ALMOST COMPLETE INTERSECTION BINOMIAL EDGE IDEALS AND THEIR REES ALGEBRAS

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Abstract. Let G be a simple graph on n vertices and J_G denote the binomial edge ideal of G in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. In this article, we compute the second graded Betti numbers of J_G, and we obtain a minimal presentation of it when G is a tree or a unicyclic graph. We classify all graphs whose binomial edge ideals are almost complete intersection, prove that they are generated by a d-sequence and that the Rees algebra of their binomial edge ideal is Cohen-Macaulay. We also obtain an explicit description of the defining ideal of the Rees algebra of those binomial edge ideals.

1. Introduction

Let G be a simple graph with vertex set $V(G) = [n] := \{1, \ldots, n\}$ and edge set $E(G)$. Villarreal in [26] defined the edge ideal of G as $I(G) = (x_ix_j \ : \ \{i, j\} \in E(G)) \subset \mathbb{K}[x_1, \ldots, x_n]$. Herzog et al. in [10] and independently Ohtani in [20] defined the binomial edge ideal of G as $J_G = (x_iy_j - x_jy_i \ : i < j \text{ and } \{i, j\} \in E(G)) \subset S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. In the recent past, researchers have been trying to understand the connection between combinatorial invariants of G and algebraic invariants of $I(G)$ and J_G. While this relation between G and $I(G)$ is well explored (see for example [1] and the references therein), the connection between the properties of G and J_G are not very well understood, see [10, 13, 15, 16, 17, 24] for a partial list. It is known that the Rees algebra of an ideal I, $\mathcal{R}(I) := \oplus_{n\geq 0} I^nt^n$, encodes a lot of asymptotic properties of I. In the case of monomial edge ideals, properties of their Rees algebra have been explored by several researchers (see [27] and the citations to this paper). In [27], Villarreal described the generators of the defining ideal of the Rees algebra of a graph. As a consequence of this, he proved that $I(G)$ is of *linear type*, i.e., the Rees algebra is isomorphic to the Symmetric algebra, if and only if G is either a tree or an odd unicyclic graph. However, nothing much is known about the Rees algebra of binomial edge ideals. In this article, we initiate such a study.

An ideal I in a standard graded polynomial ring is said to be *complete intersection* if $\mu(I) = \operatorname{ht}(I)$, where $\mu(I)$ denotes the cardinality of a minimal homogeneous generating set of I. It is said to be *almost complete intersection* if $\mu(I) = \operatorname{ht}(I) + 1$ and $I_\mathfrak{p}$ is complete intersection for all minimal primes $\mathfrak{p}$ of I. It is known that for a connected graph G, J_G is complete intersection if and only if G is a path, [6]. Rinaldo studied the Cohen-Macaulayness of certain subclasses of almost complete intersection binomial edge ideals, [23]. In this article, we characterize graphs whose binomial edge ideals are almost complete intersections. We prove that these are either a subclass of trees or a subclass of unicyclic graphs (Theorems 4.3, 4.4).

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Understanding the depth of the Rees algebra and the associated graded ring of ideals has been a long studied problem in commutative algebra. If an ideal is generated by a regular sequence in a Cohen-Macaulay local ring, then the corresponding associated graded ring and the Rees algebra are known to be Cohen-Macaulay. In general, computing the depth of these blowup algebras is a non-trivial problem. If an ideal is almost complete intersection, then the Cohen-Macaulayness of the Rees algebra and the associated graded ring are closely related by a result of Herrmann, Ribbe and Zarzuela (see Theorem 4.5). We prove that the associated graded ring, and hence, the Rees algebra of almost complete intersection binomial edge ideals are Cohen-Macaulay, (Theorem 4.7).

Another problem of interest for commutative algebraists is to compute the defining ideal of the Rees algebra. Describing the defining ideal not only gives more insight into the structure of the Rees algebra, but it also helps in understanding other homological properties and invariants associated with the Rees algebra. For example, the maximal degree occurring in a minimal generating set of the defining ideal also serves as a lower bound for one of the most important homological and computational invariant, the Castelnuovo-Mumford regularity. In general, it is quite a hard task to describe the defining ideals of Rees algebras. Huneke proved that the defining ideal of the Rees algebra of an ideal generated by a d-sequence has a linear generating set, [11] (see [21] for a simple proof). We show that homogeneous almost complete intersection ideals in polynomial rings over an infinite field are generated by a d-sequence, Proposition 4.10. As a consequence, we derive that if J_G is an almost complete intersection ideal, then J_G is generated by a d-sequence, (Corollary 4.11). We also prove that being almost complete intersection is not a necessary condition for the binomial edge ideal to be generated by a d-sequence, by showing that $J_{K_{1,n}}$ is generated by a d-sequence (Proposition 4.9). We then describe the defining ideals of the Rees algebras of almost complete intersection binomial edge ideals, (Corollary 4.13, Remark 4.14).

It is known that for an ideal I of linear type, the generators of the defining ideal of the Rees algebra can be obtained from the matrix of a minimal presentation of I [12]. For describing the generating set of the defining ideal of Rees algebras, we compute a minimal presentation of ideals. In this process, we compute the second graded Betti numbers and generators of the second syzygy of S/J_G when G is a tree or a unicyclic graph, (Theorems 3.1 - 3.7). Here we do not assume that the binomial edge ideal is almost complete intersection.

The article is organized as follows. The second section contains all the necessary definitions and notation required in the rest of the article. In Section 3, we describe the second graded Betti numbers and first syzygy of the binomial edge ideal of trees and unicyclic graphs. We study the Rees algebra of almost complete intersection binomial edge ideals in Section 4.

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2. Preliminaries

Let G be a simple graph with the vertex set $[n]$ and edge set $E(G)$. A graph on $[n]$ is said to be a *complete graph*, if $\{i, j\} \in E(G)$ for all $1 \leq i < j \leq n$. The complete graph on $[n]$ is denoted by K_n. For $A \subseteq V(G)$, $G[A]$ denotes the *induced subgraph* of G on the vertex set A, that is, for $i, j \in A$, $\{i, j\} \in E(G[A])$ if and only if $\{i, j\} \in E(G)$. For a vertex v, $G \setminus v$ denotes the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$. A subset U of $V(G)$ is said to be a *clique* if $G[U]$ is a complete graph. A vertex v of G is said to be a *simplicial vertex* if v is contained in only one maximal clique. For a vertex v,

$N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ denotes the *neighborhood* of v in G and G_v is the graph on the vertex set $V(G)$ and edge set $E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is $|N_G(v)|$. A vertex v is said to be a *pendant vertex* if $\deg_G(v) = 1$. Let $c(G)$ denote the number of components of G. A vertex v is called a *cut vertex* of G if $c(G) < c(G \setminus v)$. For an edge e in G, $G \setminus e$ is the graph on the vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. An edge e is called a *cut edge* if $c(G) < c(G \setminus e)$. Let $u, v \in V(G)$ be such that $e = \{u, v\} \notin E(G)$, then we denote by G_e, the graph on vertex set $V(G)$ and edge set $E(G_e) = E(G) \cup \{\{x, y\} : x,\ y \in N_G(u) \text{ or } x,\ y \in N_G(v)\}$. A *cycle* is a connected graph G with $\deg_G(v) = 2$ for all $v \in V(G)$. A graph is said to be a *unicyclic* graph if it contains exactly one cycle as a subgraph. A graph is a *tree* if it does not have a cycle. The *girth* of a graph G is the length of a shortest cycle in G. A *complete bipartite* graph on $m+n$ vertices, denoted by $K_{m,n}$, is the graph with the vertex set $V(K_{m,n}) = \{u_1, \dots, u_m\} \cup \{v_1, \dots, v_n\}$ and edge set $E(K_{m,n}) = \{\{u_i, v_j\} \ : \ 1 \leq i \leq m, 1 \leq j \leq n\}$. A *claw* is the complete bipartite graph $K_{1,3}$. A claw $\{u, v, w, z\}$ with center u is the graph with vertices $\{u, v, w, z\}$ and edges $\{\{u, v\}, \{u, w\}, \{u, z\}\}$. For a graph G, let $\mathcal{C}_G$ denote the set of all induced claws in G.

Let $R = \mathbb{K}[x_1, \dots, x_m]$ be a standard graded polynomial ring over a field $\mathbb{K}$ and M be a finitely generated graded R-module. Let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0,$$

be the minimal graded free resolution of M, where $R(-j)$ is the free R-module of rank 1 generated in degree j. The number $\beta_{i,j}(M)$ is called the (i, j)-th *graded Betti number* of M. Then, the exact sequence

$$\bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

is called the *minimal presentation* of M.

Let G be a graph on $[n]$. For an edge $e = \{i, j\} \in E(G)$ with $i < j$, we define $f_e = f_{i,j} = f_{j,i} := x_i y_j - x_j y_i$. For $T \subset [n]$, let $\bar{T} = [n] \setminus T$ and c_T denote the number of components of $G[\bar{T}]$. Also, let $G_1, \cdots, G_{c_T}$ be the components of $G[\bar{T}]$ and for every i, $\tilde{G}_i$ denote the complete graph on $V(G_i)$. Let $P_T(G) := (\underset{i \in T}{\cup} \{x_i, y_i\}, J_{\tilde{G}_1}, \cdots, J_{\tilde{G}_{c_T}})$. A set $T \subset [n]$ is said to have the *cut point property* if, for every $i \in T$, i is a cut vertex of graph $G[\bar{T} \cup \{i\}]$.

We recall some results on the binomial edge ideal from [10] which are used in the subsequent sections.

Theorem 2.1. *Let G be a graph on $[n]$. Then, we have the following:*
(a) (Corollary 2.2) J_G is a radical ideal.
(b) (Lemma 3.1) For $T \subset [n]$, $P_T(G)$ is a prime ideal and $\operatorname{ht}(P_T(G)) = n + |T| - c_T$.
(c) (Theorem 3.2) $J_G = \underset{T \subset [n]}{\cap} P_T(G)$.
(d) (Corollary 3.9) For $T \subset [n]$, $P_T(G)$ is a minimal prime of J_G if and only if either $T = \emptyset$ or T has the cut point property.

Mapping Cone Construction: For an edge $e = \{i, j\} \in E(G)$, We consider the following exact sequence:

$$0 \longrightarrow \frac{S}{J_{G \setminus e} : f_e}(-2) \xrightarrow{\cdot f_e} \frac{S}{J_{G \setminus e}} \longrightarrow \frac{S}{J_G} \longrightarrow 0. \tag{1}$$

By [19, Theorem 3.7], we have

$$J_{G\setminus e} : f_e = J_{(G\setminus e)_e} + (g_{P,t} : P \text{ is a path of length } s+1 \text{ between } i,j \text{ and } 0 \leq t \leq s),$$

where for a path $P : i, i_1, \ldots, i_s, j$, $g_{P,0} = y_{i_1} \cdots y_{i_s}$ and for each $1 \leq t \leq s$, $g_{P,t} = x_{i_1} \cdots x_{i_t} y_{i_{t+1}} \cdots y_{i_s}$. Let $(\mathbf{F}., d^{\mathbf{F}}.)$ and $(\mathbf{G}., d^{\mathbf{G}}.)$ be minimal S-free resolutions of $S/J_{G\setminus e}$ and $[S/(J_{G\setminus e} : f_e)](-2)$ respectively. Let $\varphi. : (\mathbf{G}., d^{\mathbf{G}}.) \longrightarrow (\mathbf{F}., d^{\mathbf{F}}.)$ be the complex morphism induced by the multiplication by f_e. The mapping cone $(\mathbf{M}(\varphi)., \delta.)$ is an S-free resolution of S/J_G such that $(\mathbf{M}(\varphi))_i = \mathbf{F}_i \oplus \mathbf{G}_{i-1}$ and the differential maps are $\delta_i(x,y) = (d_i^{\mathbf{F}}(x) + \varphi_{i-1}(y), -d_{i-1}^{\mathbf{G}}(y))$ for $x \in \mathbf{F}_i$ and $y \in \mathbf{G}_{i-1}$. It need not necessarily be a minimal free resolution. We refer the reader to [5] for more details on the mapping cone.

3. Betti numbers and Syzygy of binomial edge ideals

In this section, we describe the first graded Betti numbers and the first syzygy of binomial edge ideals of trees and unicyclic graphs. First, we compute the second graded Betti numbers of S/J_G when G is a tree.

Theorem 3.1. *Let G be a tree on $[n]$. Then,*

$$\beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \binom{n-1}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3}.$$

Proof. We prove this by induction on n. If $n = 2$, then $G = P_2$, and hence, J_G is complete intersection. Therefore, $\beta_2(S/J_G) = 0$. Hence, the assertion follows. We now assume that $n > 2$. Let $e = \{u, v\}$ be an edge such that u is a pendant vertex. The long exact sequence of Tor in degree j component corresponding to the short exact sequence (1) is:

$$\cdots \to \operatorname{Tor}_{2,j}^S\left(\frac{S}{J_{G\setminus e}}, \mathbb{K}\right) \to \operatorname{Tor}_{2,j}^S\left(\frac{S}{J_G}, \mathbb{K}\right) \to \operatorname{Tor}_{1,j}^S\left(\frac{S}{J_{G\setminus e} : f_e}(-2), \mathbb{K}\right) \to \cdots \quad (2)$$

Since e is a cut edge and u is a pendant vertex of G, $(G \setminus e)_e = (G \setminus u)_v \sqcup \{u\}$. Thus, it follows from [19, Theorem 3.7] that $J_{G\setminus e} : f_e = J_{(G\setminus u)_v}$. One can observe that

$$\operatorname{Tor}_{1,j}\left(\frac{S}{J_{(G\setminus u)_v}}(-2), \mathbb{K}\right) \simeq \operatorname{Tor}_{1,j-2}\left(\frac{S}{J_{(G\setminus u)_v}}, \mathbb{K}\right).$$

Since $G \setminus e = (G \setminus u) \sqcup \{u\}$, $J_{G\setminus e} = J_{G\setminus u}$. Therefore, by induction, we obtain

$$\beta_{2,4}(S/J_{G\setminus e}) = \binom{n-2}{2} + \sum_{w \in V(G)\setminus\{v\}} \binom{\deg_G(w)}{3} + \binom{\deg_G(v)-1}{3}$$

and $\beta_{2,j}(S/J_{G\setminus e}) = 0$ for $j \neq 4$. If $j \neq 4$, then

$$\operatorname{Tor}_{1,j-2}\left(\frac{S}{J_{(G\setminus u)_v}}, \mathbb{K}\right) = 0.$$

Hence, $\beta_{2,j}(S/J_G) = 0$, if $j \neq 4$. Since $\beta_{2,2}(S/J_{(G\setminus u)_v}) = 0$ and $\beta_{1,4}(S/J_{G\setminus e}) = 0$, we have $\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G\setminus e}) + \beta_{1,2}(S/J_{(G\setminus u)_v})$. Now, $\beta_{1,2}(S/J_{(G\setminus u)_v}) = |E((G \setminus u)_v)| = n - 2 + \binom{\deg_G(v)-1}{2}$. Hence, $\beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \binom{n-1}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3}$. $\square$

We now describe the first syzygy of binomial edge ideals of trees. To compute a minimal generating set of first syzygy, we crucially use the knowledge of the Betti numbers of J_G. A tree on $[n]$ vertices has $n-1$ edges. For convenience in writing the list of generators, we need some notation. For $A \subseteq [n]$ and $i \in A$, we define $p_A(i) = |\{j \in A \mid j \leq i\}|$. The function p_A indicates the position of an element in A when the elements are arranged in the ascending order.

Theorem 3.2. *Let G be a tree on $[n]$ vertices. Then, the first syzygy of J_G is minimally generated by elements of the form*

(a) $f_{i,j}e_{\{k,l\}} - f_{k,l}e_{\{i,j\}}$*, where* $\{i,j\},\{k,l\} \in E(G)$ *and* $\{e_{\{i,j\}} \;:\; \{i,j\} \in E(G)\}$ *is the standard basis of* $S(-2)^{n-1}$*;*

(b) $(-1)^{p_A(j)} f_{k,l}e_{\{i,j\}} + (-1)^{p_A(k)} f_{j,l}e_{\{i,k\}} + (-1)^{p_A(l)} f_{j,k}e_{\{i,l\}}$*, where* $A = \{i,j,k,l\} \in \mathcal{C}_G$ *with center at* i*.*

Proof. From Theorem 3.1, we have $\beta_2(S/J_G) = \beta_{2,4}(S/J_G) = \binom{n-1}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3}$. Therefore, the minimal presentation of J_G is of the form

$$S(-4)^{\beta_{2,4}(S/J_G)} \xrightarrow{\varphi} S(-2)^{n-1} \xrightarrow{\psi} J_G \longrightarrow 0.$$

Note that $|\mathcal{C}_G| = \sum_{v \in V(G)} \binom{\deg_G(v)}{3}$. Since $\beta_2(S/J_G) = \binom{n-1}{2} + |\mathcal{C}_G|$, we index the standard basis of $S(-4)^{\beta_2(S/J_G)}$ accordingly. Let

$$\begin{aligned} \mathcal{S}_1 &= \{E_{\{i,j\},\{k,l\}} : \{i,j\},\{k,l\} \in E(G), i<j, k<l \text{ and } (i,j) \gneq_{lex} (k,l)\} \\ \mathcal{S}_2 &= \{E^i_{\{j,k,l\}} : \{i,j,k,l\} \in \mathcal{C}_G \text{ with center at } i\} \end{aligned}$$

and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ denote the standard basis of $S(-4)^{\beta_2(S/J_G)}$. For a pair of edges $\{i,j\},\{k,l\} \in E(G)$, $f_{i,j}f_{k,l} - f_{k,l}f_{i,j} = 0$ gives a relation among the generators of J_G. Let $\{i,j,k,l\} \in \mathcal{C}_G$ be a claw with center at i. Then, it can be easily verified that for $A = \{i,j,k,l\}$,

$$(-1)^{p_A(j)} f_{k,l}f_{i,j} + (-1)^{p_A(k)} f_{j,l}f_{i,k} + (-1)^{p_A(l)} f_{j,k}f_{i,l} = 0,$$

which gives another relation among the generators of J_G. Define the maps φ and ψ as follows:

$$\begin{aligned} \varphi\big(E_{\{i,j\},\{k,l\}}\big) &= f_{i,j}e_{\{k,l\}} - f_{k,l}e_{\{i,j\}}; \\ \varphi\big(E^i_{\{j,k,l\}}\big) &= (-1)^{p_A(j)} f_{k,l}e_{\{i,j\}} + (-1)^{p_A(k)} f_{j,l}e_{\{i,k\}} + (-1)^{p_A(l)} f_{j,k}e_{\{i,l\}}; \\ \psi(e_{\{i,j\}}) &= f_{i,j}, \end{aligned}$$

where $A = \{i,j,k,l\}$. Observe that $\varphi(\mathcal{S}_1)$ is the collection of elements of type (a) in the statement of the Theorem and $\varphi(\mathcal{S}_2)$ is the collection of elements of type (b). Also, for any pair of edges $\{i,j\},\{k,l\}$ and a claw $\{u,v,w,z\}$ with u as a center, we have $\psi\big(\varphi(E_{\{i,j\},\{k,l\}})\big) = 0$ and $\psi(\varphi(E^u_{\{v,z,w\}})) = 0$. Since $\beta_{2,j} = 0$ for all $j \neq 4$, it follows that the first syzygy is generated in degree 4. Moreover, as $\beta_2(S/J_G) = \beta_{2,4}(S/J_G) = |\mathcal{S}|$, to prove the assertion, it is enough to prove that the elements of $\varphi(\mathcal{S})$ are $\mathbb{K}$-linearly independent, equivalently, the columns of the matrix of φ are $\mathbb{K}$-linearly independent. For this, note that for each $\{i,j\} \in E(G)$, the entries of the corresponding row are the coefficients of $e_{\{i,j\}}$ in the expression for the images of elements in $\mathcal{S}$ under φ. The coefficient of $e_{\{i,j\}}$ in $\varphi(E_{\{i,j\},\{k,l\}})$ or $\varphi(E_{\{k,l\},\{i,j\}})$ is $\pm f_{k,l}$. Moreover, the entry will be zero in the column corresponding to $\varphi(E_{\{u,v\},\{w,z\}})$ for $\{u,v\} \neq \{i,j\}$ and $\{w,z\} \neq \{i,j\}$. Therefore, among the first $\binom{n-1}{2}$ column entries in the row corresponding to $e_{\{i,j\}}$, there will be $(n-2)$ non-zero entries, namely the binomials corresponding to all the edges other than $\{i,j\}$. In $\varphi(E^u_{\{v,w,z\}})$, the coefficient of $e_{\{i,j\}}$ is non-zero if and only if either $i = u$ and $j \in \{v, w, z\}$ or $j = u$ and $i \in \{v, w, z\}$. If $i = u$ and $j = v$ (similarly any one of the other three), then the coefficient of $e_{\{i,j\}}$ is $\pm f_{w,z}$. It may be noted here that $f_{w,z}$ does not correspond to an edge in G. If $E^{u_1}_{\{v_1,w_1,z_1\}}$ and $E^{u_2}_{\{v_2,w_2,z_2\}}$ are two distinct basis elements $\{i, j\}$ in both the claws, then $\{u_1, v_1, w_1, z_1\} \setminus \{i, j\} \neq \{u_2, v_2, w_2, z_2\} \setminus \{i, j\}$. Hence, the corresponding coefficients of $e_{\{i,j\}}$ in $\varphi(E^{u_1}_{\{v_1,w_1,z_1\}})$ and $\varphi(E^{u_2}_{\{v_2,w_2,z_2\}})$ will not be the same. From the above discussion one concludes that in the row corresponding to $e_{\{i,j\}}$, each nonzero entry is of the form $\pm f_{k,l}$ for some $k, l \in [n]$, $\{k, l\} \neq \{i, j\}$ and no two are equal. Therefore, the entries of this row can be seen as the minimal generating set of binomial edge ideal of a graph on $[n]$, possibly different from G, and hence, they are $\mathbb{K}$-linearly independent. Therefore, the assertion follows. $\square$

We now study the first graded Betti numbers and syzygy of binomial edge ideal of unicyclic graphs. Let G be a unicyclic graph on the vertex set $[n]$ of girth m. First, we compute $\beta_2(S/J_G)$, where G is a unicyclic graph of girth 3.

Theorem 3.3. *Let G be a unicyclic graph on $[n]$ of girth 3. Let v_1, v_2, v_3 be the vertices of the cycle in G. Then,*

$$\beta_2(S/J_G) = \beta_{2,3}(S/J_G) + \beta_{2,4}(S/J_G) = 2 + \beta_{2,4}(S/J_G),$$

$$\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} - \sum_{i=1,2,3} \deg_G(v_i) + 3.$$

Proof. We prove this by induction on n. By [24, Theorem 2.2], for any graph G, $\beta_{2,3}(S/J_G) = 2\mathsf{k}_3(G)$, where $\mathsf{k}_3(G)$ denotes the number of K_3's appearing in G. If $n = 3$, then $G = K_3$, and hence, the assertion follows from [24, Theorem 2.1]. We now assume that $n > 3$. Let $e = \{u, v\}$ be an edge such that u is a pendant vertex. Since e is a cut edge and u is a pendant vertex of G, $(G \setminus e)_e = (G \setminus u)_v \sqcup \{u\}$. Thus, $J_{G \setminus e} : f_e = J_{(G \setminus u)_v}$. By [24, Theorem 2.2], we get $\beta_{2,3}(S/J_{G \setminus e}) = 2$. Therefore, by induction, we get

$$\beta_{2,4}(S/J_{G \setminus e}) = \binom{n-1}{2} + \sum_{w \in V(G) \setminus \{v\}} \binom{\deg_G(w)}{3} + \binom{\deg_G(v) - 1}{3} - \sum_{i=1,2,3} \deg_{G \setminus e}(v_i) + 3$$

and $\beta_{2,j}(S/J_{G \setminus e}) = 0$ for $j > 4$. If $j \neq 4$, then $\operatorname{Tor}_{1,j-2}\left(\frac{S}{J_{(G \setminus u)_v}}, \mathbb{K}\right) = 0$. Hence, the long exact sequence (2) gives that $\beta_{2,j}(S/J_G) = 0$, if $j > 4$. Since $\beta_{2,2}(S/J_{(G \setminus u)_v}) = 0$ and $\beta_{1,4}(S/J_{G \setminus e}) = 0$, it follows from the long exact sequence (2) that $\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G \setminus e}) + \beta_{1,2}(S/J_{(G \setminus u)_v})$. If $v = v_i$ for some i, then $\beta_{1,2}(S/J_{(G \setminus u)_v}) = |E((G \setminus u)_v)| = |E(G)| - 1 + \binom{\deg_G(v)-1}{2} - 1 = n - 2 + \binom{\deg_G(v)-1}{2}$. Moreover, for this i, $\deg_{G \setminus e}(v_i) = \deg_G(v_i) - 1$. Hence, we get the required expression for $\beta_{2,4}(S/J_G)$. If $v \neq v_i$ for all i, then $\beta_{1,2}(S/J_{(G \setminus u)_v}) = |E((G \setminus u)_v)| = n - 1 + \binom{\deg_G(v)-1}{2}$. Hence, $\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} - \sum_{i=1,2,3} \deg_G(v_i) + 3$. $\square$

We now compute the second graded Betti numbers of S/J_G when G is a unicyclic graph of girth at least 4.

Theorem 3.4. *If G is a unicyclic graph on $[n]$ of girth $m \geq 4$, then*

$$\beta_2(S/J_G) = \begin{cases} \beta_{2,4}(S/J_G), & \text{if } m = 4, \\ \beta_{2,4}(S/J_G) + \beta_{2,m}(S/J_G) & \text{if } m > 4, \end{cases}$$

where

$$\beta_{2,4}(S/J_G) = \begin{cases} \binom{n}{2} + \sum_{v\in V(G)} \binom{\deg_G(v)}{3} + 3 & \text{if } m = 4, \\ \binom{n}{2} + \sum_{v\in V(G)} \binom{\deg_G(v)}{3} & \text{if } m > 4, \end{cases}$$

and $\beta_{2,m}(S/J_G) = m - 1$, *if* $m > 4$.

Proof. Let $e = \{u, v\}$ be an edge of the cycle in G. Then, after removing the edge e, $G \setminus e$ becomes a tree. Therefore, from Theorem 3.1, we have

$$\beta_2(S/J_{G\setminus e}) = \beta_{2,4}(S/J_{G\setminus e}) = \binom{n-1}{2} + \sum_{w\in V(G)\setminus\{u,v\}} \binom{\deg_G(w)}{3} + \sum_{w\in\{u,v\}} \binom{\deg_G(w)-1}{3}.$$

Note that $(G \setminus e)_e = ((G \setminus e)_v)_u$.

It follows from [19, Theorem 3.7] that $J_{G\setminus e} : f_e = J_{((G\setminus e)_v)_u} + I$, where

$$I = (g_{P,t} : P : u, i_1, \dots, i_s, v \text{ is a path between } u \text{ and } v \text{ in } G \setminus e \text{ and } 0 \le t \le s).$$

In $G \setminus e$, there is only one path between u and v and the corresponding $g_{P,t}$ has degree $m-2$ for all t. Since $\beta_{2,2}(S/(J_{((G\setminus e)_v)_u} + I)) = 0$ and $\beta_{1,4}(S/J_{G\setminus e}) = 0$, we have $\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G\setminus e}) + \beta_{1,2}(S/(J_{((G\setminus e)_v)_u} + I))$. For $m = 4$, $I = (y_2y_3, x_2y_3, x_2x_3)$. Therefore,

$$\beta_{1,2}(S/(J_{(G\setminus e)_e} + I)) = 3 + |E((G \setminus e)_e)| = 3 + (n-1) + \binom{\deg_G(v)-1}{2} + \binom{\deg_G(u)-1}{2}.$$

Hence,

$$\beta_{2,4}(S/J_G) = \beta_{2,4}(S/J_{G\setminus e}) + \beta_{1,2}(S/(J_{(G\setminus e)_e} + I)) = \binom{n}{2} + \sum_{v\in V(G)} \binom{\deg_G(u)}{3} + 3.$$

Also, $\beta_{2,j}(S/J_{G\setminus e}) = 0$ and $\beta_{1,j-2}(S/(J_{(G\setminus e)_e} + I)) = 0$, if $j \neq 4$. Therefore, $\beta_{2,j}(S/J_G) = 0$, if $j \neq 4$. Now assume that $m > 4$. Note that for $j \neq 4$, $j \neq m$,

$$\mathrm{Tor}_{1,j-2}\left(\frac{S}{J_{((G\setminus e)_v)_u} + I}, \mathbb{K}\right) = 0 \text{ and } \dim_{\mathbb{K}}\left(\mathrm{Tor}_{1,m-2}\left(\frac{S}{J_{((G\setminus e)_v)_u} + I}, \mathbb{K}\right)\right) = m - 1.$$

Hence, it follows from the long exact sequence (2) that $\beta_{2,j}(S/J_G) = 0$, if $j \notin \{4, m\}$. Since $\beta_{1,j}(S/J_{G\setminus e}) = 0$ for $j \neq 2$, we have

$$\mathrm{Tor}_{1,m-2}\left(\frac{S}{J_{((G\setminus e)_v)_u} + I}, \mathbb{K}\right) \simeq \mathrm{Tor}_{2,m}\left(\frac{S}{J_G}, \mathbb{K}\right).$$

Thus, for $m > 4$, $\beta_{2,m}(S/J_G) = m - 1$. Now, $\beta_{1,2}(S/(J_{((G\setminus e)_v)_u}) + I) = |E(((G \setminus e)_v)_u)| = n - 1 + \binom{\deg_G(v)-1}{2} + \binom{\deg_G(u)-1}{2}$. Hence, $\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v\in V(G)} \binom{\deg_G(v)}{3}$. ☐

Now, we obtain a minimal generating set for the first syzygy of J_{C_n} for $n \geq 4$. Let $G = C_n$ be a cycle on $[n]$ with edge set $E(C_n) = \{\{i, i+1\}, \{1, n\} : 1 \le i \le n-1\}$.

Theorem 3.5. *Let C_n be the cycle on n vertices, $n \geq 4$. Let $\{e_{\{k,k+1\}}, e_{\{1,n\}} : 1 \le k \le n-1\}$ denote the standard basis of S^n and $Y = y_1 \cdots y_n$. For $i = 1, \dots, n-1$, define $b_i \in S^n$ as follows:*

$$(b_1)_k = \frac{Y}{y_k y_{k+1}} \text{ for } 1 \le k \le n-1, (b_1)_n = \frac{Y}{y_1 y_n},$$

$$\textit{for } 1 \le i \le n-2,\ (b_{i+1})_k = \begin{cases} (b_i)_k \cdot \frac{x_{i+2}}{y_{i+2}} & \textit{if } k \le i, \\ (b_i)_{i+1} \cdot \frac{x_1}{y_1} & \textit{if } k = i+1, \\ (b_i)_k \cdot \frac{x_{i+1}}{y_{i+1}} & \textit{if } k \ge i+2. \end{cases}$$

Then, the first syzygy of J_{C_n} is minimally generated by

$$\left\{f_{k,l}e_{\{i,j\}} - f_{i,j}e_{\{k,l\}} : \{i,j\},\{k,l\} \in E(C_n)\right\} \bigcup \left\{\sum_{k=1}^{n-1}(b_i)_k e_{\{k,k+1\}} - (b_i)_n e_{\{1,n\}} \ : \ 1 \le i \le n-1\right\}.$$

Proof. By [29, Corollary 16], $\beta_{2,4}(S/J_{C_n}) = \begin{cases} 9 & \text{if } n = 4; \\ \binom{n}{2} & \text{if } n > 4, \end{cases}$ $\beta_{2,n}(S/J_{C_n}) = n-1$ for $n > 4$ and $\beta_{2,j}(S/J_{C_n}) = 0$ for all $j \neq 4, n$. Therefore, the minimal presentation of J_{C_n} is

$$S(-4)^{\binom{n}{2}} \oplus S(-n)^{n-1} \longrightarrow S(-2)^n \longrightarrow J_{C_n} \longrightarrow 0. \tag{3}$$

Note that $J_{C_n} = J_{P_n} + (f_{1,n})$. Consider the following exact sequence

$$0 \longrightarrow \frac{S}{J_{P_n} : f_{1,n}}(-2) \xrightarrow{\cdot f_{1,n}} \frac{S}{J_{P_n}} \longrightarrow \frac{S}{J_{C_n}} \longrightarrow 0$$

and apply the mapping cone construction. Since J_{P_n} is complete intersection, the Koszul complex $(\mathbf{F}.,d^{\mathbf{F}}.)$ gives the minimal free resolution for S/J_{P_n}. Let $\{e_{\{i,j\},\{k,l\}} \mid \{i,j\} \neq \{k,l\} \in E(P_n)\}$ denote the standard basis of $S^{\binom{n-1}{2}}$ and $\{e_{\{j,j+1\}} \mid 1 \le j \le n-1\}$ denote the standard basis of S^{n-1}. Set $d_1^{\mathbf{F}}(e_{\{j,j+1\}}) = f_{j,j+1}$ for $1 \le j \le n-1$ and $d_2^{\mathbf{F}}(e_{\{i,j\},\{k,l\}}) = f_{k,l}e_{\{i,j\}} - f_{i,j}e_{\{k,l\}}$ for $\{i,j\} \neq \{k,l\} \in E(P_n)$. It follows from [19, Theorem 3.7] that

$$J_{P_n} : f_{1,n} = J_{P_n} + (y_2 \cdots y_{n-1}, x_2y_3 \cdots y_{n-1}, \ldots, x_2 \cdots x_{n-1}).$$

Let $(\mathbf{G}.,d^{\mathbf{G}}.)$ be the minimal resolution of $\frac{S}{(J_{P_n}:f_{1,n})}(-2)$ with the differential maps given by $d_1^{\mathbf{G}}(E_{i,i+1}) = f_{i,i+1}$ for $1 \le i \le n-1$ and $d_1^{\mathbf{G}}(E_m) = x_2 \cdots x_m y_{m+1} \cdots y_{n-1}$ for $1 \le m \le n-1$, where $\{E_{i,i+1}, E_m : 1 \le i \le n-1, 1 \le m \le n-1\}$ denotes the standard basis of G_1. Clearly the map from G_0 to F_0 in the mapping cone complex is the multiplication by $f_{1,n}$. Define the map $\varphi_1 : G_1 \longrightarrow F_1$ by

$$\begin{array}{ll} \varphi_1(E_{i,i+1}) = f_{1,n}e_{\{i,i+1\}} & 1 \le i \le n-1, \\ \varphi_1(E_m) = \sum_{k=1}^{n-1}(b_m)_k e_{\{k,k+1\}} & 1 \le m \le n-1, \end{array}$$

where $(b_m)_k$'s are as defined in the statement of the Theorem. We show that the map φ_1 satisfies the property that for all $x \in G_1$, $d_1^{\mathbf{F}}(\varphi_1(x)) = f_{1,n} \cdot d_1^{\mathbf{G}}(x)$. It is enough to prove the property for the basis elements. Clearly $d_1^{\mathbf{F}}(\varphi_1(E_{\{i,i+1\}})) = f_{1,n}f_{i,i+1} = f_{1,n} \cdot d_1^{\mathbf{G}}(E_{\{i,i+1\}})$. Now $d_1^{\mathbf{F}}(\varphi_1(E_1)) = d_1^{\mathbf{F}}\left(\sum_{k=1}^{n-1}(b_1)_k e_{\{k,k+1\}}\right) = \sum_{k=1}^{n-1} \frac{Y}{y_k y_{k+1}} f_{k,k+1}$. Note that $\frac{f_{k,k+1}}{y_k y_{k+1}} = \frac{x_k}{y_k} - \frac{x_{k+1}}{y_{k+1}}$. Now, taking the summation over k, we get $d_1^{\mathbf{F}}(\varphi_1(E_1)) = f_{1,n}(y_2 \cdots y_{n-1}) = f_{1,n} \cdot d_1^{\mathbf{G}}(E_1)$. Let $m \ge 2$. Then, $d_1^{\mathbf{F}}(\varphi_1(E_m)) = d_1^{\mathbf{F}}\left(\sum_{k=1}^{n-1}(b_m)_k e_{\{k,k+1\}}\right) = \sum_{k=1}^{n-1}(b_m)_k f_{k,k+1}$. It can be seen that

$$\begin{aligned}
\sum_{k=1}^{m-1}(b_m)_k f_{k,k+1} &= Y\left[\frac{x_2}{y_2}\cdots\frac{x_{m-1}}{y_{m-1}}\frac{x_{m+1}}{y_{m+1}}\left(\frac{x_1}{y_1}-\frac{x_m}{y_m}\right)\right],\\
(b_m)_m f_{m,m+1} &= Y\left[\frac{x_1}{y_1}\cdots\frac{x_{m-1}}{y_{m-1}}\left(\frac{x_m}{y_m}-\frac{x_{m+1}}{y_{m+1}}\right)\right],\\
\sum_{k=m+1}^{n-1}(b_m)_k f_{k,k+1} &= Y\left[\frac{x_2}{y_2}\cdots\frac{x_m}{y_m}\left(\frac{x_{m+1}}{y_{m+1}}-\frac{x_n}{y_n}\right)\right].
\end{aligned}$$

Summing up these three terms together, we get

$$d_1^{\mathbf{F}}(\varphi_1(E_m)) = Y\left[\frac{x_2}{y_2}\cdots\frac{x_m}{y_m}\left(\frac{x_1}{y_1}-\frac{x_n}{y_n}\right)\right] = x_2\cdots x_m y_{m+1}\cdots y_{n-1}f_{1,n} = f_{1,n}\cdot d_1^{\mathbf{G}}(E_m).$$

Therefore, by the mapping cone construction, we get a presentation of J_{C_n} as

$$F_2\oplus G_1 \longrightarrow F_1\oplus G_0 \longrightarrow J_{C_n} \longrightarrow 0.$$

Since $F_2\oplus G_1 \simeq S^{\binom{n}{2}+n-1}$ and $F_1\oplus G_0\simeq S^n$ whose ranks coincide with the corresponding Betti numbers of J_{C_n}, we can conclude that this is a minimal presentation. Hence, the first syzygy of J_{C_n} is minimally generated by the images of the standard basis elements under the map $\Phi : F_2\oplus G_1 \to F_1\oplus G_0$, where $\Phi = \begin{bmatrix} d_2^{\mathbf{F}} & \varphi_1 \\ 0 & -d_1^{\mathbf{G}}\end{bmatrix}$. Then, we have

$$\begin{aligned}
\Phi(e_{\{i,j\},\{k,l\}}) &= d_2^{\mathbf{F}}(e_{\{i,j\},\{k,l\}}) = f_{k,l}e_{\{i,j\}} - f_{i,j}e_{\{k,l\}} \text{ for } \{i,j\}\neq\{k,l\}\in E(P_n),\\
\Phi(E_{i,i+1}) &= (\varphi_1 - d_1^{\mathbf{G}})(E_{i,i+1}) = f_{1,n}e_{\{i,i+1\}} - f_{i,i+1}e_{\{1,n\}} \text{ for } i=1,\ldots,n-1, \text{ and}\\
\Phi(E_m) &= \varphi_1(E_m) - d_1^{\mathbf{G}}(E_1) = \sum_{k=1}^{n-1}(b_m)_k e_{\{k,k+1\}} - (b_m)_n e_{\{1,n\}} \text{ for } i=1,\ldots,n-1.
\end{aligned}$$

Hence, the assertion follows. ☐

We now describe a minimal generating set for the first syzygy of binomial edge ideals of unicyclic graphs. The syzygy structure is slightly different for unicyclic graphs of girth 3. We first deal with that case.

Theorem 3.6. *Let G be a unicyclic graph on $[n]$ of girth 3. Denote the vertices of the unique cycle of G by $v_1 < v_2 < v_3$. Let the standard basis of $S(-2)^n$ be denoted by $\{e_{\{i,j\}} : \{i,j\}\in E(G), i<j\}$. Then, the first syzygy of J_G is minimally generated by the elements of the form*

- *(a)* $x_{v_1}e_{\{v_2,v_3\}} - x_{v_2}e_{\{v_1,v_3\}} + x_{v_3}e_{\{v_1,v_2\}}, y_{v_1}e_{\{v_2,v_3\}} - y_{v_2}e_{\{v_1,v_3\}} + y_{v_3}e_{\{v_1,v_2\}}$,
- *(b)* $f_{i,j}e_{\{p,l\}} - f_{p,l}e_{\{i,j\}}$, *where* $\{\{i,j\},\{p,l\}\}\not\subset\{\{v_1,v_2\},\{v_1,v_3\},\{v_2,v_3\}\}, \{i,j\}\neq\{p,l\}$ *and* $\{i,j\},\{p,l\}\in E(G)$,
- *(c)* $(-1)^{p_A(j)}f_{k,l}e_{\{i,j\}} + (-1)^{p_A(k)}f_{j,l}e_{\{i,k\}} + (-1)^{p_A(l)}f_{j,k}e_{\{i,l\}}$, *where* $A=\{i,j,k,l\}\in\mathcal{C}_G$ *with center at* i.

Proof. We proceed by induction on $n=|V(G)|=|E(G)|$. For $n=3$, G is a complete graph i.e., J_G is the ideal generated by the set of all 2×2 minor of a 2×3 matrix. Then, it follows from Eagon-Northcott complex that the first syzygy of J_G is minimally generated by

$$\left\{x_{v_1}e_{\{v_2,v_3\}} - x_{v_2}e_{\{v_1,v_3\}} + x_{v_3}e_{\{v_1,v_2\}}, y_{v_1}e_{\{v_2,v_3\}} - y_{v_2}e_{\{v_1,v_3\}} + y_{v_3}e_{\{v_1,v_2\}}\right\}.$$

Now, we assume that $n > 3$. From Theorem 3.3, we know that the minimal presentation of J_G is of the form

$$S(-4)^{\beta_{2,4}(S/J_G)} \oplus S(-3)^{\beta_{2,3}(S/J_G)} \xrightarrow{\varphi} S(-2)^n \xrightarrow{\psi} J_G \longrightarrow 0,$$

where $\beta_{2,4}(S/J_G) = \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} - \sum_{i=1,2,3} \deg_G(v_i) + 3$ and $\beta_{2,3}(S/J_G) = 2$.

Let $e = \{u, v\}$ be an edge in G such that u is a pendant vertex of G. Since e is a cut edge and u is a pendant vertex of G, $(G \setminus e)_e = (G \setminus u)_v \sqcup \{u\}$. Thus, $J_{G\setminus e} : f_e = J_{(G\setminus u)_v}$. Since $G \setminus e$ is also a unicyclic graph having the unique cycle of girth 3 and $J_{G\setminus e} = J_{G\setminus u}$, by induction we get that the first syzygy of $J_{G\setminus e}$ is generated by elements of the form

(a) $x_{v_1} e_{\{v_2,v_3\}} - x_{v_2} e_{\{v_1,v_3\}} + x_{v_3} e_{\{v_1,v_2\}}, y_{v_1} e_{\{v_2,v_3\}} - y_{v_2} e_{\{v_1,v_3\}} + y_{v_3} e_{\{v_1,v_2\}}$,
(b) $f_{i,j} e_{\{p,l\}} - f_{p,l} e_{\{i,j\}}$, where $\{\{i,j\},\{p,l\}\} \not\subset \{\{v_1,v_2\},\{v_1,v_3\},\{v_2,v_3\}\}, \{i,j\} \neq \{p,l\}$ and $\{i,j\},\{p,l\} \in E(G \setminus e)$,
(c) $(-1)^{p_A(j)} f_{k,l} e_{\{i,j\}} + (-1)^{p_A(k)} f_{j,l} e_{\{i,k\}} + (-1)^{p_A(l)} f_{j,k} e_{\{i,l\}}$, where $A = \{i,j,k,l\} \in \mathcal{C}_{G\setminus e}$ with center at i.

Case-1: We assume that $v \neq v_i$ for all $1 \leq i \leq 3$. Now, we apply the mapping cone construction to the short exact sequence (1). Let $(\mathbf{G}., d^{\mathbf{G}}.)$ be a minimal free resolution of $[S/(J_{G\setminus e} : f_e)](-2)$. Then $G_1 \simeq S^{|E(G)|-1+\binom{\deg_G(v)-1}{2}}$. Also, let $(\mathbf{F}., d^{\mathbf{F}}.)$ be a minimal free resolution of $S/J_{G\setminus e}$. Then, $F_1 \simeq S^{|E(G)|-1}$ and $F_2 \simeq S^{\beta_2(S/J_{G\setminus e})}$. By Theorem 3.3, $\beta_2(S/J_{G\setminus e}) = 2 + \beta_{2,4}(S/J_{G\setminus e})$, where $\beta_{2,4}(S/J_{G\setminus e}) = \binom{n-1}{2} + \sum_{w \in V(G)\setminus v} \binom{\deg_G(w)}{3} + \binom{\deg_G(v)-1}{3}$. Set $\mathcal{S}_1 = \{E_{\{i,j\}} : \{i,j\} \in E(G \setminus u)\}$ and $\mathcal{S}_2 = \{E_{\{i,j\}} : i,j \in N_G(v) \setminus u\}$. Then, $|\mathcal{S}_1| = |E(G \setminus e)| = n-1$ and $|\mathcal{S}_2| = |E((G \setminus e)_e) \setminus E(G \setminus e)| = \binom{\deg_G(v)-1}{2}$. Let $\mathcal{S}_1 \cup \mathcal{S}_2$ denote the standard basis of G_1 and set $d_1^{\mathbf{G}}(E_{\{i,j\}}) = f_{i,j}$ for $E_{\{i,j\}} \in \mathcal{S}_1 \cup \mathcal{S}_2$. Also, let $\{e_{\{i,j\}} : \{i,j\} \in E(G \setminus u)\}$ be the standard basis of F_1. By the mapping cone construction, the map from G_0 to F_0 is multiplication by $f_{u,v}$. Define $\varphi_1 : G_1 \to F_1$ by

$$\varphi_1(E_{\{i,j\}}) = \begin{cases} f_{u,v} \cdot e_{\{i,j\}} & \text{if } E_{\{i,j\}} \in \mathcal{S}_1, \\ (-1)^{p_A(j)+p_A(u)+1} f_{i,u} e_{\{j,v\}} + (-1)^{p_A(i)+p_A(u)+1} f_{j,u} e_{\{i,v\}} & \text{if } E_{\{i,j\}} \in \mathcal{S}_2. \end{cases}$$

Then, to prove that φ_1 is a lifting map from G_1 to F_1 in the mapping cone construction, it is enough to show that the corresponding diagram commutes i.e., $d_1^{\mathbf{F}}(\varphi_1(x)) = f_{u,v} \cdot d_1^{\mathbf{G}}(x)$ for all $x \in G_1$. If $i,j \in N_G(v) \setminus u$, then $\{v,u,i,j\}$ is an induced claw with center v and it can be easily seen that

$$(-1)^{p_A(j)+p_A(u)+1} f_{i,u} f_{j,v} + (-1)^{p_A(i)+p_A(u)+1} f_{j,u} f_{i,v} - f_{i,j} f_{u,v} = 0.$$

Therefore, it follows that for $E_{\{i,j\}} \in \mathcal{S}_1 \cup \mathcal{S}_2$, $d_1^{\mathbf{F}}(\varphi_1(E_{\{i,j\}})) = f_{u,v} \cdot d_1^{\mathbf{G}}(E_{\{i,j\}})$. Hence, the mapping cone construction gives a S-free presentation of J_G, which is

$$F_2 \oplus G_1 \longrightarrow F_1 \oplus G_0 \longrightarrow J_G \longrightarrow 0. \tag{4}$$

Since $F_2 \oplus G_1 \simeq S^{\beta_2(S/J_G)}$ and $F_1 \oplus G_0 \simeq S^n$, the above presentation is a minimal one.

Case-2: Let $v = v_i$ for some $1 \leq i \leq 3$. Assume that $v = v_1$. Then, $\{v_2, v_3\} \in E((G \setminus e)_e) \cap E(G \setminus e)$. Hence, $\beta_{1,2}(S/J_{(G\setminus u)_v}) = \operatorname{rank} G_1 = (n-1) + \binom{\deg_G(v)-1}{2} - 1$. Also, it follows from Theorem 3.3 that

$$\beta_2(S/J_{G\setminus e}) = 2 + \binom{n-1}{2} + \sum_{x \in V(G)\setminus u} \binom{\deg_{G\setminus e}(x)}{3} - \sum_{i=1}^{3} \deg_{G\setminus e}(v_i) + 3.$$

Note that $\deg_G(v_1) = \deg_{G\setminus e}(v_1) + 1$ and $\deg_G(x) = \deg_{G\setminus e}(x)$ for all $x \neq u$ and $x \neq v$. Substituting these values in the above expression and taking summation with $\operatorname{rank} G_1$, we see that $\operatorname{rank} F_2 + \operatorname{rank} G_1 = \beta_2(S/J_G)$. Let $\mathcal{S}_1 = \{E_{\{i,j\}} : \{i,j\} \in E(G \setminus u)\}$ and $\mathcal{S}_2 = \left\{E_{\{i,j\}} : i,j \in N_G(v) \setminus u, \{i,j\} \neq \{v_2, v_3\}\right\}$. Define $\varphi_1 : G_1 \longrightarrow F_1$ as in **Case-1** and proceeding as in there, it can be proved that the mapping cone construction gives a minimal S-free presentation of J_G as in (4). The first syzygy is minimally generated by the images of the standard basis under the map $\Phi : F_2 \oplus G_1 \longrightarrow F_1 \oplus G_0$ which is given by the matrix $\begin{bmatrix} d_2^{\mathbf{F}} & \varphi_1 \\ 0 & -d_1^{\mathbf{G}} \end{bmatrix}$. Now, as done in the proof of Theorem 3.5, one concludes that the images under Φ are precisely the elements given in the assertion of the theorem. □

Theorem 3.7. *Let G be a unicyclic graph on $[n]$ of girth $m \geq 4$. Also, let the vertex set of the unique cycle in G be $\{1, \ldots, m\}$. Let $\{e_{\{i,j\}} : \{i,j\} \in E(G)\}$ denote the standard basis of S^n. Then, the first syzygy of J_G is minimally generated by elements of the form*

(a) $f_{i,j}e_{\{k,l\}} - f_{k,l}e_{\{i,j\}}$, *where* $\{i,j\}, \{k,l\} \in E(G)$ *and* $\{i,j\} \neq \{k,l\}$,
(b) $(-1)^{p_A(v)} f_{z,w}e_{\{u,v\}} + (-1)^{p_A(z)} f_{v,w}e_{\{u,z\}} + (-1)^{p_A(w)} f_{v,z}e_{\{u,w\}}$, *where* $A = \{u,v,w,z\} \in \mathcal{C}_G$ *with center at* u,
(c) $\sum_{k=1}^{m-1} (b_i)_k e_{\{k,k+1\}} - (b_i)_m e_{\{1,m\}}$, *where* $1 \leq i \leq m-1$, *and* b_i*'s are as defined in Theorem 3.5.*

Proof. We prove the assertion by induction on $n - m$. If $n = m$, then G is a cycle and the result follows from Theorem 3.5. Now, we assume that $n > m$. From Theorem 3.4, we know that the minimal presentation of J_G is of the form

$$S^{\beta_2(S/J_G)} \longrightarrow S^n \longrightarrow J_G \longrightarrow 0,$$

where

$$\beta_2(S/J_G) = \begin{cases} \beta_{2,4}(S/J_G) & \text{if } m = 4 \\ \beta_{2,4}(S/J_G) + \beta_{2,m}(S/J_G) & \text{if } m > 4, \text{ and} \end{cases}$$

$$\beta_{2,4}(S/J_G) = \begin{cases} \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} + 3 & \text{if } m = 4 \\ \binom{n}{2} + \sum_{v \in V(G)} \binom{\deg_G(v)}{3} & \text{if } m > 4 \end{cases} \quad \text{and } \beta_{2,m}(S/J_G) = m - 1.$$

Let $e = \{u, v\}$ be an edge in G such that u is a pendant vertex of G. Since e is a cut edge and u is a pendant vertex of G, $(G \setminus e)_e = (G \setminus u)_v \sqcup \{u\}$. Thus, $J_{G\setminus e} : f_e = J_{(G\setminus u)_v}$. Since $G \setminus e$ is also a unicyclic graph having the unique cycle C_m and $J_{G\setminus e} = J_{G\setminus u}$, by induction we get a minimal generating set of the first syzygy of $J_{G\setminus e}$ as

(a) $f_{i,j}e_{\{k,l\}} - f_{k,l}e_{\{i,j\}}$, where $\{i,j\}, \{k,l\} \in E(G \setminus e)$ and $\{i,j\} \neq \{k,l\}$,
(b) $(-1)^{p_A(j)} f_{k,l}e_{\{i,j\}} + (-1)^{p_A(k)} f_{j,l}e_{\{i,k\}} + (-1)^{p_A(l)} f_{j,k}e_{\{i,l\}}$, where $A = \{i,j,k,l\} \in \mathcal{C}_{G\setminus e}$ with center at i,
(c) $\sum_{k=1}^{m-1} (b_i)_k e_{\{k,k+1\}} - (b_i)_m e_{\{1,m\}}$, where $1 \leq i \leq m-1$.

Now, we apply the mapping cone construction to the short exact sequence (1). Let $(\mathbf{G}., d^{\mathbf{G}}.)$ and $(\mathbf{F}., d^{\mathbf{F}}.)$ be minimal free resolutions of $[S/(J_{G\setminus e} : f_e)](-2)$ and $S/J_{G\setminus e}$ respectively. Then, $G_1 \simeq S^{n-1+\binom{\deg_G(v)-1}{2}}$, $F_1 \simeq S^{n-1}$ and $F_2 \simeq S^{\beta_2(S/J_{G\setminus e})}$.

Denote the standard basis of G_1 by $\mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1 = \{E_{\{i,j\}} : \{i,j\} \in E(G \setminus e)\}$ and $\mathcal{S}_2 = \{E_{\{k,l\}} : k,l \in N_G(v) \setminus u\}$. Note that $|\mathcal{S}_1| = n - 1$ and $|\mathcal{S}_2| = \binom{\deg_G(v)-1}{2}$. Set $d_1^{\mathbf{G}}(E_{\{i,j\}}) = f_{i,j}$ for a basis element $E_{\{i,j\}}$. Also, let $\{e_{\{i,j\}} : \{i,j\} \in E(G \setminus e)\}$ be the standard basis of F_1. By the mapping cone construction, the map from G_0 to F_0 is given by the multiplication by f_e. Now, we define φ_1 from G_1 to F_1 by $\varphi_1(E_{\{i,j\}}) = f_e \cdot e_{\{i,j\}}$ for $E_{\{i,j\}} \in \mathcal{S}_1$ and $\varphi_1(E_{\{k,l\}}) = (-1)^{p_A(k)+p_A(u)+1} f_{u,l} e_{\{v,k\}} + (-1)^{p_A(l)+p_A(u)+1} f_{u,k} e_{\{v,l\}}$ for $E_{\{k,l\}} \in \mathcal{S}_2$. We need to prove that $d_1^{\mathbf{F}}(\varphi_1(x)) = f_e \cdot d_1^{\mathbf{G}}(x)$ for any element $x \in G_1$. For a claw $\{v,u,k,l\}$ with center at v, we have the relation $(-1)^{p_A(k)+p_A(u)+1} f_{u,l} f_{v,k} + (-1)^{p_A(l)+p_A(u)+1} f_{u,k} f_{v,l} = f_{k,l} f_{u,v}$. This yields us the equality $d_1^{\mathbf{F}}(\varphi_1(E_{\{i,j\}})) = f_{u,v} \cdot d_1^{\mathbf{G}}(E_{\{i,j\}})$ for a basis $E_{\{i,j\}}$ of G_1. So the mapping cone construction gives us a S-free presentation of J_G as

$$F_2 \oplus G_1 \longrightarrow F_1 \oplus G_0 \longrightarrow F_0 \longrightarrow J_G \longrightarrow 0.$$

Since $F_2 \oplus G_1 \simeq S^{\beta_2(S/J_G)}$ and $F_1 \oplus G_0 \simeq S^n$, this is a minimal free presentation. Hence, the first syzygy of J_G is minimally generated by the images of basis elements under the map $\Phi : F_2 \oplus G_1 \longrightarrow F_1 \oplus G_0$. Now, the assertion can be proved just as done in the proof of Theorem 3.5. $\square$

If $e = \{u,v\}$ is a cut-edge in G such that both u and v are simplicial vertices, then the mapping cone construction on the exact sequence (1) gives a minimal free resolution of S/J_G, [14, Proposition 3.2]. However, this is not a necessary condition as we see below.

Proposition 3.8. *Let $n \geq 3$. Then, the minimal free resolution of $S/J_{K_{1,n}}$ is given by the mapping cone of $S/J_{K_{1,n-1}}$ and $S/J_{K_n}(-2)$.*

Proof. Let $V(K_{1,n}) = \{1,\ldots,n,n+1\}$ with $E(K_{1,n}) = \{\{i,n+1\} : 1 \leq i \leq n\}$. For $G = K_{1,n}$ and $e = \{n,n+1\}$, note that $J_{G\setminus e} = J_{K_{1,n-1}}$ and $J_{K_{1,n-1}} : f_e = J_{K_n}$. Since $K_{1,n}$ is a tree, it follows from [6, Theorem 1.1] that $\operatorname{pd}(S/J_{K_{1,n}}) = n$. Also, by [24, Corollary 2.3], $\beta_{i,i+1}(S/J_{K_{1,n}}) = 0$ for $2 \leq i \leq n$. Since $\operatorname{reg}(S/J_{K_{1,n}}) = 2$, [25], and $\operatorname{reg}(S/J_{K_n}) = 1$, [24], $\beta_{i,i+j}(S/J_{K_{1,n}}) = 0$ for $j \neq 2$ and $\beta_{i,i+j}(S/J_{K_n}) = 0$ for $j \neq 1$. Corresponding to (1), we have the long exact sequence for all $j \geq 1$,

$$\cdots \to \operatorname{Tor}^S_{i,i+j}\left(\frac{S}{J_{K_{1,n-1}}}, \mathbb{K}\right) \to \operatorname{Tor}^S_{i,i+j}\left(\frac{S}{J_{K_{1,n}}}, \mathbb{K}\right) \to \operatorname{Tor}^S_{i-1,i+j-2}\left(\frac{S}{J_{K_n}}, \mathbb{K}\right) \to \cdots.$$

Hence, $\beta_{i,j}(S/J_{K_{1,n}}) = \beta_{i,j}(S/J_{K_{1,n-1}}) + \beta_{i-1,j-2}(S/J_{K_n})$. If $\mathbf{G}.$ denotes a minimal free resolution of $S/J_{K_n}(-2)$ and $\mathbf{F}.$ denotes a minimal free resolution of $S/J_{K_{1,n-1}}$, then the above equality implies that $\beta_i(S/J_{K_{1,n}}) = \operatorname{rank} F_i + \operatorname{rank} G_{i-1}$. Hence, the mapping cone gives a minimal free resolution of $S/J_{K_{1,n}}$. $\square$

4. Rees Algebra

Let G be a graph on $[n]$ and J_G be its binomial edge ideal. Let $R = S[T_{\{i,j\}} : \{i,j\} \in E(G)$ with $i < j]$. Let $\delta : R \to S[t]$ be the S-algebra homomorphism given by $\delta(T_{\{i,j\}}) = f_{i,j}t$. Then, $\operatorname{Im}(\delta) = \mathcal{R}(J_G)$ and $\ker(\delta)$ is called the defining ideal of $\mathcal{R}(J_G)$. We first characterize graphs whose binomial edge ideals are almost complete intersection. We begin by proving couple of simple lemmas which are useful for our main results.

Lemma 4.1. *Let I be a radical ideal in a Noetherian commutative ring A. Then, for any $f \in A$ and $n \geq 2$, $I : f = I : f^n$.*

Proof. Let $f \in A$ be an element. Observe that for any $n \geq 2$, $I : f \subset I : f^n$. Let $g \in I : f^n$. Then, $gf^n \in I$ which implies that $g^n f^n \in I$. Therefore, $gf \in \sqrt{I} = I$. Hence, $g \in I : f$. $\square$

Lemma 4.2. *If $I \subseteq A = \mathbb{K}[t_1,\ldots,t_n]$ is a homogeneous ideal such that $I = J + (a)$, where J is generated by a homogeneous regular sequence, a is a homogeneous element and $J : a = J : a^2$, then I is either a complete intersection or an almost complete intersection.*

Proof. The proof for Theorem 4.7(ii) in [8] is for the local case for the same statement, but it can be easily seen that it goes through for homogeneous ideals in A. □

We first characterize the trees whose binomial edge ideals are almost complete intersections.

Theorem 4.3. *If G is a tree which is not a path, then J_G is an almost complete intersection ideal if and only if G is obtained by adding an edge between two vertices of two paths.*

Proof. Suppose G is obtained by adding an edge e between paths P_{n_1} and P_{n_2}. Then, $J_{G\setminus e}$ is a complete intersection ideal and $J_G = J_{G\setminus e} + f_e S$. By Theorem 2.1(a), and Lemma 4.1, we get $J_{G\setminus e} : f_e^2 = J_{G\setminus e} : f_e$. Therefore, it follows from Lemma 4.2 that J_G is an almost complete intersection.

Now, assume that G is not a graph obtained by adding an edge between two paths. Therefore, either there exists a vertex v such that $\deg_G(v) \geq 4$ or there exist $z, w \in V(G)$ such that $\deg_G(z) \geq 3$, $\deg_G(w) \geq 3$ and $\{z, w\} \notin E(G)$. Let $T = \{v\}$ in the first case and $T = \{z, w\}$ in the second case. By Theorem 2.1, $\operatorname{ht}(P_T(G)) = n - c_T + |T|$. Since z and w are of degrees at least 3, $\{z, w\} \notin E(G)$ and G is a tree, $c_T \geq 5$. Hence, $\operatorname{ht}(P_T(G)) \leq n-3$. Now, if $T = \{v\}$, then $c_T \geq 4$ so that $\operatorname{ht}(P_T(G)) \leq n-3$. Note that in both cases T has the cut point property so that $P_T(G)$ is a minimal prime, by Theorem 2.1. Thus, $\operatorname{ht}(J_G) \leq n-3$. Since $\mu(J_G) = n-1$, $\mu(J_G) > \operatorname{ht}(J_G)+1$. Hence, J_G is not an almost complete intersection ideal. □

Now, we have characterized the almost complete intersection trees, we move on to graphs containing cycles.

Theorem 4.4. *Let G be a connected graph on $[n]$ which is not a tree. Then, J_G is an almost complete intersection ideal if and only if G is obtained by adding an edge between two vertices of a path or by attaching a path to each vertex of C_3.*

Proof. First assume that J_G is an almost complete intersection ideal. Therefore, $\mu(J_G) = \operatorname{ht}(J_G)+1$. Since $\operatorname{ht}(J_G) \leq n-1$, it follows that $\mu(J_G) \leq n$. Since G is not a tree, we have $\mu(J_G) = n$. Therefore, G is a unicyclic graph and $\operatorname{ht}(J_G) = n-1$. Let u be a vertex which does not belong to the unique cycle in G. If $\deg_G(u) \geq 3$, then for $T = \{u\}$, by Theorem 2.1(d), $P_T(G)$ is a minimal prime of J_G of height $\leq n-2$ which contradicts the fact that $\operatorname{ht}(J_G) = n-1$. Hence, $\deg_G(u) \leq 2$. Now, we claim that $\deg_G(u) \leq 3$, for every u belonging to vertex set of the unique cycle in G. If $\deg_G(u) \geq 4$ for such a vertex u, then $G \setminus u$ has at least three components so that for $T = \{u\}$, $P_T(G)$ is a minimal prime of J_G of height $\leq n-2$ which is a contradiction. Hence, $\deg_G(u) \leq 3$. If the girth of G is 3, then clearly it belongs to one of the categories described in the theorem. We now assume that girth of G is ≥ 4. Suppose u, v be two vertices of the unique cycle in G with $\deg_G(u) = 3$ and $\deg_G(v) = 3$. If $\{u, v\} \notin E(G)$, then for $T = \{u, v\}$, $P_T(G)$ is a minimal prime of J_G of height $\leq n-2$ which is again a contradiction. Therefore, $\{u, v\} \in E(G)$. Thus, the number of vertices of the cycle having degree three is at most 2 and if two vertices of the cycle have degree three, then they are adjacent. Therefore, G is obtained by adding an edge between two vertices of a path.

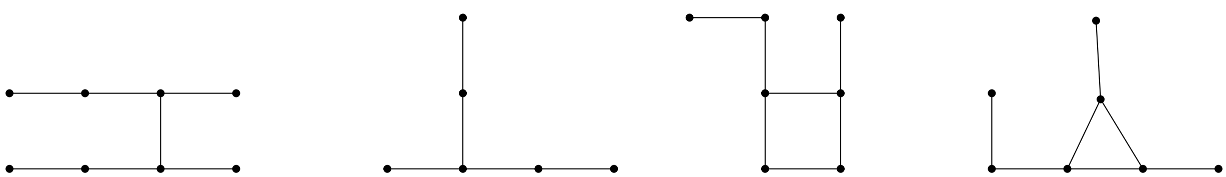
Now assume that G is a graph obtained by adding an edge between two vertices, say u and v, of a path. Let $e = \{u, v\}$. Observe that $J_{G\setminus e}$ is a complete intersection ideal. By Theorem 2.1(a) and Lemma 4.1, $J_{G\setminus e} : f_e^2 = J_{G\setminus e} : f_e$. Thus, it follows from Lemma 4.2 that J_G is an almost complete intersection ideal.

Now, suppose G is a graph obtained by adding a path to each of the vertices of a C_3. Then, by [6, Theorem 1.1], S/J_G is Cohen-Macaulay of dimension $n+1$. Therefore, $\operatorname{ht}(J_G) = n-1 = \mu(J_G)-1$. Now, we have to prove that if $\mathfrak{p}$ is a minimal prime of J_G, then $(J_G)_{\mathfrak{p}}$ is a complete intersection ideal of $S_{\mathfrak{p}}$, i.e. $\mu((J_G)_{\mathfrak{p}}) = \operatorname{ht}((J_G)_{\mathfrak{p}}) = n-1$. Let $\mathfrak{p}$ be a minimal prime of J_G. It follows from [10, Corollary 3.9] that there exists $T \subset [n]$ having cut point property such that $\mathfrak{p} = P_T(G)$. By Theorem 3.3, the minimal presentation of J_G is

$$S(-4)^{\beta_{2,4}(S/J_G)} \oplus S(-3)^{\beta_{2,3}(S/J_G)} \xrightarrow{\varphi} S(-2)^n \longrightarrow J_G \longrightarrow 0.$$

Moreover, the linear relations given in Theorem 3.6(a) show that $(x_{v_1}, y_{v_1}, x_{v_2}, y_{v_2}, x_{v_3}, y_{v_3}) \subset I_1(\varphi)$, the ideal generated by the entries of the matrix of φ. Now, if $I_1(\varphi) \subset \mathfrak{p}$, then $(x_{v_1}, y_{v_1}, x_{v_2}, y_{v_2}, x_{v_3}, y_{v_3}) \subset \mathfrak{p}$. Thus, $\{v_1, v_2, v_3\} \subset T$, which is a contradiction to the fact that T has the cut point property. Therefore, $I_1(\varphi) \not\subset \mathfrak{p}$, and hence, by [2, Lemma 1.4.8], $\mu((J_G)_{\mathfrak{p}}) \leq n-1$. If $\mu((J_G)_{\mathfrak{p}}) < n-1$, then by [18, Theorem 13.5], $\operatorname{ht}(\mathfrak{p}) < n-1$, which is a contradiction. Thus, $\mu((J_G)_{\mathfrak{p}}) = n-1$. Hence, J_G is an almost complete intersection ideal. $\square$

Below, we give representatives of four different types of graphs whose binomial edge ideals are almost complete intersection ideals.

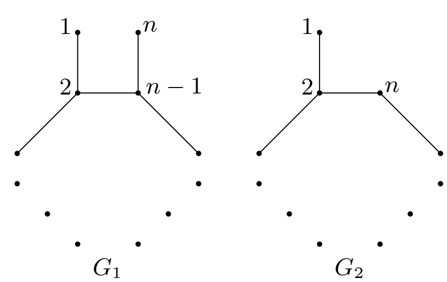


We now study the Rees algebra of almost complete intersection binomial edge ideals. We prove that they are Cohen-Macaulay and we also obtain the defining ideals of these Rees algebras. We first recall a result which characterizes the Cohen-Macaulayness of the Rees algebra and the associated graded ring.

Theorem 4.5. [9, Corollary 1.8] *Let A be a Cohen-Macaulay local (graded) ring and $I \subset A$ be a (homogeneous) almost complete intersection ideal in A. Then,*
(a) $\operatorname{gr}_A(I)$ *is Cohen-Macaulay if and only if* $\operatorname{depth}(A/I) \geq \dim(A/I) - 1$.
(b) $\mathcal{R}(I)$ *is Cohen-Macaulay if and only if* $\operatorname{ht}(I) > 0$ *and* $\operatorname{gr}_A(I)$ *is Cohen-Macaulay.*

Therefore, in our situation, to prove that $\mathcal{R}(J_G)$ is Cohen-Macaulay, it is enough to prove that $\operatorname{depth}(S/J_G) \geq \dim(S/J_G) - 1$.

4.1. **Discussion.** Suppose G is a unicyclic graph such that J_G is almost complete intersection. We may assume that G is not a cycle. If girth of G is 3, then by Theorem 4.4 and [6, Theorem 1.1], S/J_G is Cohen-Macaulay. Thus, $\operatorname{gr}_S(J_G)$ is Cohen-Macaulay, and hence, so is $\mathcal{R}(J_G)$. Now, we assume that girth of G is at least 4 and $n \geq 5$.
Let G_1 and G_2 denote graphs on the vertex set $[n]$ with edge sets given by $E(G_1) = \{\{1,2\},\{2,3\},\ldots,\{n-2,n-1\},\{n-1,n\},\{2,n-1\}\}$ and $E(G_2) = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{2,n\}\}$. If G is a unicyclic graph on $[k]$, $k \geq 5$, which is not a cycle and having an almost complete intersection binomial edge ideal, then by Theorem 4.4, G is obtained by attaching a path to each of the pendant vertices of G_1 or G_2.



Let G denote the graph obtained by identifying the vertex 1 of G_i and a pendant vertex of P_m. Then, by [22, Theorem 2.7], $\operatorname{depth}(S/J_G) = \operatorname{depth}(S_i/J_{G_i}) + \operatorname{depth}(S_P/J_{P_m}) - 2$, where S_i denotes the polynomial ring corresponding to the graph G_i and S_P denotes the polynomial ring corresponding to the graph P_m. Since J_{P_m} is generated by a regular sequence of length $m-1$, $\operatorname{depth}(S_P/J_{P_m}) = m+1$. Also $\dim(S/J_G) = n+m$. Therefore, to prove that $\operatorname{depth}(S/J_G) \geq n+m-1$, it is enough to prove that $\operatorname{depth}(S_i/J_{G_i}) \geq n$. Similarly, if G is obtained by attaching a path to each of the pendant vertices of G_1, then to prove $\operatorname{depth}(S/J_G) \geq \dim(S/J_G)-1$, it is enough to prove that $\operatorname{depth}(S_1/J_{G_1}) \geq \dim(S_1/J_{G_1})-1$. We now proceed to prove this.

Let G be a graph on $[n]$ with binomial edge ideal $J_G \subset S = \mathbb{K}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. We consider S with lexicographical order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. It follows from [10, Theorem 2.1] that $\operatorname{in}_<(J_G)$ is a squarefree monomial ideal so that by [3, Corollary 2.7], we get $\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G))$. Hence, to compute $\operatorname{depth}(S/J_G)$, we compute the depth of $S/\operatorname{in}_<(J_G)$.

Now, consider the graphs G_1 and G_2 as defined above. It follows from the labeling of the vertices of G_1 that the admissible paths in G_1 are the edges and the paths of the form $i, i-1, \ldots, 3, 2, n-1, n-2, \ldots, j$ with $2 \leq j-i \leq n-4$, [10, Section 2]. Similarly the admissible paths in G_2 are the edges and the paths of the form $i, i-1, \ldots, 3, 2, n, n-1, \ldots, j$ with $2 \leq j-i \leq n-3$. Consequently, the corresponding initial ideals are given by

$$\operatorname{in}_<(J_{G_1}) = \Big(\{x_1y_2,\ldots,x_{n-1}y_n, x_2y_{n-1}, x_ix_{j+1}\cdots x_{n-1}y_2\cdots y_{i-1}y_j : 2 \leq j-i \leq n-4\}\Big) \text{ and}$$

$$\operatorname{in}_<(J_{G_2}) = \Big(\{x_1y_2,\ldots,x_{n-1}y_n, x_2y_n, x_ix_{j+1}\cdots x_ny_2\cdots y_{i-1}y_j : 2 \leq j-i \leq n-3\}\Big).$$

We denote these monomials of degree ≥ 3 by $v_1,\ldots,v_p$. We order these monomials such that $i<j$ if either $\deg v_i < \deg v_j$ or $\deg v_i = \deg v_j$ and $v_i >_{lex} v_j$. Set $J = (x_1y_2,\ldots,x_{n-1}y_n)$, $I_0(G_1) = J + (x_2y_{n-1})$, $I_0(G_2) = J + (x_2y_n)$ and, for $1 \leq k \leq p$, $I_k(G_i) = I_{k-1}(G_i) + (v_k)$ for $i=1,2$. Then $I_p(G_i) = \operatorname{in}_<(J_{G_i})$ for $i=1,2$. We now compute the projective dimension, equivalently depth, of these ideals.

Lemma 4.6. *For $0 \leq k \leq p$ and $i = 1,2$, $\operatorname{pd}(S/I_k(G_i)) \leq n$.*

Proof. We prove the assertion by induction on k. If $k=0$, then consider the following exact sequences:

$$0 \longrightarrow \frac{S}{J:(x_2y_{n-1})}(-2) \xrightarrow{\cdot x_2y_{n-1}} \frac{S}{J} \longrightarrow \frac{S}{I_0(G_1)} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{S}{J:(x_2y_n)}(-2) \xrightarrow{\cdot x_2y_n} \frac{S}{J} \longrightarrow \frac{S}{I_0(G_2)} \longrightarrow 0.$$

Note that J is generated by a regular sequence of length $n-1$. Moreover

$$\begin{aligned} J : x_2y_{n-1} &= (x_1y_2, y_3, x_3y_4, \ldots, x_{n-3}y_{n-2}, x_{n-2}, x_{n-1}y_n) \text{ and} \\ J : x_2y_n &= (x_1y_2, y_3, x_3y_4, \ldots, x_{n-3}y_{n-2}, x_{n-2}y_{n-1}, x_{n-1}) \end{aligned}$$

which are generated by regular sequences of length $n-1$. Therefore

$$\operatorname{pd}(S/J) = \operatorname{pd}(S/(J:x_2y_{n-1})) = \operatorname{pd}(S/(J:x_2y_n)) = n-1.$$

Hence, it follows from the long exact sequence of Tor that $\operatorname{pd}(S/I_0(G_i)) \leq n$ for $i=1,2$. Now, assume that $k>0$ and $\operatorname{pd}(S/I_{k-1}(G_i)) \leq n$ for $i=1,2$. For $i=1,2$, consider the short exact sequences

$$0 \longrightarrow \frac{S}{I_{k-1}(G_i) : (v_k)}(-\deg v_k) \xrightarrow{\cdot v_k} \frac{S}{I_{k-1}(G_i)} \longrightarrow \frac{S}{I_k(G_i)} \longrightarrow 0. \tag{5}$$

We first prove the assertion for G_1. It can be seen that the monomials v_k's are of the form

$$v_k = \begin{cases} x_2x_{j+1}\cdots x_{n-1}y_j & \text{for } 4 \leq j \leq n-2, \\ x_iy_2\cdots y_{i-1}y_{n-1} & \text{for } 3 \leq i \leq n-3, \\ x_ix_{j+1}\cdots x_{n-1}y_2\cdots y_{i-1}y_j & \text{for } 3 \leq i; j \leq n-2 \text{ and } 2 \leq j-i. \end{cases}$$

If $v_i = x_2x_{j+1}\cdots x_{n-1}y_j$ for some $4 \leq j \leq n-2$, then

$$\begin{aligned} I_{k-1}(G_1) : v_k &= (I_0(G_1) : v_k) + (v_1, \dots, v_{k-1}) : v_k \\ &= (x_1y_2, x_3y_4, \dots, x_{j-2}y_{j-1}, x_jy_{j+1}, x_{j-1}, y_3, y_{j+2}, \dots, y_n) + (v_1, \dots, v_{k-1}) : v_k. \end{aligned}$$

It can be seen that $(v_1, \dots, v_{k-1}) : v_k \subseteq (I_0(G_1) : v_k) + (y_{j+1})$ and $y_{j+1}v_k \in (v_1, \dots, v_{k-1})$. Hence,

$$I_{k-1}(G_1) : v_k = (x_1y_2, x_3y_4, \dots, x_{j-2}y_{j-1}, x_{j-1}, y_3, y_{j+1}, \dots, y_n).$$

This is a regular sequence of length $n-1$. The proof that $I_{k-1}(G_1) : v_k$ is generated by a regular sequence of length $n-1$ if v_k is of the other two types is similar. Therefore $\operatorname{pd}(S/(I_{k-1}(G_1) : v_k)) = n-1$. Hence, it follows from the short exact sequence (5) that $\operatorname{pd}(S/I_k(G_1)) \leq n$.

In a similar manner, using the short exact sequence (5) and the colon ideal, one can prove that $\operatorname{pd}(S/I_k(G_2)) \leq n$. □

We now show that the associated graded ring and the Rees algebra of almost complete intersections binomial edge ideals are Cohen-Macaulay.

Theorem 4.7. *If G is a graph such that J_G is an almost complete intersection ideal, then $\operatorname{gr}_S(J_G)$ and $\mathcal{R}(J_G)$ are Cohen-Macaulay.*

Proof. Suppose J_G is an almost complete intersection ideal. By Theorem 4.5(b), it is enough to prove that $\operatorname{gr}_S(J_G)$ is Cohen-Macaulay, if one wants to prove that $\mathcal{R}(J_G)$ is Cohen-Macaulay. Now, $\operatorname{gr}_S(J_G)$ is Cohen-Macaulay if $\operatorname{depth}(S/J_G) \geq \dim(S/J_G) - 1$, by Theorem 4.5(a). If G is a tree, then it follows from [6, Theorem 1.1] and Theorem 4.3 that $\operatorname{depth}(S/J_G) = n+1 = \dim(S/J_G) - 1$. If $G = C_n$, then it follows from [28, Theorem 4.5] that $\operatorname{depth}(S/J_{C_n}) = \dim(S/J_{C_n}) - 1$. Now, we assume that G is a unicyclic graph other than cycle. It follows from Discussion 4.1 that it is enough to prove that $\operatorname{depth}(S_i/J_{G_i}) \geq n$ for $i = 1, 2$. From [3, Corollary 2.7], we get $\operatorname{depth}(S_i/J_{G_i}) = \operatorname{depth}(S_i/\operatorname{in}_>(J_{G_i}))$. It follows from Lemma 4.6 that $\operatorname{depth}(S_i/\operatorname{in}_>(J_{G_i})) = \operatorname{depth}(S_i/I_p(G_i)) \geq n$. This completes the proof. □

We now study binomial edge ideals which are of linear type. Since complete intersections are of linear type, binomial edge ideals of paths are of linear type. Now, we show that the $J_{K_{1,n}}$ is of linear type. For this purpose, recall the definition of d-sequence.

Definition 4.8. *Let A be a commutative ring. Set $d_0 = 0$. A sequence of elements $d_1, \dots, d_n$ is said to be a d-sequence if $(d_0, d_1, \dots, d_i) : d_{i+1}d_j = (d_0, d_1, \dots, d_i) : d_j$ for all $0 \leq i \leq n-1$ and for all $j \geq i+1$.*

We refer the reader to the book [12] by Swanson and Huneke for more properties of d-sequences.

Proposition 4.9. *The binomial edge ideal of $K_{1,n}$ is of linear type.*

Proof. Let $K_{1,n}$ denote the graph on $[n+1]$ with the edge set $\{\{i, n+1\} \ : \ 1 \leq i \leq n\}$. We claim that $J_{K_{1,n}}$ is generated by the d-sequence $d_1, d_2, \ldots, d_n$, where $d_i = x_i y_{n+1} - x_{n+1} y_i$. Let $1 \leq i \leq n-1$ and K_{i+1} denote the complete graph on the vertex set $\{1, \ldots, i, n+1\}$. Then, for $j \geq i+1$,

$$(d_0, d_1, \ldots, d_i) : d_{i+1} d_j = ((d_0, d_1, \ldots, d_i) : d_{i+1}) : d_j = J_{K_{i+1}} : d_j = J_{K_{i+1}},$$

also $(d_0, d_1, \ldots, d_i) : d_j = J_{K_{i+1}}$, where the last two equalities follow from [19, Theorem 3.7]. Therefore, $J_{K_{1,n}}$ is generated by a d-sequence. Hence, by [12, Corollary 5.5.5], $J_{K_{1,n}}$ is of linear type. □

We now prove that in the polynomial ring over an infinite field, almost complete intersection homogeneous ideals are generated by d-sequences.

Proposition 4.10. *If $I \subset A = \mathbb{K}[t_1, \ldots, t_n]$ is a homogeneous almost complete intersection, where $\mathbb{K}$ is infinite, then I is generated by a homogeneous d-sequence $f_1, \ldots, f_{h+1}$ such that $f_1, \ldots, f_h$ is a regular sequence, where $h = \operatorname{ht}(I)$.*

Proof. Since I is an almost complete intersection ideal, by [4, Proposition 5.1(i)], there exists a homogeneous system of generators $\{f_1, \ldots, f_{h+1}\}$ of I such that $f_1, \ldots, f_h$ is a regular sequence. Let $J = (f_1, \ldots, f_h)$. Since A is regular, J is unmixed. It follows from [4, Proposition 5.1(ii)] and the proof of [8, Theorem 4.7] that $J : f_{h+1} = J : f_{h+1}^2$. Therefore, $f_1, \ldots, f_{h+1}$ is a homogeneous d-sequence. □

In the above Lemma, the assumption that $\mathbb{K}$ is infinite is required in Proposition 5.1 of [4]. We assume that $\mathbb{K}$ is infinite for the following result as well.

Corollary 4.11. *Let G be a graph on $[n]$. If J_G is an almost complete intersection ideal, then J_G is generated by a d-sequence. In particular, J_G is of linear type.*

Proof. If J_G is an almost complete intersection, then it follows from Proposition 4.10 that J_G is generated by a d-sequence. The second assertion that J_G is of linear type is a consequence of [11, Theorem 3.1]. □

If G is a tree or a unicyclic graph of girth ≥ 4 such that J_G is an almost complete intersection, then one can show that the minimal generators consisting of the binomials corresponding to the edges of G form a d-sequence.

Remark 4.12. Suppose G is a tree such that J_G is almost complete intersection. Then, by Theorem 4.3, G is obtained by adding an edge between two paths, say P_{n_1} and P_{n_2}. Let e denote the edge between P_{n_1} and P_{n_2}. Note that $G \setminus e$ is the disjoint union of two paths. Assume now that G is a unicyclic graph with unique cycle C_m, $m \geq 4$, such that J_G is almost complete intersection. Then, by Theorem 4.4, G is obtained by adding an edge e between two vertices of a path. Thus, in both the cases, $J_{G\setminus e}$ is complete intersection, by [6, Corollary 1.2]. Since $J_{G\setminus e}$ is a radical ideal, by Lemma 4.1, $J_{G\setminus e} : f_e^2 = J_{G\setminus e} : f_e$. Hence, J_G is generated by a d-sequence. It may also be observed that we do not require the assumption that $\mathbb{K}$ is infinite in this case.

If G is obtained by adding a path each to the vertices of a C_3, then, it can be seen that $J_{G\setminus e}$ is not a complete intersection for any edge $e \in E(G)$. Thus, the binomials corresponding to the edges of G do not form a d-sequence with first $n-1$ of them forming a regular sequence.

But at the same time, Proposition 4.10 ensures the existence of such a generating set. We have not been able to explicitly construct one such.

As a consequence of Remark 4.12, we obtain the defining ideal of the Rees algebra of binomial edge ideals of cycles.

Corollary 4.13. *Let $\varphi : S[T_{\{1,n\}}, T_{\{i,i+1\}} : i = 1, \dots, n-1] \longrightarrow \mathcal{R}(J_{C_n})$ be the map defined by $\varphi(T_{\{i,j\}}) = f_{i,j}t$. The defining ideal of $\mathcal{R}(J_{C_n})$, the kernel of φ, is minimally generated by*

$$\left\{f_{i,j}T_{\{k,l\}} - f_{k,l}T_{\{i,j\}} : \{i,j\} \neq \{k,l\} \in E(G)\right\} \cup \left\{\sum_{k=1}^{n-1}(b_i)_k T_{\{k,k+1\}} - (b_i)_n T_{\{1,n\}} : 1 \leq i \leq n-1\right\},$$

where b_i's are as defined in Theorem 3.5.

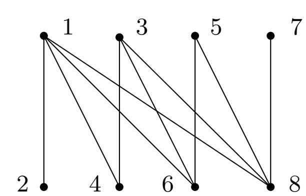
Proof. Let

$$S(-4)^{\binom{n}{2}} \oplus S(-n)^{n-1} \xrightarrow{\phi} S(-2)^n \longrightarrow J_{C_n} \longrightarrow 0$$

be the minimal presentation of J_{C_n} given in the proof of Theorem 3.5. Since J_{C_n} is of linear type (Remark 4.12), it follows from [12, Exercise 5.23] that the defining ideal of $\mathcal{R}(J_{C_n})$ is generated by TA, where A is the matrix of ϕ and $T = [T_{\{1,2\}}, \dots, T_{\{n-1,n\}}, T_{\{1,n\}}]$. Hence, the assertion follows directly from Theorem 3.5. $\square$

Remark 4.14. Suppose G is a unicyclic graph of girth $m \geq 4$ or a tree. If J_G is almost complete intersection, then by Remark 4.12, J_G is of linear type. Therefore, as in Corollary 4.13, we can conclude that the defining ideal of $\mathcal{R}(J_G)$ is generated by TA, where T is the matrix consisting of variables and A is the matrix of the presentation of J_G. Hence, we obtain a minimal set of generators for the defining ideal of $\mathcal{R}(J_G)$ by replacing the $e_{\{i,j\}}$'s by $T_{\{i,j\}}$'s in the list of generators given in the statements in Theorems 3.2, 3.7. In a similar manner, using Proposition 4.9 and using a minimal presentation of $J_{K_{1,n}}$, one can obtain the minimal generators of the defining ideal of the Rees algebra, $\mathcal{R}(J_{K_{1,n}})$. If $\mathbb{K}$ is infinite, then one can derive similar conclusions for unicyclic graphs of girth 3 as well.

Remark 4.15. We have shown that if G is a tree with an almost complete intersection binomial edge ideal J_G, then J_G is of linear type. It would be interesting to know whether binomial edge ideals of trees, or more generally all bipartite graphs, are of of linear type. Here we give an example to show that J_G need not be of linear type for all bipartite graphs. Let G be the graph as given on the right. Then, it can be seen (for example, using Macaulay 2 [7]) that the defining ideal of J_G is not of linear type. If $\delta : S[T_{\{i,j\}} : \{i,j\} \in E(G)] \longrightarrow \mathcal{R}(J_G)$ is the map given by $\delta(T_{\{i,j\}}) = f_{i,j}t$, then $x_8T_{\{1,6\}}T_{\{3,4\}} - x_6T_{\{1,8\}}T_{\{3,4\}} + x_8T_{\{1,4\}}T_{\{3,6\}} - x_4T_{\{1,8\}}T_{\{3,6\}} - x_6T_{\{1,4\}}T_{\{3,8\}} + x_4T_{\{1,6\}}T_{\{3,8\}}$ is a minimal generator of $\ker(\delta)$.



It will be interesting to obtain an answer to:

Question 4.16. *Classify all bipartite graphs whose binomial edge ideals are of linear type.*

Note that the above bipartite graph is not a tree. We have enough experimental evidence to pose the following conjecture:

Conjecture 4.17. *(a) If G is a tree or a unicyclic graph, then J_G is of linear type.*
(b) $\mathcal{R}_s(J_{C_n}) = \mathcal{R}(J_{C_n})$, where $\mathcal{R}_s(J_{C_n})$ denote the symbolic Rees algebra of J_{C_n}.

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