$

be the tautological section defining the trivialization of $\gamma_L^* L$. The isomorphism of $\gamma_{L*}\eta$ with $(\gamma_{L*}\eta) \otimes L$ is given by the direct image of the homomorphism

$$\eta \xrightarrow{\otimes s} \eta \otimes \gamma_L^* L \tag{2.13}$$

defined by tensoring with the section s; note that by the projection formula, $\gamma_{L*}(\eta \otimes \gamma_L^* L) = (\gamma_{L*}\eta) \otimes L$. The resulting isomorphism

$$\rho : \gamma_{L*}\eta \longrightarrow (\gamma_{L*}\eta) \otimes L \tag{2.14}$$

clearly preserves the decompositions of $(\gamma_{L*}\eta)_{x_i}$ and $((\gamma_{L*}\eta) \otimes L)_{x_i}$ obtained from (2.11). From this it follows immediately that the parabolic vector bundle E_* constructed in (2.12) lies in $PS(L)$. This proves the first part of the lemma.

Note that $f_L(E_*) = E$, where f_L is the map in (2.6). Therefore, f_L is surjective. From the isomorphism in (2.13), and the fact that ρ in (2.14) is constructed from (2.13), we conclude the following:

$$\rho(S) = S \otimes L_{x_i}$$

for some line $S \subset (\gamma_{L*}\eta)_{x_i}$ if and only if S is one of the two lines in (2.11). Therefore, $PS(L)$ is a disjoint union of copies of $S(L)$, and the copies are parametrized by the finite set

$$\prod_{i=1}^{\ell} \gamma_L^{-1}(x_i)$$

which has cardinality 2^ℓ. This completes the proof of the lemma. □

Lemma 2.1 has the following corollary.

Corollary 2.2. *Let L be as in Lemma* 2.1. *The cohomology algebra* $H^*(PS(L), \mathbb{Q})$ *is isomorphic to the direct sum* $H^*(S(L), \mathbb{Q})^{\oplus 2^\ell}$.

3. Action on tangent space at a fixed point

The holomorphic tangent bundle of PM_ξ is denoted as $T(PM_\xi)$. Let $L \longrightarrow X$ be any nontrivial line bundle in Γ. Take any parabolic vector bundle $E_* \in PS(L)$ (defined in (2.1)). Let

$$d\widetilde{\phi}_L(E_*) : T(PM_\xi)_{E_*} \longrightarrow T(PM_\xi)_{E_*}$$

be the differential of the map $\widetilde{\phi}_L$ in (1.4), where $T(PM_\xi)_{E_*}$ is the tangent space to PM_ξ at the point E_*.

Lemma 3.1. *Let* $L \in \Gamma$ *be a nontrivial line bundle on* X. *The eigenvalues of the differential* $d\widetilde{\phi}_L(E_*)$ *are* ± 1, *and the multiplicity of the eigenvalue* -1 *is* $2(g-1) + \ell$.

Proof. Since $\widetilde{\phi}_L \circ \widetilde{\phi}_L = \widetilde{\phi}_{L\otimes L} = \mathrm{Id}_{PM_\xi}$, only ± 1 can be the eigenvalues of $d\widetilde{\phi}_L(E_*)$. As we noted earlier, the condition in (1.2) ensures that given any stable vector bundle $E \in M_\xi$, any choice of lines in the fibers of E over $\{x_i\}_{i=1}^{\ell}$ defines a stable parabolic vector bundle lying in PM_ξ. Therefore, all elements of PM_ξ are of the form $(E; y_1, \ldots, y_\ell)$, where $E \in M_\xi$, and $y_i \in \mathbb{P}(E_{x_i})$. Hence the surjective map γ in (2.2) has the property that each fiber of it is isomorphic to the Cartesian product $(\mathbb{CP}^1)^\ell$.

For $i \in [1, \ell]$, let

$$\zeta_i \longrightarrow PM_\xi$$

be the line bundle whose fiber over any point

$$(E; y_1, \ldots, y_\ell) \in PM_\xi$$

is the line $\mathrm{Hom}_{\mathbb{C}}(y_i, E_{x_i}/y_i)$. The tangent bundle $T(PM_\xi)$ fits in the following short exact sequence of vector bundles

$$0 \longrightarrow \bigoplus_{i=1}^{\ell} \zeta_i \longrightarrow T(PM_\xi) \xrightarrow{d\gamma} \gamma^* TM_\xi \longrightarrow 0, \tag{3.1}$$

where $d\gamma$ is the differential of γ, and TM_ξ is the holomorphic tangent bundle of M_ξ.

The action of $d\widetilde{\phi}_L$ on $T(PM_\xi)$ preserves the exact sequence in (3.1); the action of $d\widetilde{\phi}_L$ on TM_ξ coincides with the action of the differential $d\phi_L$, where ϕ_L is the automorphism in (2.3).

For any $E \in M_\xi$, the multiplicity of the eigenvalue -1 of $d\phi_L(E)$ is $2(g-1)$ (see [3, Lemma 3.1]). The automorphism $d\widetilde{\phi}_L$ acts on each line bundle ζ_i as multiplication by -1. Therefore, the multiplicity of the eigenvalue -1 of $d\widetilde{\phi}_L(E_*)$ is $2(g-1)+\ell$. This completes the proof of the lemma. □

For any nontrivial line bundle $L \in \Gamma \setminus \{O_X\}$, and for any $E_* \in PS(L)$ (see (2.1)), the *degree shift* at E_* for L is defined to be

$$\pi(L, E_*) := \sum_j m_j b_j, \tag{3.2}$$

where $\exp(2\pi\sqrt{-1}b_j)$, $0 \leqslant b_j < 1$, are the eigenvalues of $d\widetilde{\phi}_L(E_*)$ with multiplicity m_j.

Lemma 3.1 has the following corollary.

Corollary 3.2. *Let L be as in Lemma* 3.1. *The degree shift $\pi(L) := \pi(L, E_*)$ is $g + \frac{\ell-2}{2}$.*

4. Cohomology of PM_ξ/Γ

Let PM denote the moduli space of stable parabolic vector bundles of rank two and degree one with parabolic structure on D with parabolic weights $\{\alpha_1^{(i)}, \alpha_2^{(i)}\}$ at each $x_i \in D$. We have a natural inclusion

$$PM_\xi \subset PM.$$

Note that PM_ξ is the fiber over ξ of the projection $PM \longrightarrow \mathrm{Pic}^1(X)$ defined by $E_* \longrightarrow \bigwedge^2 E$, where E is the vector bundle underlying the parabolic bundle E_*.

Since $\mathrm{rank}(E)$ and $\mathrm{degree}(E)$ are coprime for any $E_* \in PM_\xi$, there exists a universal parabolic vector bundle $\mathcal{U}$ on $X \times PM_\xi$ [4, Proposition 3.2]. Recall that a universal parabolic vector bundle $\mathcal{U}$ is a vector bundle U on $X \times PM_\xi$ together with a line sub-bundle

$$U|_{\{x_i\}\times PM_\xi} =: U^1_{x_i} \supset U^2_{x_i} \supset 0 \tag{4.1}$$

on $\{x_i\} \times PM_\xi$ for each $x_i \in D$. It is universal in the sense that for any point $m \in PM_\xi$, the parabolic vector bundle represented by m is the one defined by the vector bundle $U|_{X\times\{m\}}$ equipped the parabolic structure over x_i given the filtration in (4.1) restricted to $x_i \times m$. Any two universal parabolic vector bundles over $X \times PM_\xi$ differ by tensoring with a line bundle pulled back from PM_ξ. Let

$$a_2\big(\mathbb{P}(U)\big) = c_2\big(\mathcal{E}nd(U)\big) = 4c_2(U) - c_1(U)^2 \in H^4(X \times PM_\xi, \mathbb{Q}) \tag{4.2}$$

be the characteristic class of the projective bundle $\mathbb{P}(U)$. Since any two universal parabolic vector bundles over $X \times PM_\xi$ differ by tensoring with a line bundle, it follows that $a_2(\mathbb{P}(U))$ is independent of the choice of the universal parabolic vector bundle.

Let A and B be any two manifolds. Then any element $\alpha \in H^*(A \times B, \mathbb{Q})$ induces a linear map

$$\sigma(\alpha) : H_*(A, \mathbb{Q}) \longrightarrow H^*(B, \mathbb{Q}) \tag{4.3}$$

which is known as the *slant product*. The Poincaré polynomial of A is defined to be

$$P_t(A) := \sum_{i=0}^{\dim A} \dim\bigl(H^i(A, \mathbb{Q})\bigr) t^i.$$

For each $0 \leqslant r \leqslant 2$, the cohomology class $a_2(\mathbb{P}(U))$ in (4.2) induces a slant product

$$\sigma_r\bigl(a_2\bigl(\mathbb{P}(U)\bigr)\bigr) : H_r(X, \mathbb{Q}) \longrightarrow H^{4-r}(PM_\xi, \mathbb{Q}). \tag{4.4}$$

The cohomology algebra $H^*(PM_\xi, \mathbb{Q})$ is generated by

- the Chern classes $c_1(\mathcal{H}om(\mathcal{U}^2_{x_i}, U^1_{x_i}/U^2_{x_i}))$ (see (4.1)) together with
- the image

  $$\bigoplus_{r=0}^{2} \sigma_r\bigl(a_2\bigl(\mathbb{P}(U)\bigr)\bigr)\bigl(H_r(X, \mathbb{Q})\bigr),$$

  where $\sigma_r(a_2(\mathbb{P}(U)))$ is constructed in (4.4).

(See [1, Theorem 1.5].)

Note that the above generators are independent of the choice of universal parabolic vector bundle because any two universal bundles differ by tensoring with a line bundle. Consider the action of Γ on PM_ξ given by $\widetilde{\phi}$ constructed in (1.5). Let

$$\chi : PM_\xi \longrightarrow PM_\xi/\Gamma$$

be the quotient map. Let

$$\chi^* : H^*(PM_\xi/\Gamma, \mathbb{Q}) \longrightarrow H^*(PM_\xi, \mathbb{Q}) \tag{4.5}$$

be the corresponding homomorphism of cohomologies.

Proposition 4.1. *The homomorphism* χ^* *in* (4.5) *is an isomorphism.*

Proof. Let $J_X = \mathrm{Pic}^0(X)$ be the Jacobian of X. The group Γ acts on $PM_\xi \times J_X$ by the rule

$$(E_*, \alpha) \cdot \gamma = \bigl(E_* \otimes \gamma, \gamma^{-1} \otimes \alpha\bigr).$$

Consider the moduli space PM defined at the beginning of the section. We have a natural surjective map

$$\pi : PM_\xi \times J_X \longrightarrow PM \tag{4.6}$$

defined by $(E_*, \alpha) \longmapsto E_* \otimes \alpha$. This map π is a principal Γ-bundle (see the proof of Theorem 1.5 in [1, pp. 12–13]). The Poincaré polynomial of J_X is

$$P_t(J_X) = (1+t)^{2g}. \tag{4.7}$$

We have

$$P_t(PM_\xi) \cdot (1+t)^{2g} = P_t(PM) \tag{4.8}$$

(see [7, Remark 3.11]). From (4.7) and (4.8), the following equality of Poincaré polynomials is deduced:

$$P_t(PM_\xi) \cdot P_t(J_X) = P_t(PM). \tag{4.9}$$

Since π in (4.6) is a principal Γ-bundle, from (4.9) it follows that the action of Γ on $H^*(PM_\xi \times J_X, \mathbb{Q})$ is trivial. Hence the action of Γ on $H^*(PM_\xi, \mathbb{Q})$ is trivial. This completes the proof of the proposition. $\square$

5. Chen–Ruan cohomology ring

The Chen–Ruan cohomology of a smooth orbifold was introduced in [5]. Consider the smooth orbifold PM_ξ/Γ (the orbifold is smooth means that PM_ξ is smooth). Since the group Γ is abelian, for any $L \in \Gamma$, the fixed point set $PS(L)$ (defined in (2.1)) is preserved by the action of Γ on PM_ξ.

The Chen–Ruan cohomology group of PM_ξ/Γ is

$$H^j_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q}) := \bigoplus_{L\in\Gamma} H^{j-2\pi(L)}\big(PS(L)/\Gamma, \mathbb{Q}\big),$$

where $j \geqslant 0$, and the degree shift $\pi(L)$ is computed in Corollary 3.2; for the trivial line bundle $\mathcal{O}_X$, we have $\pi(\mathcal{O}_X) = 0$. From Corollary 2.2,

$$H^*_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q}) = H^*(PM_\xi/\Gamma, \mathbb{Q}) \oplus \left(\bigoplus_{L\in\Gamma\setminus\mathcal{O}_X} \big(H^{*-2\pi(L)}\big(S(L)/\Gamma, \mathbb{Q}\big)\big)^{\oplus 2^\ell}\right). \tag{5.1}$$

The group operation "+" of $H^*_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q})$ is uniquely determined by the condition that (5.1) is an isomorphism of groups. The Chen–Ruan product "$\cup$" is a new multiplicative structure on $H^*_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q})$ such that $(H^*_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q}), +, \cup)$ becomes a ring; it is called the *Chen–Ruan cohomology ring* of the orbifold PM_ξ/Γ.

The fixed point locus $PS(L)$ is a smooth compact sub-manifold of PM_ξ of real dimension $2(g-1)$, and it has 2^ℓ connected components and each is diffeomorphic to $(S(L)) = \mathrm{Prym}_\xi/\mathrm{Gal}(\gamma_L)$ (see (2.10) and Lemma 2.1). Let ω be a differential form on $PS(L)/\Gamma$. Let $\widetilde{\omega}$ be the Γ-invariant differential form on $PS(L)$ obtained by pulling back ω. So

$$\widetilde{\omega} = (\widetilde{\omega}_1, \ldots, \widetilde{\omega}_{2^\ell}),$$

where $\widetilde{\omega}_j$ is a differential form on the j-th copy of $S(L)$ in $PS(L)$ (recall that $PS(L)$ is 2^ℓ copies of $S(L)$). For any $2(g-1)$-form ω on $PS(L)/\Gamma$, the orbifold integration of ω is defined as follows:

$$\int^{\mathrm{orb}}_{PS(L)/\Gamma} \omega := \frac{1}{|\Gamma|} \int_{PS(L)} \widetilde{\omega} = \frac{1}{|\Gamma|} \sum_{i=j}^{2^\ell} \int_{S(L)} \widetilde{\omega}_j, \tag{5.2}$$

where $|\Gamma| = 2^{2g}$ is the order of the finite group Γ.

The real dimension of PM_ξ/Γ is

$$2d = 2\big(3(g-1) + \ell\big). \tag{5.3}$$

Definition 5.1. (Cf. [5].) For any integer $0 \leqslant n \leqslant 2d$, the Chen–Ruan Poincaré pairing

$$\langle\, ,\rangle_{\mathrm{CR}} : H^n_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q}) \times H^{2d-n}_{\mathrm{CR}}(PM_\xi/\Gamma, \mathbb{Q}) \longrightarrow \mathbb{C}$$

is defined by taking the direct sum of

$$\langle\, ,\rangle^{(L_1,L_2)} : H^{n-2\pi(L_1)}\big(PS(L_1)/\Gamma, \mathbb{Q}\big) \times H^{2d-n-2\pi(L_2)}\big(PS(L_2)/\Gamma, \mathbb{Q}\big) \longrightarrow \mathbb{C}$$

which are constructed as follows:

(1) If $L_1 = L_2 = L$, then

$$\langle\alpha, \beta\rangle^{(L,L)} = \sum_{i=1}^{2^\ell} \int^{\mathrm{orb}}_{S(L)/\Gamma} \alpha_i \wedge \beta_i .$$

(2) If $L_1 \neq L_2$, then

$$\langle\alpha, \beta\rangle^{(L_1,L_2)} = 0,$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_{2^\ell}) \in \big(H^{n-2\pi(L_1)}\big(S(L_1)/\Gamma, \mathbb{Q}\big)\big)^{\oplus 2^\ell} \quad \text{and}$$
$$\beta = (\beta_1, \ldots, \beta_{2^\ell}) \in \big(H^{2d-n-2\pi(L_2)}\big(S(L_2)/\Gamma, \mathbb{Q}\big)\big)^{\oplus 2^\ell} .$$

The Chen–Ruan product

$$\alpha \cup \beta \in H^{p+q}_{\mathrm{CR}}\big(PS(L_1 \otimes L_2)/\Gamma, \mathbb{Q}\big) \tag{5.4}$$

is defined by the relation

$$\langle\alpha \cup \beta, \gamma\rangle_{\mathrm{CR}} = \int^{\mathrm{orb}}_{\widetilde{\mathcal{S}}/\Gamma} \widetilde{e}_1^*(\alpha) \wedge \widetilde{e}_2^*(\beta) \wedge \widetilde{e}_3^*(\gamma) \wedge c_{\mathrm{top}}(\widetilde{\mathcal{F}}_{L_1,L_2}) \tag{5.5}$$

for all $\gamma = (\gamma_1, \ldots, \gamma_{2^\ell}) \in H^{2d-(p+q)}_{\mathrm{CR}}(PS(L_3), \mathbb{Q})$, where $L_3 = L_1 \otimes L_2$,

$$\widetilde{\mathcal{S}} := \bigcap_{i=1}^{3} PS(L_i) \tag{5.6}$$

and $\widetilde{e}_i : \widetilde{\mathcal{S}}/\Gamma \longrightarrow PS(L_i)/\Gamma$, $i = 1, 2, 3$, are the canonical inclusions. Here $c_{\mathrm{top}}(\widetilde{\mathcal{F}}_{L_1,L_2})$ is the top Chern class of the orbifold obstruction bundle $\widetilde{\mathcal{F}}_{L_1,L_2}$ on $\widetilde{\mathcal{S}}/\Gamma$ (see [3, Section 4, p. 13]); the rank of $\widetilde{\mathcal{F}}_{L_1,L_2}$ is the following:

$$\mathrm{rank}(\widetilde{\mathcal{F}}_{L_1,L_2}) = \dim_{\mathbb{R}}(\widetilde{\mathcal{S}}) - \dim_{\mathbb{R}}(PM_\xi) + \sum_{j=1}^{3} \pi(L_j).$$

We extend product structure "∪" to $H^*_{\mathrm{CR}}(PM_\xi, \mathbb{Q})$ by $\mathbb{R}$-linearity.

Let

$$\mathcal{S} := \bigcap_{i=1}^{3} S(L_i).$$

The intersection $\widetilde{\mathcal{S}}$ in (5.6) is 2^ℓ copies of $\mathcal{S}$. The intersection $\mathcal{S}$ is described in [3].

Note that γ (2.2) is a Γ-equivariant morphism which sends $\widetilde{\mathcal{S}}$ to $\mathcal{S}$ (see (2.5)). Hence we have

$$\widetilde{\mathcal{F}}_{L_1,L_2}|_{\mathcal{S}/\Gamma} \cong \mathcal{F}_{L_1,L_2}, \tag{5.7}$$

where $\mathcal{F}_{L_1,L_2}$ is the orbifold obstruction bundle on $\mathcal{S}/\Gamma$.

From (5.2) and (5.7),

$$\langle \alpha \cup \beta, \gamma \rangle_{\mathrm{CR}} = \sum_{i=1}^{2^\ell} \langle \alpha_i \cup \beta_i, \gamma_i \rangle_{\mathrm{CR}}. \tag{5.8}$$

The Chen–Ruan product $\alpha_i \cup \beta_i$ for the orbifold M_ξ/Γ are explicitly computed in [3, Section 6]. We have

$$\alpha \cup \beta = (\alpha_1 \cup \beta_1, \ldots, \alpha_{2^\ell} \cup \beta_{2^\ell}).$$

Note that if $L_1 = L_2 = \mathcal{O}_X$, then the Chen–Ruan product $\cup$ is the ordinary cup product on $H^*(PM_\xi/\Gamma, \mathbb{Q})$.

References

[1] I. Biswas, N. Raghavendra, Canonical generators of the cohomology of moduli of parabolic bundles on curves, Math. Ann. 306 (1996) 1–14.
[2] I. Biswas, Parabolic ample bundles, Math. Ann. 307 (1997) 511–529.
[3] I. Biswas, M. Poddar, Chen–Ruan cohomology of some moduli spaces, Int. Math. Res. Not. 2008 (2008), doi:10.1093/imrn/rnn104, article ID rnn104.
[4] H.U. Boden, K. Yokogawa, Rationality of moduli spaces of parabolic bundles, J. Lond. Math. Soc. 59 (1999) 461–478.
[5] W. Chen, Y. Ruan, A new cohomology theory of orbifold, Comm. Math. Phys. 248 (2004) 1–31.
[6] V.B. Mehta, C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann. 248 (1980) 205–239.
[7] N. Nitsure, Cohomology of moduli of parabolic vector bundles, Proc. Indian Acad. Sci. Math. Sci. 95 (1986) 61–77.