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A review of infinite matrices and their applications

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ABSTRACT

Infinite matrices, the forerunner and a main constituent of many branches of classical mathematics (infinite quadratic forms, integral equations, differential equations, etc.) and of the modern operator theory, is revisited to demonstrate its deep influence on the development of many branches of mathematics, classical and modern, replete with applications. This review does not claim to be exhaustive, but attempts to present research by the authors in a variety of applications. These include the theory of infinite and related finite matrices, such as sections or truncations and their relationship to the linear operator theory on separable and sequence spaces. Matrices considered here have special structures like diagonal dominance, tridiagonal, sign distributions, etc. and are frequently nonsingular. Moreover, diagonally dominant finite and infinite matrices occur largely in numerical solutions of elliptic partial differential equations. The main focus is the theoretical and computational aspects concerning infinite linear algebraic and differential systems, using techniques like conformal mapping, iterations, truncations, etc. to derive estimates based solutions. Particular attention is paid to computable precise error estimates, and explicit lower and upper bounds. Topics include Bessel's, Mathieu equations, viscous fluid flow, simply and doubly connected regions, digital dynamics, eigenvalues of the Laplacian, etc. Also presented are results in generalized inverses and semi-infinite linear programming.

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1. Introduction

In mathematical formulation of physical problems and their solutions, infinite matrices arise more naturally than finite matrices. In the past, infinite matrices played an important role in the theory of summability of sequences and series. The book "Infinite Matrices," by Cooke [9] is perhaps the first one to deal with infinite matrices. It also gives a clear indication of the role of infinite matrices as applied to topics like quantum mechanics, spectral theory and linear operators in the context of functional abstract Hilbert spaces. An excellent detailed account of the colorful history of infinite matrices is given by Bernkopf [6]. Infinite matrices and determinants were introduced into analysis by Poincare in 1884 in the discussion of the well known Hill's equation. In 1906, Hilbert used infinite quadratic forms (which are equivalent to infinite matrices) to discuss the solutions of Fredholm integral equations. Within a few years, many theorems fundamental to the theory of abstract operators on function spaces were discovered although they were expressed in special matrix terms. In 1929, John von Neumann showed that an abstract approach was powerful and preferable to using infinite matrices as a tool for the study of operators. The tools of functional analysis establish in a limited number of cases, qualitative results such as existence, uniqueness and in some cases justification of truncation, but the results are of generally little use in deriving explicit, computable and meaningful error bounds needed for approximate or computational work. A main source for infinite matrices has been including solutions of ordinary and partial differential equations. In this review, we concentrate on particular structures for matrices, finite and infinite. Infinite matrices occur in numerous applications. A brief list, in addition to those presented in this review, includes inversion of Laplace Transforms, biorthogonal infinite sets, generalized moments, interpolation, Sobolev spaces, signal processing, time series, ill-posed problems and birth–death problems. In particular, we deal with diagonally dominant matrices (full and tridiagonal), sign distributions, etc. The advantage of such matrices lies in knowing estimates for the inverse elements replete with an error analysis which also includes lower and upper bounds. This review reflects the interests of the authors and is intended to demonstrate, although in a limited context, the influence of the discussion on a variety of problems in applied mathematics and physics. In what follows, our main aim is to get meaningful, computable error bounds in which process, existence and uniqueness of solutions of infinite systems fall out as a bonus. Sections 2–8 consist of theoretical considerations, while Section 9 gives a variety of applications. As our analysis of various problems start with finite matrices obtained by truncating infinite matrices, Section 2, discusses nonsingularity conditions and provides some easily computable estimates for upper and lower bounds concerning the inverse and its elements. Infinite linear algebraic systems $A\tilde{x} = \tilde{b}$ are considered in Section 3, where analytical results and error estimates for the the truncated system and the infinite system are given. Section 4 discusses the linear eigenvalue problem $A\tilde{x} = \lambda\tilde{x}$ for the infinite case, where the eigenvalues are boxed in non overlapping intervals using estimates for the inverse. This enables us to calculate any eigenvalue to any required degree of accuracy. Linear differential systems of the form $d\tilde{x}/dt = A\tilde{x} + \tilde{f}$ constitute discussions in Section 5. Here also, truncations are used to establish existence, uniqueness and approximations with error analysis, under stated conditions on A, and $\tilde{f}$. Section 6 considers iterative methods for $T\tilde{x} = \tilde{v}$, where $\tilde{x}, \tilde{v}$ belong to ℓ_∞ and T is a linear bounded operator on ℓ_∞. To our knowledge, iteration techniques used here for infinite systems open up avenues for further research. A novel computational technique involving determinants and their estimates to locate eigenvalues is described in Section 7. This intricate technique is used later on when discussing solutions of Mathieu equations. Some interesting results on nonuniqueness of the Moore–Penrose inverse are given in Section 8. A number of applications are given in Section 9. These include classical conformal mapping problems, fluid and gas flow in pipes, eigenvalues of Mathieu equations, iteration techniques for Bessel's equation, vibrating membrane with a hole, differential equations on semi-infinite intervals, semi-infinite linear programming, groundwater flow problem and determination of eigenvalues of the Laplacian in an elliptic region. Other linear problems include analysis of solutions on semi-infinite intervals of the differential equation $y'' + f(x)y = 0$, with $f(x) \to \infty$ [30] and using finite difference methods [8].

2. Finite matrices and their inverses

In this section we give three different criteria for an $n \times n$ matrix $A = (a_{ij})$ to be nonsingular. We also give in these cases some estimates for the inverse elements of $A^{-1}\left(=\frac{A_{ji}}{\det A}\right)$ where A_{ji} represents the cofactor of a_{ij} and $\det A$ is the determinant of A.

(a) Diagonal dominance

Writing

$$\sigma_i |a_{ii}| = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad 0 \leqslant \sigma_i < 1, \; i = 1, 2, \ldots, n. \tag{2.1}$$

A is called diagonally row dominant (d.d.), if $\sigma_i \leqslant 1$ and strictly diagonally row dominant (s.d.d), if $\sigma_i < 1, i = 1, \ldots, n$. Similar definitions hold good for columns. There are a number of proofs showing that A is nonsingular if A is s.d.d. A is a tri-diagonal matrix denoted by $\{a_i, b_i, c_i\}$, where $b_i, i = 1, \ldots, n$ denote diagonal elements and $a_i, i = 2 \ldots, n$ denote the lower diagonal elements and $c_i, i = 1, \ldots, n-1$ denote the upper diagonal elements.

For the finite linear system

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, n \tag{2.2}$$

or

$$A\tilde{x} = \tilde{b},$$

we have by Cramer's rule,

$$\tilde{x} = A^{-1}\tilde{b}, \qquad x_j = \sum_{k=1}^{n} \frac{A_{kj}}{\det A} b_k, \quad j = 1, 2, \ldots, n. \tag{2.3}$$

There have been many generalizations of the concept of diagonal dominance and an equivalent theorem is given by Varga [43]:

Let $N = \{1, 2, \ldots, n\}$ and

$$J = \left\{ i \in N : |a_{ii}| > \sum_{j\neq i} |a_{ij}| \right\} \neq \emptyset. \tag{2.4}$$

If $J = N$, then A is strictly row d.d. and hence $\det A$ does not vanish [3]. If A is irreducible, then A is nonsingular [41].

Using (2.1) with $\sigma_i < 1, i = 1, 2, \ldots n$, we have the following results [15]:

$$\begin{aligned} &\det A \neq 0, \\ &|A_{ij}| \leqslant \sigma_j |A_{ii}|, \quad i \neq j \\ &\left|\frac{A_{ii}}{\det A}\right| \leqslant \frac{1}{|a_{ii}|(1-\sigma_i)}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2.5}$$

Setting $\alpha = \min_k\{|a_{kk}(1-\sigma_k)|\}$, Varah [42] establishes that

$$\|A^{-1}\|_\infty = \max_i \sum_{k=1}^{n} \frac{|A_{ki}|}{\det A} < \frac{1}{\alpha}. \tag{2.6}$$

Further, if $a_{ii} > 0, a_{ij} \leqslant 0, i \neq j$, then A is nonsingular and $A^{-1} \geqslant 0$ so that A is an M-matrix.

(b) A chain condition

If for the matrix $A = (a_{ij})$, an $n \times n$ diagonally dominant matrix, which satisfies (2.4), there exists, for each $i \notin J$, a sequence of nonzero elements of the form $a_{i,i_1}, a_{i,i_2}, \ldots, a_{i,i_j}$ with $j \in J$, then A is nonsingular

[24]. The proof of this result depends on an earlier result by Ky Fan, who has shown that for a matrix $A(= a_{ij})$ and an M-matrix $B(= b_{ij})$ where $|b_{ii}| \leqslant |a_{ii}|$ for all $i \in N$ and $|a_{ij}| \leqslant |b_{ij}|$ for $i \neq j$, the inequality $\det A \geqslant \det B$ holds. This result extends to cases where A is reducible and satisfies the chain condition. We will use this chain condition later in the application to digital circuit dynamics.

(c) Tridiagonal matrices

We consider here diagonally dominant tridiagonal matrices $A = \{a_i, b_i, c_i\}$ with

$$\mu_i|b_i| = |a_i| + |c_i|, \qquad 0 \leqslant \mu_i \leqslant 1, \quad i = 1, 2, \ldots, n, \quad a_i, b_i, c_i \neq 0. \tag{2.7}$$

Ostrowski [15] gave the following upper bounds for the inverse elements of a full s.d.d. $n \times n$ matrix $B(= b_{ij})$

$$\left|\frac{B_{jj}}{\det B}\right| \leqslant \frac{1}{|b_{jj}|(1 - \mu_j)}$$

and

$$|B_{ij}| \leqslant \lambda_j |B_{ii}|,$$

where

$$\lambda_i|b_{ii}| = \sum_{j \neq i} |b_{ij}|, \qquad 0 \leqslant \lambda_i \leqslant 1, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

In [32], Shivakumar and Ji improvee Ostrowski's bounds and further gave lower bounds for finite tridiagonal s.d.d. matrices. The improved bounds for $i \leqslant j$ are given by

$$\left(\prod_{k=i+1}^{j} \frac{|a_k|}{|b_k|(1 + \mu_k)}\right) |A_{ii}| \leqslant |A_{ij}| \leqslant \left(\prod_{k=i+1}^{j} \mu_k\right) |A_{ii}|, \tag{2.9}$$

$$\left|\frac{a_j}{a_i}\right| \left(\prod_{k=i}^{j-1} \frac{|a_k|}{|b_k|(1 + \mu_k)}\right) |A_{jj}| \leqslant |A_{ij}| \leqslant \left|\frac{a_j}{a_i}\right| \left(\prod_{k=i}^{j-1} \mu_k\right) |A_{jj}| \tag{2.10}$$

with similar results for $i \geqslant j$.

Further

$$\frac{1}{|b_i| + |a_i|\mu_{i-1} + |c_i|\mu_{i+1}} \leqslant \left|\frac{A_{ii}}{\det A}\right| \leqslant \frac{1}{|b_i| - |a_i|\mu_{i-1} - |c_i|\mu_{i+1}}, \tag{2.11}$$

where $\mu_0 = \mu_{n+1} = 0$.

For inverse elements of A, we have for $i \leqslant j$,

$$\frac{\prod_{k=i+1}^{j} |a_k|}{\prod_{k=i}^{j} |b_k|(1 + \mu_k)} \leqslant \left|\frac{A_{ij}}{detA}\right| \leqslant \frac{\prod_{k=i+1}^{j} \mu_k}{|b_i| - |a_i|\mu_{i-1} - |c_i|\mu_{i+1}} \tag{2.12}$$

with similar results for $i \geqslant j$.

In many applications, we need an upper bound for $\|A^{-1}\|_\infty$. This is given by

$$\|A^{-1}\|_\infty \leqslant \frac{1}{\delta(1 - \mu)},$$

where

$$\mu = \sup_k\{\mu_k\}, \quad \delta = \inf_k\{|b_k|\}.$$

Note that the inequality fails if some of the μ's are 1.

This result has been extended where the tridiagonal matrix is an infinite matrix. As an example, we quote the following theorem [32]:

For the matrix A defined above,

$$\|A^{-1}\|_\infty < \frac{2\left(r^{n+1} - r^{\frac{n+1}{2}}\right)}{\delta r^3}, \tag{2.13}$$

where $\mu_k < 1, \mu_i = 1$, for $i \neq k, q = \min_l \left\{ \frac{|a_l|}{|b_l|}, \frac{|c_l|}{|b_l|} \right\}$ and $r = \frac{(1+q)}{q} > 2, \delta = \inf_k\{|b_k|\}$.

(d) Some sign patterns

A new nonsingularity criteria for an $n \times n$ real matrix A, based on sign distributions and some stated conditions on the elements of A is given. This work is motivated by the solution of Poisson's equation for doubly connected regions (see Section 9.2). Nonsingularity related to M-matrices and positive matrices, with fixed sign distributions, have been extensively studied by Fiedler, Fan, Householder, Drew and Johnson (for details of see [35]).

We now give a set of earlier verifiable sufficient conditions for a matrix A to be nonsingular based on sign distribution and some easily computable sufficient conditions on the elements of A. We consider the sign distribution for the $m \times m$ matrix A, which ensures nonsingularity of A are given as follows [35]:

We consider the sign distribution for the $m \times m$ matrix A, given as follows [35]: $a_{ij} > 0$ for $i \leqslant j$ and

$$(-1)^{i+j} a_{ij} > 0 \quad \text{for } i > j, \; i, j = 1, 2, \ldots, m. \tag{2.14}$$

For convenience we define for $i \leqslant j$,

$$\nu_{ij} = a_{ij} - \sum_{k=j+1}^{m} a_{ik} \tag{2.15}$$

and

$$\mu_{ij} = a_{ij} - \sum_{k=1}^{i-1} a_{ij} \tag{2.16}$$

and for $j \leqslant i < l < m$,

$$\omega_{ij}^{l} = \min \left\{ |a_{ij}| - \left| \sum_{k=i+1}^{l} a_{kj} \right|, \left| \sum_{k=i+1}^{l} a_{kj} \right| - |a_{l+1\,j}| \right\} \tag{2.17}$$

and

$$\delta_{ijl} = \left| \sum_{k=j+1}^{l} a_{ik} \right| - |a_{ij}| \quad \text{for } j < l \leqslant i. \tag{2.18}$$

3. Linear algebraic systems

We are concerned here with the infinite system of linear equations [21]:

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, \infty, \tag{3.1}$$

or alternatively

$$A\tilde{x} = \tilde{b}, \tag{3.2}$$

where the infinite matrix $A = (a_{ij})$, is s.d.d., i.e,

$$\sigma_i |a_{ii}| = \sum_{j=1}^{\infty} |a_{ij}|; \quad 0 \leqslant \sigma_i < 1; \; i = 12, \ldots, \infty \tag{3.3}$$

and the sequence $\{b_i\}$ is bounded. We establish sufficient conditions which guarantee the existence and uniqueness of a bounded solution to (3.1). The infinite case is approached by finite truncations and use of results in Section 2. In classical analysis, linear equations in infinite matrices occur in interpolation, sequence spaces, summability and the present case occurs in solution of elliptic partial differential equations in multiply connected regions. Our approach is to initially develop estimates for the truncated system

$$\sum_{j=1}^{n} a_{ij}x_j^{(n)} = b_i, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

or alternatively

$$A^{(n)}\tilde{x}^{(n)} = \tilde{b}^{(n)}. \tag{3.5}$$

The following inequalities are established:

$$|x_j^{(n+1)} - x_j^{(n)}| \leqslant P\sigma_{n+1} + Q|a_{n+1,n+1}|^{-1} \tag{3.6}$$

for some positive constraints P and Q, and for p, q, being positive integers. We also have

$$|x_j^{(q)} - x_j^{(p)}| \leqslant P \sum_{i=p+1}^{\infty} \sigma_{n+1} + Q \sum_{i=p+1}^{q} |a_{n+1,n+1}|^{-1}. \tag{3.7}$$

An estimate for the solution for (3.4) is given by (on using results from Section 2),

$$|\tilde{x}^{(n)}| \leqslant \prod_{k=1}^{n} \frac{1+\sigma_k}{1-\sigma_k} \sum_{k=1}^{n} \frac{|b_k|}{|a_{kk}|(1+\sigma_k)}. \tag{3.8}$$

Under the conditions

$$\sum_{i=1}^{\infty} \frac{1}{|a_i i|} < \infty, \qquad \sum_{j=1, j\neq i}^{\infty} |a_{ij}| \leqslant M \tag{3.9}$$

for some $M > 0$ and all i, for the infinite system (3.1), we get similar results as we let n tends infinity. A by-product of the analysis establishes existence and uniqueness of a bounded solution to (1.1). An earlier notable result on the existence of a solution to such infinite system was given by Polya [22], which excluded discussion of uniqueness.

4. Linear eigenvalue problem

Our main concern here is the problem of determining the location of eigenvalues for diagonally dominant infinite matrices and establishes upper and lower bounds of eigenvalues. These matrices occur in (a) solutions of elliptic partial differential equations (b) solutions of second order linear differential equations (see Section 9.8).

Given a matrix $A = (a_{ij}), i, j \in \mathbb{N}$, a space of infinite sequences X and $\tilde{x} = (x_i), i \in \mathbb{N}$, we define $A\tilde{x}$ by $(A\tilde{x})_i = \sum_{j=1}^{\infty} a_{ij}x_j$, provided this series converges for each $i \in \mathbb{N}$. We define the domain of A by $D(A) = \{\tilde{x} \in X : A\tilde{x}$ exists and $A\tilde{x} \in X\}$. We define an eigenvalue of A to be any scalar λ for which $A\tilde{x} = \lambda\tilde{x}$ for some $0 \neq x \in D(A)$.

We first give the result for the eigenvalue problem for matrix operators. We make the following assumptions on A as an operator on ℓ_1.

E(1): $a_{ii} \neq 0, i \in \mathbb{N}$ and $|a_{ii}| \to \infty$ as $i \to \infty$.
E(2): There exists a $\rho, 0 \leqslant \rho < 1$, such that for each $j \in \mathbb{N}$

$$Q_j = \sum_{i=1, i\neq j}^{\infty} |a_{ij}| = \rho_j |a_{jj}|, \quad 0 \leqslant \rho_j \leqslant \rho.$$

E(3): For all $i, j in \mathbb{N}, i \neq j, |a_{ii} - a_{jj}| \geqslant Q_i + Q_j$.
E(4): For all $i\ in \mathbb{N}, \sup\{|a_{ij}| : j \in \mathbb{N}\} < \infty$.

The Gershgorin disks are defined as $C_i = \{z \in \mathbb{C} : |z - a_{ii}| \leqslant Q_i\}, i \in \mathbb{N}$, where Q_j are as defined above. E(2) represents uniform strict column diagonal dominance. Conditions E(2), E(1) imply that A^{-1} is compact. E(3) is equivalent to the (almost) disjointness of the Gershgorin disks C_i (the intersection of two disks consists of at most one point). Finally, E(4) implies that A is a closed linear operator on ℓ_1.

For an operator A on ℓ_1 with E(1) and E(2), it is shown in [28, Theorem 2] that A is an operator with dense domain, $R(A) = \ell_1$, A is compact on ℓ_1 and $\|A^{-1}\|$ is compact on ℓ_1, and $\|A^{-1}\|_1 \leqslant (1-e)^{-1}(\inf_i |a_{ii}|)^{-1}$. The main result for an operator A on ℓ_1 with E(1)–E(4), is that the operator A consists of a discrete countable set of non zero eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ such that $|\lambda_k| \to \infty$ as $k \to \infty$. Similar results are also derived for operators on ℓ_∞. For an operator A on ℓ_∞ consider:

E(5): There exists a $\sigma, 0 \leqslant \sigma < 1$, such that for each $i \in \mathbb{N}$,

$$P_i = \sum_{j=1, j\neq i}^{\infty} |a_{ij}| = \sigma_i |a_{ii}|, \quad 0 \leqslant \sigma_i \leqslant \sigma.$$

E(6): For all $i, j \in \mathbb{N}, i \neq j, |a_{ii} - a_{jj}| \geqslant P_i + P_j$.
E(7): For all j, $\sup_j \{|a_{ij}| : i \in \mathbb{N}\} < \infty$.

Analogous to the above, let $R_i = \{z \in \mathbb{C} : |z - a_{ii}| \leqslant P_i\}, i \in \mathbb{N}$.

We also note that E(5) implies uniform strict row diagonal dominance and E(6) is equivalent to the (almost) disjointness of the Gershgorin disks R_i.

For an operator A on ℓ_∞ with E(1), E(5), E(7), it is proved that [28] A is a closed operator, $R(A) = \ell_\infty, A^{-1}$ is compact on ℓ_∞ and $\|A^{-1}\|_\infty \leqslant (1-\sigma)^{-1}(\inf_i |a_{ii}|)^{-1}$. Also, the spectrum of A consists of discrete countable set of nonzero eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ such that $|\lambda_k| \to \infty$ as $k \to \infty$. These results give a complete description of all the eigenvalues for a (matrix) operator on ℓ_1 and ℓ_∞.

Further, in [28], truncation of the system $A\tilde{x} = \tilde{y}$ and corresponding error analysis are studied in detail.

5. Linear differential systems

We consider the infinite system

$$\frac{d}{dt} x_i(t) = \sum_{j=1}^{\infty} a_{ij} x_j(t) + f_i(t), \quad t \geqslant 0,\ x_i(0) = y_i,\ i = 1, 2, \ldots, \tag{5.1}$$

which can be written as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + \tilde{f}(t), \quad \dot{\tilde{x}}(0) = \tilde{y} \tag{5.2}$$

is of considerable theoretical and applied interest. In particular, such systems occur frequently in the theory of stochastic processes, perturbation theory of quantum mechanics, degradation of polynomials, infinite ladder network theory, etc. Arley and Brochsenius 1944, Bellman 1947, Shaw 1973 are the notable ones who have studied problem (5.2). In particular, if A is a bounded operator on ℓ_1, then the convergence of a truncated system has been established. However, none of these papers yields explicit error bounds for such a truncation, where [26] provides such error bounds.

We consider (5.2) rewritten as $\dot{\tilde{x}} = A\tilde{x} + \tilde{f}, \tilde{x}(0) = \tilde{y}$, where $\tilde{x}, \tilde{y}$ and f are infinite column vectors in X, and A is a constant infinite matrix defining a bounded operator on X, where X is ℓ_1, c_0 (*the space of real convergent sequences converging to zero* or ℓ_∞). Consider the following assumptions for homogeneous systems ($\tilde{f} = 0$):

DS(1): $M = \sum_{i=1}^{\infty} |y_i| < \infty$.
DS(2): $\alpha = \sup\{\sum_{i=1}^{\infty} |a_{ij}|, j = 1, 2, \ldots, \infty\} < \infty$.
DS(3): $\gamma_n = \sup\{\sum_{i=n+1}^{\infty} |a_{ij}|, j = 1, 2, \ldots, n\} \to 0$ as $n \to \infty$.
DS(4): $\delta_n = \sup\{\sum_{i=1}^{n} |a_{ij}|, j = 1, 2, \ldots, n\} \to 0$ as $n \to \infty$.

In the above, DS(1) states that $y \in \ell_1$. DS(2) is equivalent to the statement that A is bounded on ℓ_1. DS(3) states that the sums in DS(2) converge uniformly below the main diagonal; it is a condition involving only the entries of A below the main diagonal. DS(4) is a condition involving only the entries of A above the main diagonal; it is somewhat less natural than DS(3).

In [26, Theorem 1] for the homogeneous linear system of differential equations as given above, suppose that DS(1), DS(2) and either DS(3) or DS(4) hold. It is then shown that $\lim_{n\to\infty} x^{(n)}(t) = x(t)$ in l_1-norm uniformly in t on compact subsets of $[0,\infty)$, with explicit error bounds as given below:

$$\sum_{i=1}^{n} |x_i(t) - x_i^{(n)}(t)| \leqslant \alpha t e^{\alpha t}\left[\frac{1}{2}\gamma_n M t + \sum_{k=n+1}^{\infty} |y_k|\right], \tag{5.3}$$

$$\sum_{i=n+1}^{\infty} |x_i(t)| \leqslant e^{\alpha t}\left[\gamma_n M t + \sum_{k=n+1}^{\infty} |y_k|\right], \tag{5.4}$$

$$\|\tilde{x}(t) - \tilde{x}^{(n)}(t)\| \leqslant e^{\alpha t}\left[\left(1+\frac{1}{2}\alpha t\right)\gamma_n M t + (1+\alpha t)\sum_{k=n+1}^{\infty} |y_k|\right] \tag{5.5}$$

for DS(3) (with the right hand side converging to zero as $n \to \infty$) and

$$\sum_{i=1}^{n} |x_i(t) - x_i^{(n)}(t)| \leqslant \delta_n M t e^{\alpha t} \tag{5.6}$$

for DS(4) (with the right hand side converging to zero as $n \to \infty$.)

For the nonhomogeneous system, consider the assumption

DS(5): $f_i(t)$ is continuous on $[0,\infty)$, $i = 1, 2, \ldots$, and $\|f(t)\| = \sum_{i=1}^{\infty} |f_i(t)|$ converges uniformly in t on compact subsets of $[0,\infty)$.

Define $L(t) = \sup\{\|f(\tau)\| : 0 \leqslant \tau \leqslant t\}$.

DS(5) is equivalent to the statement that $f(t)$ is continuous from $[0,\infty)$ into ℓ_1 and this implies that $L(t) < \infty$ for all $t \geqslant 0$.

For the nonhomogeneous case we have the following result [26, Theorem 2], if DS(2), DS(5) and either DS(3) or $DS(4)$ hold, then $\lim_{n\to\infty} \tilde{x}^{(n)}(t) = \tilde{x}(t)$ in l_1 norm uniformly in t on compact subsets of $[0,\infty)$, with explicit error bounds as given below:

$$\|\tilde{x}(t) - \tilde{x}^{(n)}(t)\| \leqslant \frac{1}{2}t^2 e^{\alpha t}\gamma_n L(t) + t e^{\alpha} t \sup\left\{\sum_{k=n+1}^{\infty} |f_k(\tau)| : 0 \leqslant \tau \leqslant t\right\} \tag{5.7}$$

for DS(3) (with the right hand side converging to zero as $n \to \infty$) and

$$\sum_{i=1}^{n} |x_i(t) - x_i^{(n)}(t)| \leqslant \delta_n L(t)\alpha^{-2}[\alpha t e^{\alpha t} + (1 - e^{\alpha} t)] \tag{5.8}$$

for $DS(4)$ (with the right hand side converging to zero as $n \to \infty$).

Similar results hold for systems on l_1 [26, Theorems 4 and 6] and for systems on c_0 [26, Theorems 3 and 5].

The analysis above concerns A, being a constant infinite matrix defining a bounded operator on E, where E is ℓ_1, c_0 or ℓ_∞. Explicit error bounds are obtained for the approximation of the solution of the infinite system by the solutions of finite truncation systems.

6. An iterative method

Iterative methods for linear equations in finite matrices have been the subject of very vast literature which is still growing due to the advent of finite elements. Since all methods involve the nonsingularity of the matrix, d.d. matrices have played a major role. Our interest here is to discuss an iterative method for such infinite systems which we believe to be one of the first such attempts.

For the finite system $\mathfrak{I}$.d.d. matrices were introduced by Chew and Shivakumar [27] where s.d.d. matrices as well as irreducibly d.d. matrices were subsets. Many results were derived for the convergence of the iterative procedures with bounds for the relaxation parameters. Details can be found in [29].

In this section, we consider an infinite system of equations of the form $T\tilde{x} = \tilde{v}$, where $\tilde{x}, \tilde{v} \in \ell_\infty$, and T is a linear (possibly unbounded) operator on ℓ_∞. Suppose $T = (t_{ij})$ satisfies:

IM(1): There exists $\eta > 0$ such that $|t_{ii}| \geqslant \eta$ for all $i = 1, 2, \ldots$
IM(2): There exists σ with $0 \leqslant \sigma < 1$ such that

$$\sum_{j=1, j\neq i} \infty |t_{ij}| = \sigma_i |t_{ii}|, \quad \text{where } 0 \leqslant \sigma_i < \sigma < 1 \text{ for all } i = 1, 2, \ldots$$

IM(3): $\sum_{j=1}^{i-1} \frac{|t_{ij}|}{|t_{ii}|} \to 0 \text{as} i \to \infty$
IM(4): Either $|t_{ii}| \to \infty$ as $i \to \infty$ or $v \in c_0$.
We have the following result.

Theorem 6.1 [29, Theorem 1]. *Let $v \in \ell_\infty$ and let T satisfy* IM*(1) and* IM*(2). Then T has a bounded inverse and the equation $Tx = v$ has a unique ℓ_∞ solution.*

Let $T = D + F$, where D is the main diagonal of T and F is the off-diagonal of T. Let A be defined by $A = -D^{-1}F$ and $b = D^{-1}v$. Then $Tx = v$ is equivalent to the system $x = Ax + b, b \in \ell_\infty$, where A is a bounded linear operator on ℓ_∞.

This leads naturally into the iterative scheme:

$$x^{(p+1)} = x^{(p)} + b, \quad x(0) = b, \quad p = 0, 1, 2, \ldots \tag{6.1}$$

Consider the following iterations and truncations:

$$x^{p+1} = Ax^{(p)} + b, \quad x^{(0)} = b, \tag{6.2}$$
$$x^{(p+1,n)} = A^{(n)}x^{(p,n)} + b, \quad x^{(0,n)} = b \tag{6.3}$$

for $p = 0, 1, 2, \ldots$ and $n = 1, 2, 3 \ldots$, where $A_{(n)}$ is the infinite matrix $(a_{ij}^{(n)})$ defined by

$$a_{ij}^{(n)} = \begin{cases} a_{ij} & \text{if } l \leqslant i, j \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Then we have the following result.

Theorem 6.2 [29, Theorem 2]. *Suppose that T satisfies* IM(1)–IM(4). *Then A and b satisfy the following*:

(i) $\|A\| = \sup_{i\geqslant 1} \sum_{j=1}^{\infty} |a_{ij}| \leqslant \sigma < 1 i \geqslant 1$,
(ii) $\sum_{j=1}^{i-1} |a_{ij}| \to 0$ *as* $i \to \infty$ *and*
(iii) $b = (b_i) \in c_0$.

Also, the iterates as defined above, satisfy:

$$\|x^{(p)} - x^{(p,n)}\| \leqslant \beta\gamma_n \sum_{k=0}^{p-1} (k+1)\sigma^k + \beta_n\mu_n \sum_{k=0}^{p-1} \sigma^k \to 0 \quad \text{as } n \to \infty. \tag{6.5}$$

It can also be shown that

Corollary 6.3 [29, Corollary 2]

$$\|\tilde{x} - \tilde{x}^{(p,n)}\| \leqslant \sigma^{p+1}(1-\sigma)^{-1}\beta + \beta\gamma_n(1-\sigma)^{-2} + \beta_n\mu_n(1-\sigma)^{-1} \to 0 \tag{6.6}$$

as $n \to \infty$.

An application of the above in the recurrence relations of the Bessel's functions is given in Section 9.5.

7. A computational technique

A powerful computational technique for the infinite system $A\tilde{x} = \lambda\tilde{x}$, is by using a truncated matrix $G^{(1,k)}$ (defined below). The technique is to box the eigenvalues and then use a simple bisection method to give the value of λ_n to any required degree of accuracy.

We will now consider a matrix $A = (a_{ij})$ acting on ℓ_∞ satisfying E(2), E(5), E(6) and E(7), of Section 4 together with the conditions:

$$0 < a_{ii} < a_{i+1,i+1}, \quad i \in \mathbb{N},\ a_{ij} = 0 \text{ if } |i-j| \geqslant 2,\ i,j \in \mathbb{N}$$

and

$$a_{i,i+1}a_{i+1,i} > 0, \quad i \in \mathbb{N}.$$

Suppose λ satisfies $a_{nn} - P_n \leqslant \lambda \leqslant a_{nn} + P_n$, where P_i are as defined in section 4. Let $G = A - \lambda I$. Let $G^{(1,k)}$ denote the truncated matrix of A obtained from A by taking only the first k rows and k columns. We then have [28, Section 8]:

$$\det G^{(1,k)} = (a_{11} - \lambda)\det G^{(2,k)} - a_{12}a_{21}\det G^{(3,k)} \tag{7.1}$$

$$= [(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}]\det G^{(3,k)} - (a_{11} - \lambda)a_{23}a_{32}\det G^{(4,k)} \tag{7.2}$$

so that

$$\det G^{(1,k)} = p_s \det G^{(s,k)} - p_{s-1}a_{s-1,s}a_{s,s-1}\det G^{(s+1,k)}, \tag{7.3}$$

where

$$p_s = p_{s-1}(a_{s-1,s-1} - \lambda) - p_{s-2}a_{s-1,s-2}a_{s-2,s-1}$$

and

$$p_0 = 0; \quad p_1 = 1.$$

If we set

$$Q_{s,k} = \frac{p_s}{p_{s-1}} - a_{s-1,s}a_{s,s-1}\frac{\det G^{(s+1,k)}}{\det G^{(s,k)}}, \tag{7.4}$$

we then have

$$\det G^{(1,k)} = p_{s-1}Q_{s,k}\det G^{(s,k)}. \tag{7.5}$$

We have the following cases:

Case (i): If p_{s-1} and p_s have opposite signs, then $Q_{s,k} < 0$ and $\det G^{(1,k)}$ has the same sign as $-p_{s-1}$.

Case (ii): If p_{s-1} and p_s have the same sign and

$$\frac{p_s}{p_{s-1}} > a_{s,s-1}a_{s-1,s}(a_{ss} - \lambda - |a_{s,s+1}|)^{-1},$$

we then have $Q_{s,k} > 0$ and $\det G^{(1,k)}$ has the same sign as p_{s-1}.

Case (iii): If p_{s-1} and p_s have the same sign and

$$\frac{p_s}{p_{s-1}} < a_{s,s-1}a_{s-1,s}(a_{ss} - \lambda - |a_{s,s+1}|)^{-1},$$

then, $Q_{s,k} < 0$, and $\det G^{(1,k)}$ have the same sign as $-p_{s-1}$. We can use the method of bisection to establish both upper and lower bounds for λ_n for any degree of accuracy.

Application to Mathieu function of the above technique forms the discussion in Section 9.4.

8. On the non-uniquness of the Moore–Penrose inverse and group inverse of infinite matrices

Our intention here is not to propose a theory of generalized inverses of infinite matrices, but only to point out an interesting result discovered very recently [38]. There, the author gives an example of an invertible infinite matrix V which has infinitely many classical inverses (which are automatically Moore–Penrose inverses) and also has a Moore–Penrose inverse which is not a classical inverse. Thus it follows that there are infinitely many Moore–Penrose inverses of V. In [39], the author shows that the very same infinite matrix considered earlier has group inverses that are Moore–Penrose inverses, too. In [20], the authors employ a novel notion of convergence of infinite series to show that results similar to the ones mentioned above hold good for infinite matrices over a finite field. These results are interesting in that, for infinite matrices, it follows that the set of generalized inverses is properly larger than the set of classical inverses. It might be remembered that these are not true for finite matrices with real or complex entries. It also follows that for a "symmetric" infinite matrix, the Moore–Penrose inverse need not be the group inverse, unlike the finite symmetric matrix case, where they coincide.

First we give the definitions.

Definition 8.1. Let $A \in \mathbb{R}^{m\times n}$. A matrix $X \in \mathbb{R}^{n\times m}$ is called the Moore–Penrose inverse of A if it satisfies the following Penrose equations:

$$AXA = A; \quad XAX = X;$$
$$(AX)^T = AX; \quad (XA)^T = XA.$$

It is well known [5] that the Moore–Penrose inverse exists for any matrix A and that it is unique. It is denoted by $A^\dagger$. Let $A \in \mathbb{R}^{n\times n}$. $X \in \mathbb{R}^{n\times n}$ is called the group inverse of A if X satisfies $AXA = A$; $XAX = X$; and$AX = XA$. The group inverse when it exists, is denoted by $A^\#$. In contrast to the Moore–Penrose inverse of a matrix, which always exists, not all matrices have the group inverse. A necessary and sufficient condition that the group inverse exists is that $rank(A) = rank(A^2)$ [5].

It is well known that for infinite matrices multiplication is non-associative. Hence we also impose associativity in the first two Penrose equations and rewrite them as

$$(AX)A = A(XA) = A,$$
$$(XA)X = X(AX) = X.$$

We call X as a Moore–Penrose inverse of an infinite matrix A if X satisfies the two equations as above and the last two Penrose equations. Similarly, for group inverses of infinite matrices, we demand associativity in the first two equations of definition.

Suppose that the infinite matrix A can be viewed as a bounded operator between Hilbert spaces. Then it is well known that A has a bounded Moore–Penrose inverse $A^\dagger$ iff its range, $R(A)$ is closed [17]. For operators between Banach spaces we have a similar result. However, in such a case, the last two equations involving adjoint, should be rewritten in terms of projections on certain subspaces. (See [16] for details.)

In this section, our approach will be purely algebraic, in the sense that infinite matrices are not viewed as operators on some vector spaces. We use the "basis" $\{e^n : n \in \mathbb{N}\}$, where $e^n = (0, 0, \ldots, 1, 0, \ldots)$ with 1 appearing in the nth coordinate. With this notation we consider the infinite matrix V such that $V(e^1) = e^2$ and $V(e^n) = e^{n-1} + e^{n+1}, n \geqslant 2$. The next result states that V has infinitely many algebraic inverses.

Theorem 8.2 [38, Theorem 0.1]. *Let U and W be infinite matrices defined by $U(e^1) = e^2 - e^4 + e^6 - e^8 + \cdots$; $U(e^2) = e^1$ and $U(e^{n+1}) = e^n - U(e^{n-1}), n \geqslant 2$ and $W(e^1) = (e^2 - e^4 + e^6 - e^8 + \cdots) - (e^1 - e^3 + e^5 - e^7 + \cdots)$; $W(e^2) = e^1$ and $W(e^{n+1}) = e^n - W(e^{n-1}), n \geqslant 2$. Then $UV = VU = I, WV = VW = I$ and V has infinitely many algebraic inverses.*

The fact that V has a Moore–Penrose inverse which is not a classical inverse is the next result.

Theorem 8.3 [38, Theorem 0.3]. *Let Z be the infinite matrix defined by $Z(e^1) = e^4 - 2e^6 + 3e^8 - 4e^{10} + \cdots, Z(e^2) = e^1$ and $Z(e^{n+1}) = e^n - Z(e^{n-1}), n \geqslant 2$. Then Z is not a classical inverse of V but Z is a Moore–Penrose inverse of V.*

Remark 8.4. It follows that the matrix Z defined above satisfies $ZV = I$, whereas, we have $VZ(e^1) \neq e^1$, so that $ZV \neq VZ$. Thus Z is not a group inverse of V. Recall that a finite matrix is called range hermitian, if $R(A) = R(A^*)$, where $R(A)$ denotes the range space of A. It is well known [5] that a matrix A is range hermitian iff $A^\dagger = A^\#$. As a corollary, if $A \in \mathbb{R}^{n\times n}$ is symmetric, then its Moore–Penrose inverse and the group inverse coincide. Theorem 8.3 shows that that the situation is different in the case of infinite matrices. We have a Moore–Penrose inverse that is not a group inverse, even though the infinite matrix V is "symmetric".

The next two results show that there are group inverses of V that also turn out to be Moore–Penrose inverses.

Theorem 8.5 [39, Theorem 3.6]. *For $\alpha \in \mathbb{R}$, let Y_α be the infinite matrix defined by*

$$\begin{aligned} Y_\alpha(e^1) &= \alpha(e^2 - 2e^4 + 3e^6 \cdots) + (e^4 - 2e^6 + 3e^8 - \cdots),\\ Y_\alpha(e^2) &= \alpha(e^1 - e^3 + e^5 - \cdots) + (e^3 - e^5 + e^7 - \cdots),\\ Y_\alpha(e^{2n}) &= (-1)^n\{(e^3 - e^5 + \cdots + (-1)^n e^{2n-1}) + (n-1)e^1 - nY_\alpha(e^2)\} \end{aligned}$$

for $n \geqslant 2$ and

$$Y_\alpha(e^{2n+1}) = e^{2n} - Y_\alpha(e^{2n-1}), \quad n \geqslant 1.$$

For the sake of convenience, let Y denote Y_α. Then Y is a group inverse of V, but not a classical inverse.

Theorem 8.6 [39, Theorem 3.7]. *Let Y be defined as above. Then Y is a Moore–Penrose inverse of V.*

Finally, we consider the case of infinite matrices over a finite field. There are two essential differences between generalized inverses of real or complex matrices and generalized inverses of matrices over finite fields. One is that the Moore–Penrose inverse of a real or complex matrix always exists (and it is unique), whereas, it need not exist for a matrix over a finite field. For example, the matrix $\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}$ over the finite field $\mathbb{F}_2 = \{0, 1\}$ does not have a Moore–Penrose inverse. Another difference is that there are only finitely many generalized inverses. We refer to [20] (and the references cited therein) for a history of generalized matrices over finite fields. We observe that a new notion of convergence of an infinite series was employed in the proofs of the results in [20].

Theorem 8.7 [20, Theorem 3.6]. *Let Z be the infinite matrix over $\mathbb{F}_2$ defined by*

$$Z(e^1) = e^4 + e^8 + e^{12} + \cdots, \quad Z(e^2) = e^1$$

and

$$Z(e^{n+1}) = e^n + Z(e^{n-1}), \quad n \geqslant 2.$$

Then Z is not a classical inverse of V nor a group inverse of V, but Z is a Moore–Penrose inverse of V.

Theorem 8.8 [20, Theorems 4.6 and 4.7]. *Let Y be the infinite matrix defined by*

$$\begin{aligned} Y(e^1) &= e^4 + e^8 + e^{12} + \cdots,\\ Y(e^2) &= e^3 + e^5 + e^7 + \cdots,\\ Y(e^{2n}) &= \begin{cases} e^{2n+1} + e^{2n+3} + e^{2n+5} + \cdots & \text{if } n \text{ is odd}\\ e^1 + e^3 + \cdots + e^{2n-1} & \text{if } n \text{ is even}\end{cases} \end{aligned}$$

for $n \geqslant 2$ *and*

$$Y(e^{2n+1}) = e^{2n} + Y(e^{2n-1}) \quad \textit{for } n \geqslant 1.$$

Then Y *is a group inverse of* V, *a Moore–Penrose inverse of* V, *but not a classical inverse of* V.

9. Applications

9.1. Digital circuit dynamics

We consider a real weakly d.d. matrix A and establish upper and lower bounds for the minimal eigenvalue of A, and its corresponding eigenvector, and for the entries of inverse of A. The results are applied to find meaningful two-sided bounds for both ℓ_1-norm and the weighted Perron-Norms of the solution $\tilde{x}(t)$ to the linear differential system $\dot{x} = -Ax$, $\tilde{x}(0) = x_0 > 0$. These systems occur in R–C electrical circuits and a detailed analysis of a model for the transient behavior of digital circuits is given in [36].

We consider J and N as defined in section 2.1, where A is an $n \times n$ matrix with the conditions

D(1): For all $i, j \in N$, with $i \neq j, a_{ij} \leqslant 0$ and $a_{ii} > 0$.
D(2): For all $i \in N, |a_{ii}| \geqslant \sum_{j \neq i} |a_{ij}|$ and $J \neq \emptyset$.
D(3): A is irreducible.

Under the conditions D(1)–D(3) and $\dot{\tilde{x}} = -A\tilde{x}, \tilde{x}(0) = 0$, it is proved that for all $t \geqslant 0$,

$$\sum_{i=1}^{n} x_i(t) = e^{-qt} \sum_{i=1}^{n} z_i x_{0,i}, \tag{9.1}$$

where $q = q(\widetilde{A})$ and $z = (z_1, z_2, \ldots, z_n)^T$ is the positive eigenvector of $\widetilde{A}^T$ corresponding to q.

The following inequalities also hold:

$$\frac{z_{\min}}{z_{\max}} \|x_0\|_1 e^{-qMt} \leqslant \|x(t)\|_1 \leqslant \frac{z_{\max}}{z_{\min}} \|x_0\|_1 e^{-qmt}, \tag{9.2}$$

where $z_{\min} \leqslant z_i \leqslant z_{\max}$ for all $i = 1, 2, \ldots, n$. In connection with digital circuits, we have the following: If $v = (v_1(t), \ldots, v_n(t))^T$ denotes the vector of node voltages, then the transient evolution of an R-C circuit (under certain conditions) is described by the equation

$$C \frac{d\tilde{v}(t)}{dt} = -Gv + J, \tag{9.3}$$

where C is a diagonal matrix, J is a given vector and G is a given matrix of conductances. The crucial performance of the digital circuit is the high operating speed that is measured by the quantity

$$T(\epsilon) = \sup \left\{ t : \frac{\|x(t)\|}{\|x_0\|} = \epsilon \right\}. \tag{9.4}$$

It can be shown that [34] the following inequalities (as a consequence of the inequalities above) hold:

$$\frac{z_{\min}}{z_{\max}} e^{-qMT_1} \leqslant \epsilon = \frac{\|x(T_1(\epsilon))\|_1}{\|x_0\|_1} \leqslant \frac{z_{\max}}{z_{\min}} e^{-qmT_1} \tag{9.5}$$

for a fixed value T_1. This implies that

$$\frac{1}{qM} \ln \frac{z_{\min}}{\epsilon x_{\max}} \leqslant T_1(t) \leqslant \frac{1}{qm} \ln \frac{z_{\max}}{\epsilon z_{\min}}. \tag{9.6}$$

9.2. Conformal mapping of doubly connected regions

Solution of a large number of problems in modern technology, such as leakage of gas in a graphite brick of a gas-cooled nuclear reactor, analysis of stresses in solid propellant rocket grain, simultaneous flow of oil and gas in concentric pipes, and microwave theory, hinges critically on the possibility of conformal transformation of a doubly connected region into a circular annulus.

If D is a doubly connected region of the z-plane, then the frontier of D consists of two disjoint continua C_0 and C_1. It is well known [7] that D can be mapped one to one conformally onto a circular annulus. Moreover, if a and b are the radii of two concentric circles of the annulus, then the modulus of D given by b/a is a number uniquely determined by D. The difficulties involved in finding such a mapping function and estimating the modulus of D are described in [11]. In fact, studies concerning specific regions are very few in the literature. In this section, a direct method of reducing the mapping problem of the region between a circle and curvilinear polygon of n sides to an infinite system of linear algebraic equations is given. The truncated system of linear algebraic equations turns out to be a s.d.d. system.

Let the Jordan curves C_0 and C_1 bound externally and internally a doubly connected region D in the z-plane. Then the mapping function $w(z) = e^{[\log z+\phi(z)]}, z = x + iy = re^{i\theta}$, which is unique except for an arbitrary rotation, maps $D + \partial D$ onto the annulus $0 \leqslant a \leqslant |w| \leqslant b < \infty$, where the ratio b/a is unique and $\phi(z)$ is regular in D. If $\phi(z)$ has the form $\phi(z) = \sum_{-\infty}^{\infty} c_n z^n$, we then have

$$\log(z\bar{z}) + \phi(z) + \bar{\phi}(\bar{z}) = \begin{cases} \log b^2 & \text{if } z \in C_0, \\ \log a^2 & \text{if } z \in C_1. \end{cases}$$

The requirement that $\phi(z)$ satisfies the conditions given above is equivalent to solving the system of infinite linear equations

$$\sum_{q=1}^{\infty} A_{pq} x_q = R_p, \quad p = 1, 2, \ldots$$

for suitable numbers A_{pq} and R_p. It can be shown that if $x_q^{(N)}$ is the solution of

$$\sum_{q=1}^{N} A_{pq} x_q = R_p, \quad p = 1, 2, \ldots, N,$$

then $\lim_{N\to\infty} x_q^{(N)} = x_q$. We refer to [25] for the details and numerical examples. We also refer to [23] for a similar procedure for the solution of the Poisson's equation describing a fluid flow problem.

9.3. Fluid flow in pipes

In this section, we consider the problem that arises from the idea that two fluids could be transported with one fluid inside a pipe of cross-section E and the other flowing in an annular domain D in the xy plane bounded internally by C_2 and externally by C_1. The flow velocity $w(x, y)$ satisfies the Poisson's equation:

$$w_{xx} + w_{yy} = -P/\mu \quad \text{in } D, \tag{9.7}$$

(P, μ being positive constants), with the boundary conditions:

$$w = 0 \text{ on } C_1 \quad \text{and} \quad w = 0 \text{ on } C_2.$$

In this problem we are concerned with the rate of flow given by

$$R = \iint w \, dx \, dy. \tag{9.8}$$

It can be shown that using the conformal mapping function

$$z = \frac{c}{1-\zeta}, \quad \zeta = \xi + i\eta, \tag{9.9}$$

we get

$$w = -\frac{P}{u\mu} z\bar{z} + \phi(z) + \bar{\phi}(\bar{z}). \tag{9.10}$$

We get an infinite series expression for w whose coefficients satisfy an infinite system of algebraic equations. These equations have been shown to have a unique solution. We refer to [23,27,31] for the details. In these the following cases are considered.

(a) C_0 and C_1 being eccentric circles.
(b) C_0 and C_1 being confocal ellipses.
(c) C_0 being a circle and C_1 being an ellipse.
(d) C_0 and C_1 being two ellipses.

Calculation of the rate of the flow suggests that the flow is a maximum when the inner boundary has the least parameter and the outer boundary has the largest perimeter for a given area of flow.

We refer to [14] for an application of the ideas given above in studying fluid flow in a pipe system where the inside of the outer pipe has a lining of porous media. This has been shown to have applications in the cholesterol problem in arteries [14].

9.4. Mathieu equation

Here we consider a Mathieu equation (also see Section 7)

$$\frac{d^2y}{dx^2} + (\lambda - 2q\cos 2x)y = 0 \tag{9.11}$$

for a given q with the boundary conditions: $y(0) = y(\pi/2) = 0$. Our interest is in the case when two consecutive eigenvalues merge and become equal for some values of the parameter q. This pair of merging points is called a double point for that value of q. It is well known that for real double points to occur, the parameter q must attain some pure imaginary value. In [37], the authors developed an algorithm to compute some special double points. Theoretically, the method can achieve any required accuracy. We refer to [37] for the details. We briefly present the main results, here.

Using an even solution of the form $y(x) = \sum_{r=1}^{\infty} x_r q^{-r} \sin 2rx$, the Mathieu equation is equivalent to the system of infinite linear algebraic equations given by $Bx = \lambda x$, where $B = (b_{ij})$ is an infinite tridiagonal matrix given by

$$b_{ij} = \begin{cases} -Q & \text{if } j = i-1, \quad i \geqslant 2, \\ 4i^2 & \text{if } j = i, \\ 1 & \text{if } j = i+1, \quad i \geqslant 1. \end{cases}$$

Here $Q = -\lambda^2 > 0$. Then we have the following result.

Theorem 9.1 [37, Theorem 3.1]. *There exists a unique double point in the interval* $[4, 17]$.

It is also shown that there is no double point in the interval $[17, 37]$ [37, Theorem 4.1]. They also present an algorithm for computing the double points. In fact, we have $\lambda \approx 11.20, Q \approx 48.09$.

The problem of determining the bounds for the width of the instability intervals in the Mathieu equation has also been studied [33]. We present the main result, here. We consider the following Boundary Conditions: $y'(0) = y'(\pi/2) = 0$, and $y(0) = y(\pi/2) = 0$, with the corresponding eigenvalues denoted by $\{a_{2n}\}_0^\infty$ and $\{b_{2n}\}_0^\infty$, respectively. In this connection, the following inequalities are well known. $a_0 < b_1 < a_1 < b_2 < a_2 < b_3 < a_3 < \cdots$. We give upper and lower bound for $a_{2n} - b_{2n}$.

Theorem 9.2 [33, Theorem 2]. *For* $n \geqslant \max\left\{\frac{h^2+1}{2}, 3\right\}$,

$$a_{2n} - b_{2n} \leqslant \frac{8h^{4n}}{4^{2n}[(2n-1)!]^2 k_-} \tag{9.12}$$

and

$$a_{2n} - b_{2n} \geqslant \frac{8h^{4n}}{4^{2n}[(2n-1)!]^2 k_+}\left[1 - \frac{h^4}{8(2n-1-h^2)^2}\right], \tag{9.13}$$

where

$$k_{\pm} = \left(1 \pm \frac{3h^2}{4n^2}\right) \prod_{k=1}^{n-1} \left(1 \pm \frac{3h^2}{4n^2 - 4k^2}\right)^2.$$

9.5. Bessel's equation

The Bessel functions, $J_n(x)$, satisfy the well-known recurrence relation:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \quad n = 0, 1, 2, \ldots, \; 0 < x < 2.$$

Treating $J_0(x)$ as given and solving for $J_n(x) = x_n, n = 1, 2, \ldots$, we get a system of equations satisfying (the conditions considered in Section 6): IM(1), IM(3) and IM(4), and will satisfy $IM(2)$ iff $0 < x < 2$. For instance, choosing $x = 0.6$, yields the following system of equations:

$$\begin{aligned} &10x_1 - 3x_2 = 3J_0(0.6), \\ &-3x_{n-1} + 10nx_n - 3xn + 1 = 0, \quad n \geqslant 2, \end{aligned}$$

where $3J_0(0.6) \simeq 2.73601459$ [1]. From Corollary 6.3, Section 6, it can be verified that

$$\|\tilde{x} - \tilde{x}^{(p,n)}\| \leqslant \left\{ \frac{3}{7}(0.3)^{(p+12)} + \frac{9}{49(n+1)} \right\} J_0(0.6).$$

Thus for an error of less than 0.01, we choose $n = 16$ and $p = 6$. For the purpose of comparison, if we take $p = 16$, then for instance we have the following: 9.99555051E−07 as against 9.9956E−07 in [1] and 9.99555051E−07 as against 9.9956E−07 in [1] for $J_6(0.6)$ and 1.61393490E−12 as against 1.61396824E−12 in [1] for $J_{10}(0.6)$. We refer to [29] for more details and comparison with known results.

9.6. Vibrating membrane with a hole

In this section, we discuss the behaviour of the minimal eigenvalue λ of the Dirichlet Laplacian in an annulus. Let D_1 be a disc on $\mathbb{R}^2$, with origin at the center, of radius 1, $D_2 \subset D_1$ be a disc of radius $a < 1$, the center $(h, 0)$ of which is at a distance h from the origin. Let $\lambda(h)$ denote the minimum Dirichlet eigenvalue of the Laplacian in the annulus $D := D_h := D_1 \backslash D_2$. The following conjecture is proposed in [19]:

Conjecture. *The minimal eigenvalue $\lambda(h)$ is a monotonic decreasing function of h on the interval $0 \leqslant h \leqslant (1 - a)$. In particular, $\lambda(0) > \lambda(h), h > 0$.*

The above conjecture is supported by the following lemmas.

Lemma 9.3. *We have*

$$\frac{d\lambda}{dh} = \int_S u_N^2 N_1 \, ds,$$

where N is the unit normal to $S = S_h$, pointing into the annulus D_h, N_1 is the projection of N onto x_1-axis, u_N is the normal derivative of u, and $u(x) = u(x_1, x_2)$ is the normalized $L^2(D)$ eigenfunction corresponding to the first eigenvalue λ.

Let $D(r)$ denote the disc $|x| \leqslant r$ and $\mu(r)$ be the first Dirichlet eigenvalue of the Laplacian in $D_1 \backslash D_r$. Then we have,

Lemma 9.4. $\mu(a - h) < \lambda(h) < \mu(a + h), 0 < h < 1 - a.$

The conjecture is also substantiated by numerical results. For details we refer to [19].

9.7. Groundwater flow

Here, we are concerned with the problem of finding the hydraulic head, ϕ, in a non-homogeneous porous medium, the region being bounded between two vertical impermeable boundaries, bounded on top by a sloping sinusoidal curve and unbounded in depth. The hydraulic conductivity $K(z)$ is modelled as $K(z) = e^{dz}$, supported by some data available from Atomic Energy of Canada Ltd. We develop a method that reduced the problem to that of solving an infinite system of linear equations. The present method yields a Grammian matrix which is positive definite, and the truncation of this system yields an approximate solution that provides the best match with the given values on the top boundary. We refer to [36] for the details. We only present a brief outline, here.

We require ϕ to satisfy the equation:

$$\nabla \cdot (e^{dz} \nabla \phi(x, z)) = 0,$$

where ∇ is the vector differential operator

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial z}.$$

The domain under consideration is given by

$$0 < x < L, \quad -\infty < z < g(x) = -\left[\frac{ax}{L} + V\sin\left(\frac{2\pi nx}{L}\right)\right], \tag{9.14}$$

where $d \geqslant 0, L \geqslant 0, a \geqslant 0$ and V are constants and n is a positive integer. The boundary conditions are given by

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \left.\frac{\partial \phi}{\partial x}\right|_{x=L} = 0, \tag{9.15}$$

$\phi(x, z)$ is bounded on $z \leqslant g(x)$, and $\phi(x, z) = z$ on $z = g(x)$. The determination of ϕ reduces to the problem of solving the infinite system of linear algebraic equations:

$$\sum_{m=0}^{\infty} b_{km}\alpha_m = c_k, \quad k = 0, 1, 2, \ldots, \tag{9.16}$$

where b_{km} are given by means of certain integrals. The infinite matrix $B = (b_{km})$ is the Grammian of a set of functions which arise in the study. The numbers b_{km} become difficult to evaluate for large values of k and m by numerical integration. The authors use an approach using modified Bessel functions, which gives analytical expressions for b_{km}. They also present numerical approximations and estimates for the error.

9.8. Semi-infinite linear programming

Infinite linear programming problems are linear optimization problems where, in general there are infinitely (possibly uncountable) many variables and constraints related linearly. There are many problems arising from real world situations that can be modelled as infinite linear programs. The bottleneck problem of Bellman in economics, infinite games, continuous network flows, to name a few. We refer to the excellent book [2] for an exposition of infinite linear programs, a simplex type method of solution and applications. Semi-infinite linear programs are a subclass of infinite programs, wherein the number of variables is finite with infinitely many constraints in the form of equations or inequalities. Semi-infinite programs have been shown to have applications in a number of areas that include robot trajectory planning, eigenvalue computations, vibrating membrane problems and statistical design problems. For more details we refer to the two excellent reviews [10,18] on semi-infinite programming.

Let us recall that a vector space V is said to be partially ordered if there is a relation denoted '$\leqslant$' defined on V that satisfies reflexivity, antisymmetry and transitivity and which is also compatible with the vector space operations of addition and scalar multiplication. Let H_1 and H_2 be real Hilbert spaces with H_2 partially ordered, $A : H_1 \to H_2$ be bounded linear, $c \in H_1$ and $a, b \in H_2$ with $a \leqslant b$. Consider the linear programming problem called interval linear program $ILP(a, b, c, A)$:

$$\begin{aligned}&\text{Maximize} && \langle c, x\rangle\\ &\text{subject to} && a \leqslant Ax \leqslant b.\end{aligned}$$

Such problems were investigated by Ben–Israel and Charnes [4] in the case of finite dimensional real Euclidean spaces and explicit optimal solutions were obtained using the techniques of generalized inverses of matrices. These results were later extended to the general case and several algorithms were proposed for interval linear programs. Some of these results were extended to certain infinite dimensional spaces [12,13].

The objective of this section is to present an algorithm for the countable semi-infinite interval linear program denoted $SILP(a, b, c, A)$ of the type:

$$\begin{aligned}&\text{Maximize} && \langle c, x\rangle\\ &\text{subject to} && a \leqslant Ax \leqslant b,\end{aligned}$$

where $c \in \mathbb{R}^m$, H, a real separable partially ordered Hilbert space, $a, b \in H$ with $a \leqslant b$ and $A : \mathbb{R}^m \longrightarrow H$ is a (bounded) linear map. Finite dimensional truncations of the above problems are solved as finite dimensional interval programs. This generates a sequence $\{x^k\}$ with each x^k being optimal for the truncated problem at the kth stage. This sequence is shown to converge to an optimal solution of the problem $SILP(a, b, c, A)$. We also show how this idea can be used to obtain optimal solutions for continuous semi-infinite linear programs and to obtain approximate optimal solutions to doubly infinite interval linear programs.

We will quickly go through the necessary definitions and results needed for proving our main result. We will state our results generally and then specialize later. Let H_1 and H_2 be real Hilbert spaces with H_2 partially ordered, $A : H_1 \to H_2$ be linear, $c \in H_1$ and $a, b \in H_2$ with $a \leqslant b$. Consider the interval linear program $ILP(a, b, c, A)$:

$$\begin{aligned}&\text{Maximize} && \langle c, x\rangle\\ &\text{subject to} && a \leqslant Ax \leqslant b.\end{aligned}$$

Definition 9.5. Set $F = \{x : a \leqslant Ax \leqslant b\}$. A vector x^* is said to be feasible for $ILP(a, b, c, A)$ if $x^* \in F$. The problem $ILP(a, b, c, A)$ is said to be feasible if $F \neq \emptyset$. A feasible vector x^* is said to be optimal if $\langle c, (x^* - x)\rangle \geqslant 0$ for every $x \in F$. $ILP(a, b, c, A)$ is said to be bounded if $sup\{\langle c, x\rangle : x \in F\} < \infty$. If $ILP(a, b, c, A)$ is bounded, then the value α of the problem is defined as $\alpha = sup\{\langle c, x\rangle : x \in F\}$.

We will assume throughout the paper that $F \neq \emptyset$. Boundedness is then characterized by Lemma 9.6, to follow. Let $A : H_1 \longrightarrow H_2$ be linear. A linear map $T : H_2 \longrightarrow H_1$ is called a {1}-inverse of A if $ATA = A$.

We recall that A bounded linear map A has a bounded {1}-inverse iff $R(A)$, the range space of A is closed [16].

Lemma 9.6 [12, Lemma 2]. *Suppose H_1 and H_2 are real Hilbert spaces, H_2 is partially ordered and $ILP(a, b, c, A)$ is feasible. If $ILP(a, b, c, A)$ is bounded then $c \perp N(A)$. The converse holds if A has a bounded {1}-inverse and $[a, b] = \{z \in H_2 : a \leqslant z \leqslant b\}$ is bounded.*

9.9. Finite dimensional approximation scheme

Let H be a partially ordered real Hilbert space with an orthonormal basis $\{u^n : n \in \mathbb{N}\}$ such that $u^n \in \mathscr{C}\forall n$, where $\mathscr{C}$ is the positive cone in H. Let $\{S_n\}$ be a sequence of subspaces in H such that $\{u^1, u^2, \ldots, u^n\}$ is an orthonormal basis for S_n. Then $\overline{\cup_{n=1}^{\infty} S_n} = H$. Let $T_n : H \longrightarrow H$ be the orthogonal projection of H onto S_n. Then for $x = (x_1, x_2, \ldots) \in H$, $T_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Then the sequence $\{T_n\}$ converges strongly to the identity operator on H.

Consider the semi-infinite linear program $SILP(a, b, c, A)$. We will assume that this problem is bounded and that the columns of A are linearly independent. Then $SILP(a, b, c, A)$ is bounded, by Lemma 9.6. Define the k^{th} subproblem $SILP(T_k a, T_k b, c, T_k A)$ denoted by $SILP_k$ by

$$\begin{aligned} &\text{Maximize} \quad \langle c, x\rangle \\ &\text{subject to} \quad T_k a \leqslant T_k A x \leqslant T_k b. \end{aligned}$$

$SILP_k$ has k interval inequalities in the m unknowns x_i, $i = 1, 2, \ldots, m$. In view of the remarks given above, it follows that $SILP(T_k a, T_k b, c, T_k A)$ is bounded whenever $k \geqslant m$.

We now state a convergence result for the sequence of optimal solutions of the finite dimensional subproblems. The proof will appear elsewhere [40].

Theorem 9.7. *Let $\{x^k\}$ be a sequence of optimal solutions for $SILP_k$. Then $\{x^k\}$ converges to an optimal solution of $SILP(a, b, c, A)$.*

We next present a simple example to illustrate Theorem 9.7. A detailed study of various examples showing how the above theorem can be used to obtain optimal solutions even for a continuous semi-infinite linear program, is presently underway [40].

Example. Let $H = \ell^2, A : \mathbb{R}^2 \to \ell^2$ be defined by

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{9} \\ \vdots & \vdots \end{pmatrix}$$

$a = \left(0, -1, -\frac{1}{4}, -\frac{1}{9}, \ldots\right)$, $b = (1, 0, 0, \ldots)$ and $c = (1, 0)$. Consider the semi-infinite program $SILP(a, b, c, A)$ given by

$$\begin{aligned} &\text{Maximize} \quad x_1 \\ &\text{subject to } 0 \leqslant x_1 \leqslant 1, \\ &\tfrac{-1}{k^2} \leqslant \tfrac{x_1}{k} + \tfrac{x_2}{k^2} \leqslant 0, \quad k = 1, 2, 3, \ldots \end{aligned}$$

Clearly $c \perp N(A)$, as $N(A) = \{0\}$ and so the problem is bounded, by Lemma 9.6. Rewriting the second set of inequalities, we get

$$\frac{-1}{k} \leqslant x_1 + \frac{x_2}{k} \leqslant 0, \quad k = 1, 2, 3, \ldots$$

Then $x_1 = 0$, by passing to limit as $k \to \infty$. Thus $(0, \alpha)$, $-1 \leqslant \alpha \leqslant 0$ is an optimal solution for $SILP(a, b, c, A)$ with optimal value 0.

Now we consider the finite truncations. Let $S_k = \{e^1, e^2, \ldots, e^k\}$ where $\{e^1, e^2, e^3, \ldots\}$ is the standard orthonormal basis for ℓ^2. Define $T_k = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ where I_k is the identity map of order $k \times k$, $k \geqslant 2$. Thus $R(T_k A) = S_k$. The subproblem $SILP(T_k a, T_k b, c, T_k A)$ is

$$\begin{aligned} &\text{Maximize} \quad x_1 \\ &\text{subject to} \quad 0 \leqslant x_1 \leqslant 1, \\ &\frac{-1}{(l-1)^2} \leqslant \frac{x_1}{l-1} + \frac{x_2}{(l-1)^2} \leqslant 0, \quad l = 2, 3, \ldots, k. \end{aligned}$$

Clearly, the subproblem $SILP(T_ka, T_kb, c, T_kA)$ is bounded and the optimal solution is $x^k = \left(\frac{1}{k-2}, \frac{-(k-1)}{k-2}\right)$. This converges to $(0, -1)$ in the limit as $k \to \infty$.

9.10. Approximate optimal solutions to a doubly infinite linear program

In this section, we show how we can obtain approximate optimal solutions to continuous infinite linear programming problems. With the notations earlier, assume that $ILP(a, b, c, A)$ is bounded. Let c^k denote the vector in H whose first k components are the same as the vector c and whose other components are zero. Let A_k denote the matrix with k columns and infinitely many rows whose k columns are precisely the same as the first k columns of A. Consider the problem:

$$\begin{aligned} &\text{Maximize} \quad \langle c^k, u\rangle \\ &\text{subject to} \quad a \leqslant A_k u \leqslant b \end{aligned}$$

for $u \in \mathbb{R}^k$. This problem is $SILP(a, b, c^k, A_k)$ for each k. Suppose that the columns of A_k are linearly independent for all k. Denoting $SILP(a, b, c^k, A_k)$ by P_k, we solve P_k using Theorem 9.7. This generates a sequence $\{u^k\}$. Let $u^k = (u_1, u_2, \ldots, u_k)^T$ be an optimal solution for P_k. Let $u = (u^k, 0)^T$. Then $A_{k+1}u = A_k u^k$. Since $a \leqslant A_n u^k \leqslant b$, u is feasible for P_{k+1}. Let $v(P_k)$ denote the value of problem P_k, viz., $v(P_k) := sup\{\langle c^k, u\rangle : a \leqslant A_k u \leqslant b.\}$ Then $v(P_{k+1}) \geqslant v(P_k)$ as $\langle c^{k+1}, u\rangle \geqslant \langle c^k, u\rangle$. Let $x^k = (u^k, 0, 0 \cdots) \in H$. Then x^k is feasible for $ILP(a, b, c, A)$ and $\langle c, x^k\rangle = \langle c^k, x^k\rangle$, the value of P_k. Thus the sequence $\{\langle c, x^k\rangle\}$ is an increasing sequence bounded above by the value of $ILP(a, b, c, A)$ and is hence convergent. It follows that x^k is weakly convergent. However, unlike the semi-infinite case, it need not be convergent. (In the following example, it turns out that x^k is convergent.) So, we have optimal value convergence but not optimal solution convergence. Hence our method yields only an approximate optimal solution for a continuous linear program. It will be interesting to study how good is this approximation.

Example. Let $A : \ell^2 \to \ell^2$ be defined by $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \cdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. Let $c = \left(1, \frac{1}{4}, \frac{1}{9}, \ldots\right)$, $a = \mathbf{0}$ and $b = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. The problem $ILP(a, b, c, A)$ is

$$\begin{aligned} &\text{Maximize} \quad x_1 + \frac{x_2}{4} + \frac{x_3}{9} + \cdots \\ &\text{subject to} \quad 0 \leqslant \sum_{i=1}^{l} \frac{x_i}{i} \leqslant \frac{1}{l}, \quad l = 1, 2, 3, \ldots. \end{aligned}$$

Clearly $A \in BL(\ell^2)$ and $N(A) = \{0\}$. Therefore $c \perp N(A)$, i.e., the problem $ILP(a, b, c, A)$ is bounded. Consider the kth subproblem P_k:

$$\begin{aligned} &\text{Maximize} \quad \langle c^k, u\rangle \\ &\text{subject to} \quad a \leqslant A_r u \leqslant b, \quad r = 1, 2, \ldots, k, \end{aligned}$$

where $c^k = (1, 1/4, 1/9, \ldots, 1/k^2)$, and $A_k : \mathbb{R}^k \to \ell^2$ is defined by $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$. Clearly, the subproblem is bounded. By the finite dimensional scheme, the optimal solution of the subproblem P_k is found to be $x^k = \left(1, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{k-1}\right)$ which converges to $x^* = \left(1, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\right)$. The optimal value converges to 6.450. We conclude by observing that since A is invertible, it is possible to solve the original problem by the methods in [12], directly.

9.11. Eigenvalues of the Laplacian on an elliptic domain

The importance of eigenvalue problems concerning the Laplacian is well documented in classical and modern literatures. Finding the eigenvalues for various geometries of the domains has posed many challenges which have included infinite systems of algebraic equations (see Section 9.4), asymptotic methods, integral equations,finite element methods, etc. Details of earlier work can be found in [44]. The eigenvalue problems of the Laplacian is represented by the Helmholtz equations, Telegraph equations or the equations of the vibrating membrane and is given by

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda^2 u = 0 \quad \text{in } D,\\
&u = 0 \quad \text{on } \partial D,
\end{aligned}$$

where D is a plane region bounded by smooth curve ∂D. The eigenvalue k_n and corresponding eigenfunctions u_n describe the natural modes of vibration of a membrane. According to the maximum principle, k_n must be positive for a nontrivial solution to exist. Further $k_n, n = 1, 2, \ldots$ are ordered satisfying

$$0 < k_1 < k_2 < \cdots < k_n < \cdots$$

Using complex variables $z = x + iy, \bar{z} = x - iy$, the problem becomes

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\lambda^2}{4} u = 0 \quad \text{in } D \text{ and } u = 0 \text{ on } C \tag{9.17}$$

and $u = u(z, \bar{z})$, It is well known that the general solution to (9.17) is given by

$$u = \left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) dt \right\} + \text{conjugate}, \tag{9.18}$$

where $f_0(z)$ is an arbitrary holomorphic function which can be expressed as

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{9.19}$$

J_0 is the Bessel's function of the first kind of order 0, which is given by a series representation as,

$$J_0\left(\lambda\sqrt{\bar{z}(z-t)}\right) = \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k \frac{\bar{z}^k (z-t)^k}{k!k!}. \tag{9.20}$$

On substituting for $f_0(z)$ in the general solution, we get the general solution to the Helmholtz equation as given by

$$\begin{aligned}
u &= 2a_0 J_0(\lambda\sqrt{z\bar{z}})\\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} A_{n,k} \left((z+\bar{z})^n + \sum_{m=1}^{n/2} (-1)^m \frac{n}{m} \binom{n-m-1}{m-1} (z+\bar{z})^{n-2m} (z\bar{z})^m a_n \right) (z\bar{z})^k.
\end{aligned}$$

demonstrating that the general solution of (9.17) without boundary conditions can be expressed in terms of power of $z\bar{z}$ and $(z + \bar{z})$.

In our case, we consider the domain bounded by the ellipse represented by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

which can be expressed correspondingly in the complex plane by

$$(z+\bar{z})^2 = a + bz\bar{z}, \tag{9.21}$$

where $a = \frac{4\alpha^2\beta^2}{\beta^2-\alpha^2}$, $b = \frac{4\alpha^2}{\alpha^2-\beta^2}$.

After considerable manipulation, we get the value of u on the ellipse as,

$$\begin{aligned} u = {} & 2a_0 + \sum_{n=1}^{\infty} A_{2n,0} b_{0,n} a_n \\ & + \sum_{k=1}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k \frac{2a_0}{k!k!} (z\bar{z})^k \\ & + \sum_{k=1}^{\infty} \left[\sum_{n=1}^{k} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{n} A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k \\ & + \sum_{k=1}^{\infty} \left[\sum_{n=k+1}^{\infty} \left(A_{2n,k} b_{0,n} + \sum_{l=1}^{k} A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k, \end{aligned} \tag{9.22}$$

For $u = 0$ on the elliptic boundary, we equate the powers of $z\bar{z}$ to zero where we arrive at an infinite system of linear equations of the form

$$\sum_{k=0}^{\infty} d_{kn} a_n = 0, \quad n = 0, 1, 2, \ldots \infty, \tag{9.23}$$

where d_{kn}'s are know polynomials of λ^2. In [44], the infinite system is truncated to $n \times n$ system and numerical vaues were calculate and compared to existing results in literature. The method derived here provides a procedure to numerically calculate the eigenvalues.

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