

A note on perfect scalar fields

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We derive a condition on the Lagrangian density describing a generic, single, non-canonical scalar field, by demanding that the intrinsic, non-adiabatic pressure perturbation associated with the scalar field vanishes identically. Based on the analogy with perfect fluids, we refer to such fields as perfect scalar fields. It is common knowledge that models that depend only on the kinetic energy of the scalar field (often referred to as pure kinetic models) possess no non-adiabatic pressure perturbation. While we are able to construct models that seemingly depend on the scalar field and also do not contain any non-adiabatic pressure perturbation, we find that all such models that we construct allow a redefinition of the field under which they reduce to pure kinetic models. We show that, if a perfect scalar field drives inflation, then, in such situations, the first slow roll parameter will always be a monotonically decreasing function of time. We point out that this behavior implies that these scalar fields can not lead to features in the inflationary, scalar perturbation spectrum.

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I. INTRODUCTION

In cosmology, the sources of matter that drive the expansion of the universe are often considered to be either fluids or, in particular, while considering the inflationary epoch or late time acceleration, as scalar fields. Almost always, the fluids are assumed to be perfect, i.e. it is assumed that they do not possess any intrinsic, non-adiabatic pressure perturbation. In contrast, generically, the non-adiabatic pressure perturbation proves to be non-zero for scalar fields. It is well known that iso-curvature (i.e. the extrinsic, non-adiabatic pressure) perturbations are always present when one considers multiple fields and/or fluids. In the context of inflation, it is often said that single scalar field models are adiabatic (see any of the following texts [1] or reviews [2]). While this is true, it is not because the non-adiabatic pressure perturbation is identically zero for the single field models, but because they decay exponentially for cosmological modes after they leave the Hubble radius during inflation (for a discussion on specific cases, see Refs. [3]; for a generic discussion, see Ref. [4]). It is interesting to enquire whether there exist scalar field models wherein the intrinsic, non-adiabatic pressure perturbation vanishes exactly (say, at the first order in the perturbations), as in the case of perfect fluids.

In this paper, using the standard definition of the non-adiabatic pressure perturbation associated with the scalar fields, we shall obtain a condition on the Lagrangian density describing the scalar field by demanding that the non-adiabatic pressure perturbation is identically zero at the first order in the perturbation theory. We shall also discuss specific examples of scalar field models that satisfy this condition. Following the convention with fluids, we shall call these fields as perfect scalar fields. It is well known that models that depend only on the kinetic energy of the scalar field behave as perfect fluids. We are also able to construct models which seem to depend on the scalar field, and possess no non-adiabatic pressure perturbation. However, we find that all such models that we consider reduce to pure kinetic models after a suitable redefinition of the field. Interestingly, we shall show that, if the perfect scalar fields are used to drive inflation, they can not lead to features in the inflationary, scalar perturbation spectrum.

This paper is organized as follows. In the following section, we shall rapidly outline essential, *linear*, cosmological perturbation theory. We shall highlight the key equations, and point out the standard definition of the intrinsic, non-adiabatic pressure perturbation. In Sec. III, by demanding that the non-adiabatic pressure perturbation associated with the scalar field vanishes exactly, we shall obtain a condition on the Lagrangian density of the scalar field. We shall discuss a few specific examples of Lagrangian densities that satisfy this condition in Sec. IV. In Sec. V, we shall show that, if scalar fields with a vanishing non-adiabatic pressure perturbation are responsible for inflation then, in such cases, the first slow roll parameter is always a monotonically decreasing function of time. We shall argue that this behavior points to the fact that such inflatons will not lead to features in the primordial scalar perturbation

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spectrum. Finally, we shall close with a few concluding remarks in Sec. VI.

A few words on our conventions and notations are in order before we proceed. We shall set $c = 1$, but shall display G explicitly. We shall work in a spatially flat, $(3 + 1)$ -dimensional Friedmann background, and we shall adopt the metric signature of $(+, -, -, -)$. Note that, while the Greek indices μ or ν shall denote the spacetime coordinates, the Latin indices i or j shall denote the spatial coordinates. The sub-script k shall refer to the Fourier component of the perturbations. We shall express all the quantities in terms of either the cosmic or the conformal time coordinates. While an overdot shall denote differentiation with respect to the cosmic time, an overprime shall denote differentiation with respect to the conformal time. It is handy to note that, for any given function, say, f , $\dot{f} = (f'/a)$ and $\ddot{f} = [(f''/a^2) - (f' a'/a^3)]$, where a is the scale factor associated with the Friedmann metric.

II. SCALAR PERTURBATIONS IN A FRIEDMANN UNIVERSE—KEY EQUATIONS AND DEFINITIONS

In this section, using the equations governing the evolution of the scalar perturbations induced by an arbitrary matter field and the standard definition of the non-adiabatic pressure perturbation, we shall arrive at an expression for the non-adiabatic pressure perturbation associated with the scalar field in terms of the adiabatic and the effective speeds of sound associated with the perturbations.

In $(3 + 1)$ -dimensions, when no perturbations are present, the spatially flat Friedmann universe is described by the line element

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta) (d\eta^2 - d\mathbf{x}^2), \quad (1)$$

where t is the cosmic time, $a(t)$ is the scale factor, and $\eta = \int [dt/a(t)]$ denotes the conformal time. If ρ and p denote the energy density and the pressure of the smooth component of the matter field that is driving the expansion, then the Einstein's equations for the above line-element lead to the following Friedmann equations for the scale factor $a(t)$:

$$H^2 = \left(\frac{8\pi G}{3}\right) \rho \quad \text{and} \quad \left(\frac{\ddot{a}}{a}\right) = -\left(\frac{4\pi G}{3}\right) (\rho + 3p), \quad (2)$$

where $H = (\dot{a}/a)$ is the Hubble parameter.

A. Equations governing the scalar perturbations and the definition of the non-adiabatic pressure perturbation

If we now take into account the scalar perturbations to the background metric (1), then the Friedmann line-element, in general, can be written as [1, 2]

$$ds^2 = (1 + 2A) dt^2 - 2a(t) (\partial_i B) dt dx^i - a^2(t) [(1 - 2\psi) \delta_{ij} + 2(\partial_i \partial_j E) dx^i dx^j], \quad (3)$$

where A , B , ψ and E are the scalar functions that describe the perturbations. The gauge-invariant Bardeen variables that characterize the two degrees of freedom describing the scalar perturbations are given by [5]

$$\Phi \equiv A + \left[a(B - a\dot{E}) \right] \quad \text{and} \quad \Psi \equiv \psi - \left[aH(B - a\dot{E}) \right]. \quad (4)$$

In the absence of anisotropic stresses, as it is in the case of the scalar field sources that we shall be interested in, it can be readily shown that, at the linear order in the perturbations, the non-diagonal, spatial components of the Einstein's equations lead to the relation: $\Phi = \Psi$. The remaining first order Einstein's equations then reduce to [1, 2]

$$\left(\frac{1}{a^2}\right) \nabla^2 \Phi - 3H (\dot{\Phi} + H\Phi) = (4\pi G) \left[\delta\rho + (\dot{\rho}a)(B - a\dot{E}) \right] = (4\pi G) \widehat{\delta\rho}, \quad (5)$$

$$\partial_i (\dot{\Phi} + H\Phi) = (4\pi G) \partial_i \left[\delta\sigma + [(\rho + p)a](B - a\dot{E}) \right] = (4\pi G) (\partial_i \widehat{\delta\sigma}), \quad (6)$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2) \Phi = (4\pi G) \left[\delta p + (\dot{p}a)(B - a\dot{E}) \right] = (4\pi G) \widehat{\delta p}, \quad (7)$$

where $\delta\rho$, $\delta\sigma$ and δp denote the perturbations at the linear order in the energy density, flux, and the pressure of the matter field, respectively, while the quantities $\widehat{\delta\rho}$, $\widehat{\delta\sigma}$ and $\widehat{\delta p}$ represent the corresponding gauge-invariant quantities.

The first and the third of the above first order Einstein equations can be combined to lead to the following differential equation for the Bardeen potential Φ [1, 2]:

$$\Phi'' + 3\mathcal{H}(1 + c_A^2)\Phi' - c_A^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_A^2)\mathcal{H}^2]\Phi = (4\pi G a^2) \delta p^{\text{NA}}, \quad (8)$$

where $\mathcal{H} = (H a)$ is the conformal Hubble parameter. In arriving at this equation, we have changed over to the conformal time coordinate, and have made use of the following standard definition of the non-adiabatic pressure perturbation δp^{NA} [6]:

$$\delta p^{\text{NA}} = (\widehat{\delta p} - c_A^2 \widehat{\delta \rho}) = (\delta p - c_A^2 \delta \rho), \quad (9)$$

where $c_A \equiv \sqrt{p'/\rho'}$ denotes the adiabatic speed of the perturbations.

B. Perturbations induced by a generic scalar field

Consider a generic scalar field ϕ that is described by the action [7]

$$S[\phi] = \int d^4x \sqrt{-g} \mathcal{L}(X, \phi), \quad (10)$$

where X is a term that describes the kinetic energy of the scalar field, and is defined as

$$X = \left(\frac{1}{2}\right) (\partial_\mu \phi \partial^\mu \phi). \quad (11)$$

Let us assume that the Lagrangian density \mathcal{L} is an arbitrary function of the kinetic term X and the field ϕ . The stress-energy tensor associated with the above action can be written as

$$T_\nu^\mu = \left(\frac{\partial \mathcal{L}}{\partial X}\right) (\partial^\mu \phi \partial_\nu \phi) - \delta_\nu^\mu \mathcal{L}. \quad (12)$$

When no perturbations are present, the energy density and the pressure associated with the homogeneous scalar field are given by

$$\rho = \left(\frac{\partial \mathcal{L}}{\partial X}\right) (2X) - \mathcal{L} \quad \text{and} \quad p = \mathcal{L}, \quad (13)$$

with $X = (\dot{\phi}^2/2)$. If we now denote the perturbation in the scalar field as $\delta\phi$, then, the perturbations in the energy density, the momentum flux and the pressure of the scalar field can be obtained to be [7, 8]

$$\delta\rho = \left[\left(\frac{\partial \mathcal{L}}{\partial X}\right) + (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X^2}\right)\right] \dot{\phi} (\delta\dot{\phi} - \dot{\phi} A) - \left[\left(\frac{\partial \mathcal{L}}{\partial \phi}\right) - (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X \partial \phi}\right)\right] \delta\phi, \quad (14)$$

$$\delta\sigma = \left(\frac{\partial \mathcal{L}}{\partial X}\right) (\dot{\phi} \delta\phi), \quad (15)$$

$$\delta p = \left(\frac{\partial \mathcal{L}}{\partial X}\right) \dot{\phi} (\delta\dot{\phi} - \dot{\phi} A) + \left(\frac{\partial \mathcal{L}}{\partial \phi}\right) \delta\phi. \quad (16)$$

The gauge-invariant perturbation in the scalar field, say, $\delta\varphi$, is given by [1, 2]

$$\delta\varphi = \left[\delta\phi + (\dot{\phi} a) (B - a \dot{E})\right]. \quad (17)$$

The gauge-invariant perturbations in the energy density, the momentum flux and the pressure of the scalar field can be expressed in terms of the gauge-invariant perturbation in the scalar field $\delta\varphi$ and the Bardeen potential Φ as follows:

$$\widehat{\delta\rho} = \left[\left(\frac{\partial \mathcal{L}}{\partial X}\right) + (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X^2}\right)\right] \dot{\phi} (\delta\dot{\varphi} - \dot{\phi} \Phi) - \left[\left(\frac{\partial \mathcal{L}}{\partial \phi}\right) - (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X \partial \phi}\right)\right] \delta\varphi, \quad (18)$$

$$\widehat{\delta\sigma} = \left(\frac{\partial \mathcal{L}}{\partial X}\right) (\dot{\phi} \delta\varphi), \quad (19)$$

$$\widehat{\delta p} = \left(\frac{\partial \mathcal{L}}{\partial X}\right) \dot{\phi} (\delta\dot{\varphi} - \dot{\phi} \Phi) + \left(\frac{\partial \mathcal{L}}{\partial \phi}\right) \delta\varphi. \quad (20)$$

Our goal now is to arrive at an expression for the non-adiabatic pressure perturbation δp^{NA} for the scalar field in terms of the Bardeen potential Φ . Algebraically, we find that, one useful way would be to derive the equivalent of the Bardeen equation (8) for the case of the scalar field. On substituting the above expressions for $\widehat{\delta\rho}$ and $\widehat{\delta p}$ in the first order Einstein's equations (5) and (7), we find that the Bardeen potential Φ induced by the perturbations in the scalar field satisfies the following differential equation:

$$\Phi'' + 3\mathcal{H} (1 + c_{\text{A}}^2) \Phi' - c_{\text{A}}^2 \nabla^2 \Phi + [2\mathcal{H}' + (1 + 3c_{\text{A}}^2) \mathcal{H}^2] \Phi = (c_{\text{s}}^2 - c_{\text{A}}^2) \nabla^2 \Phi, \quad (21)$$

where the quantity c_{s} is often referred to as the effective speed of the perturbations, and is given by [7]

$$c_{\text{s}}^2 = \left[\frac{(\partial\mathcal{L}/\partial X)}{(\partial\mathcal{L}/\partial X) + (2X) (\partial^2\mathcal{L}/\partial X^2)} \right]. \quad (22)$$

It should be mentioned that, in arriving at the above equation for the Bardeen potential Φ , we have made use of the following equation of motion for the background scalar field ϕ :

$$\left[\left(\frac{\partial\mathcal{L}}{\partial X} \right) + (2X) \left(\frac{\partial^2\mathcal{L}}{\partial X^2} \right) \right] \ddot{\phi} + \left[(3H) \left(\frac{\partial\mathcal{L}}{\partial X} \right) + \dot{\phi} \left(\frac{\partial^2\mathcal{L}}{\partial X \partial \phi} \right) \right] \dot{\phi} - \left(\frac{\partial\mathcal{L}}{\partial \phi} \right) = 0, \quad (23)$$

which, in turn, can be derived from the equation describing the conservation of the background energy density, viz.

$$\dot{\rho} + (3H) (\rho + p) = 0, \quad (24)$$

and using the expressions for ρ and p from Eq. (13). Upon comparing the equations (8) and (21), it is straightforward to see that the non-adiabatic pressure perturbation δp^{NA} for the scalar field can be expressed as

$$\delta p^{\text{NA}} = \left(\frac{c_{\text{s}}^2 - c_{\text{A}}^2}{4\pi G a^2} \right) \nabla^2 \Phi. \quad (25)$$

This result for δp^{NA} can be obtained in a more straightforward manner by first noticing that the quantities $\widehat{\delta\rho}$, $\widehat{\delta\sigma}$ and $\widehat{\delta p}$ as given by the expressions (18), (19) and (20), respectively, are related as follows:

$$\widehat{\delta p} = \left[c_{\text{s}}^2 \widehat{\delta\rho} + (3H) (c_{\text{s}}^2 - c_{\text{A}}^2) \widehat{\delta\sigma} \right]. \quad (26)$$

This relation, in turn, allows us to write δp^{NA} as

$$\delta p^{\text{NA}} = \left(\widehat{\delta p} - c_{\text{A}}^2 \widehat{\delta\rho} \right) = (c_{\text{s}}^2 - c_{\text{A}}^2) \left[\widehat{\delta\rho} + (3H) \widehat{\delta\sigma} \right]. \quad (27)$$

Also, the first order Einstein's equations (5) and (6) can be combined to arrive at

$$\left[\widehat{\delta\rho} + (3H) \widehat{\delta\sigma} \right] = \left(\frac{1}{4\pi G a^2} \right) \nabla^2 \Phi. \quad (28)$$

Evidently, these last two equations immediately lead to the expression (25) for δp^{NA} .

The reason for the appearance of the quantity c_{s}^2 in this expression for δp^{NA} can be understood as follows. Let us work in a gauge wherein $\delta\phi = 0$, a choice of coordinates that is often referred to as the comoving gauge [2]. In such a gauge, no flux of energy arises, i.e. ${}^{(\text{C})}\delta\sigma = 0$ [cf. Eq. (15)], and we have used the super-script (C) to denote the fact that we are working in the comoving gauge. It is clear from the expressions (14) and (16) for $\delta\rho$ and δp that, in the comoving gauge, we have

$${}^{(\text{C})}\delta p = c_{\text{s}}^2 {}^{(\text{C})}\delta\rho. \quad (29)$$

Therefore, the non-adiabatic pressure perturbation δp^{NA} in this gauge is given by

$$\delta p^{\text{NA}} = [{}^{(\text{C})}\delta p - c_{\text{A}}^2 {}^{(\text{C})}\delta\rho] = (c_{\text{s}}^2 - c_{\text{A}}^2) {}^{(\text{C})}\delta\rho \quad (30)$$

and, since ${}^{(\text{C})}\delta\sigma = 0$, the relation (28) leads to (25), as required.

III. CONDITION FOR VANISHING NON-ADIABATIC PRESSURE PERTURBATION

In Fourier space, $\nabla^2 \Phi \propto (k^2 \Phi_k)$. It is then clear from Eq. (25) that the non-adiabatic pressure perturbation δp^{NA} will vanish in the super Hubble limit (i.e. as $k \rightarrow 0$) for all scalar fields [3, 4]. This occurs, for instance, when the modes are well outside the Hubble radius during the inflationary epoch [3]. It is interesting to enquire whether the non-adiabatic pressure perturbation δp^{NA} vanishes *identically* for any scalar field model. In such a case, the scalar field will behave *exactly* like a perfect fluid described by an equation of state, at least at the first order in the perturbations. It is evident from Eq. (25) that such a behavior will be possible if and only if $c_s^2 = c_A^2$. Based on this condition, we shall now arrive at the corresponding condition on the Lagrangian density describing the scalar field.

Using Eq. (24) that describes the conservation of energy, and the expressions (13) for the background energy density and pressure associated with the scalar field, the adiabatic speed of the perturbations c_A^2 can be expressed as

$$c_A^2 = \left(\frac{\dot{p}}{\dot{\rho}} \right) = - \left(\frac{\dot{p}}{(3H)(\rho+p)} \right) = - \left(\frac{\ddot{\phi} (\partial \mathcal{L} / \partial X) + (\partial \mathcal{L} / \partial \phi)}{(3H\dot{\phi}) (\partial \mathcal{L} / \partial X)} \right). \quad (31)$$

The equation of motion (23) governing the field, then allows us to arrive at the following expression for the adiabatic speed of the perturbations:

$$c_A^2 = - \left((3H\dot{\phi}) \left(\frac{\partial \mathcal{L}}{\partial X} \right) \left[\left(\frac{\partial \mathcal{L}}{\partial X} \right) + (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \right] \right)^{-1} \\ \times \left[2 \left(\frac{\partial \mathcal{L}}{\partial X} \right) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) - (3H\dot{\phi}) \left(\frac{\partial \mathcal{L}}{\partial X} \right)^2 + (2X) \left(\frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) - (2X) \left(\frac{\partial \mathcal{L}}{\partial X} \right) \left(\frac{\partial^2 \mathcal{L}}{\partial X \partial \phi} \right) \right]. \quad (32)$$

From the definition (22) of c_s^2 , it is then straightforward to show that the condition $c_A^2 = c_s^2$ implies that

$$\left(\frac{\partial \mathcal{L}}{\partial X} \right) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) + X \left(\frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) - X \left(\frac{\partial \mathcal{L}}{\partial X} \right) \left(\frac{\partial^2 \mathcal{L}}{\partial X \partial \phi} \right) = 0. \quad (33)$$

This condition can be further simplified to be

$$\frac{\partial}{\partial X} \left[\left(\frac{1}{Y} \right) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) \right] = 0, \quad (34)$$

where we have defined Y as

$$Y = X \left(\frac{\partial \mathcal{L}}{\partial X} \right). \quad (35)$$

As we had mentioned before, based on the analogy with fluids, we shall refer to scalar fields that satisfy the condition (34) as perfect scalar fields. Interestingly, the condition (33) above has been arrived at earlier while considering a completely different problem involving the stationary configurations of scalar fields [9]. This seems to indicate some relation between stationarity and a vanishing non-adiabatic pressure perturbation, which invites further investigation.

It is straightforward to show that, if a given Lagrangian density, say, $\mathcal{L}_1(X, \phi)$, satisfies the condition (34), then the Lagrangian densities, say, $\mathcal{L}_2(X, \phi)$ and $\mathcal{L}_3(X, \phi)$, that are related to $\mathcal{L}_1(X, \phi)$ as follows:

$$\mathcal{L}_2(X, \phi) = C_1 \exp [C_2 \mathcal{L}_1(X, \phi)] \quad \text{and} \quad \mathcal{L}_3(X, \phi) = C_3 \log [C_4 \mathcal{L}_1(X, \phi)], \quad (36)$$

where C_1 , C_2 , C_3 and C_4 are constants, also satisfy the same condition. Though these three Lagrangian densities satisfy the condition (34), it may be worthwhile to note that, they, in fact, lead to different evolution equations for the background as well as the perturbations.

IV. EXAMPLES OF PERFECT SCALAR FIELDS

In this section, we shall construct examples of perfect scalar fields that satisfy the condition (34). We shall first construct purely kinetic models that do not depend on the scalar field directly, and then consider models that also seemingly depend on the scalar field.

A. Pure kinetic models

Note that any Lagrangian density \mathcal{L} that does not explicitly depend on the scalar field ϕ , i.e. when $(\partial\mathcal{L}/\partial\phi) = 0$, naturally satisfies the condition (34). These Lagrangian densities are often referred to as pure kinetic models. It is well known that pure kinetic models can be described as barotropic, perfect fluids which, by definition, do not possess any non-adiabatic pressure perturbation (in this context, see Refs. [10, 11]). Such models were originally considered in the context of inflation [7], and they have been resurrected more recently as a potential candidate of dark energy [12]. We shall now discuss a couple of specific examples of these models.

1. A scalar field mimicking a perfect fluid with a constant equation of state

Consider a Lagrangian density of the following form:

$$\mathcal{L}(X) = V_0 X^\alpha, \quad (37)$$

where V_0 is a constant potential, and α is a real number. Such a Lagrangian density clearly satisfies the condition (34). The background energy density and pressure corresponding to this Lagrangian density can be obtained to be [cf. Eq. (13)]

$$\rho = (2\alpha - 1) V_0 X^\alpha \quad \text{and} \quad p = V_0 X^\alpha, \quad (38)$$

so that the resulting equation of state $w \equiv (p/\rho)$ is a constant, and is given by

$$w = \left(\frac{1}{2\alpha - 1} \right). \quad (39)$$

In other words, the scalar field described by the Lagrangian density (37) essentially behaves like a perfect fluid with a constant equation of state. It is also useful to note that, in such a case, the two speeds of sound simplify to [cf. Eqs. (22) and (32)]: $c_A^2 = c_s^2 = w$.

2. Models that behave as the Chaplygin gas

Consider a tachyon that is governed by the Lagrangian density

$$\mathcal{L}(X) = - \left(V_0 \sqrt{1 - 2X} \right), \quad (40)$$

where V_0 is a constant potential. Needless to add, this Lagrangian density also satisfies the condition (34). The background energy density and pressure corresponding to the above Lagrangian density are given by

$$\rho = \left(\frac{V_0}{\sqrt{1 - 2X}} \right) \quad \text{and} \quad p = - \left(V_0 \sqrt{1 - 2X} \right), \quad (41)$$

so that we can write

$$p = - \left(\frac{V_0^2}{\rho} \right). \quad (42)$$

This is the equation of state that describes the Chaplygin gas [13–16] and, in such a case, we obtain that

$$c_A^2 = c_s^2 = (1 - 2X) = -w. \quad (43)$$

Upon using the conservation equation (24) for the background energy density, it can be readily shown that the equation of state parameter of the Chaplygin gas evolves as a function of the scale factor in the following fashion:

$$w(a) = - \left(\frac{V_0^2}{V_0^2 + (\mathcal{A}/a^6)} \right), \quad (44)$$

where \mathcal{A} is a positive constant.

A more generic form of the Lagrangian density (40) that satisfies the condition (34) is given by

$$\mathcal{L}(X) = - \left(V_0 \left[1 - (X/V_0)^{[(1+\alpha)/2\alpha]} \right]^{\alpha/(1+\alpha)} \right), \quad (45)$$

where V_0 is again a constant, while α is a real number. It is straightforward to show that, for this Lagrangian density, the homogeneous energy density and pressure are given by

$$\rho = \left(V_0 \left[1 - (X/V_0)^{[(1+\alpha)/2\alpha]} \right]^{-1/(1+\alpha)} \right) \quad \text{and} \quad p = - \left(V_0 \left[1 - (X/V_0)^{[(1+\alpha)/2\alpha]} \right]^{\alpha/(1+\alpha)} \right), \quad (46)$$

so that, we have

$$p = - \left(\frac{V_0^{(1+\alpha)}}{\rho^\alpha} \right), \quad (47)$$

which is the equation of state of the generalized Chaplygin gas. Over the last few years, such models have been considered in the literature as possible candidates for dark energy [14–16]. In this case, the equation of state parameter w evolves as

$$w(a) = - \left(\frac{V_0^{(1+\alpha)}}{V_0^{(1+\alpha)} + (\mathcal{A}/a)^{[3(1+\alpha)]}} \right), \quad (48)$$

where \mathcal{A} is again a positive constant. Also, we find that

$$c_A^2 = c_s^2 = \alpha \left[1 - (X/V_0)^{[(1+\alpha)/2\alpha]} \right]. \quad (49)$$

3. Another model

Now, consider a Lagrangian density of the following form:

$$\mathcal{L}(X) = -V_0 \left(\sqrt{X} - 1 \right)^{-\alpha}, \quad (50)$$

where, again, V_0 is a constant, and α is a real number. The background energy density and pressure corresponding to the above Lagrangian density are found to be

$$\rho = V_0 \left[(1 + \alpha) \sqrt{X} - 1 \right] \left(\sqrt{X} - 1 \right)^{-(1+\alpha)} \quad \text{and} \quad p = -V_0 \left(\sqrt{X} - 1 \right)^{-\alpha}. \quad (51)$$

In such a case, it can be shown that the equation of state parameter w evolves with the scale factor as

$$w(a) = - \left(\frac{1}{(1 + \alpha) + (\alpha/\mathcal{A}) a^{-[3/(1+\alpha)]}} \right), \quad (52)$$

where, as before, \mathcal{A} is a positive constant, while

$$c_A^2 = c_s^2 = \left(\frac{1 - \sqrt{X}}{(1 + \alpha) \sqrt{X}} \right). \quad (53)$$

Note that the expression (52) for $w(a)$ implies that, for positive α , $w(a)$ evolves from zero to an asymptotic value of $-(1 + \alpha)^{-1}$. For an appropriate choice of the constants \mathcal{A} and α , it is then possible to achieve $w(a)$ at the present epoch to be less than $-(1/3)$. Hence, the above Lagrangian density may be considered as a possible model of dark energy.

B. Masquerading scalar fields

We find that we can construct three types of models that *seemingly* depend on the scalar field and also satisfy the condition (34).

The first type are models wherein the Lagrangian density of the scalar field can be expressed as a sum of the kinetic and the potential terms as follows:

$$\mathcal{L}(X, \phi) = f(X) - V(\phi). \quad (54)$$

In such a situation, the condition (34) immediately restricts the function $f(X)$ to be

$$f(X) = \alpha \log (X/X_0), \quad (55)$$

where α and X_0 are positive constants. Moreover, the energy density and pressure associated with the homogeneous scalar field are given by

$$\rho = \alpha [2 - \log (X/X_0)] + V(\phi) \quad \text{and} \quad p = \alpha \log (X/X_0) - V(\phi), \quad (56)$$

and we find that the equation of state parameter evolves as

$$w(a) = -1 + \left(\frac{2\alpha}{\mathcal{A} - 6\alpha \log a} \right), \quad (57)$$

where \mathcal{A} is a suitably chosen positive constant. Also, we obtain that $c_A^2 = c_s^2 = -1$.

The second type of models are wherein we can write the Lagrangian density as a product of the kinetic and the potential terms in the following fashion [7, 15, 16]:

$$\mathcal{L}(X, \phi) = f(X) V(\phi). \quad (58)$$

In such cases, the condition (34) restricts the function $f(X)$ to be X^α , where α is a real number. The smooth energy density and pressure associated with this Lagrangian density can be obtained to be

$$\rho = (2\alpha - 1) [X^\alpha V(\phi)] \quad \text{and} \quad p = [X^\alpha V(\phi)]. \quad (59)$$

The resulting equation of state is evidently a constant, and is given by

$$w = \left(\frac{1}{2\alpha - 1} \right). \quad (60)$$

Obviously, this example is a more general case of the Lagrangian density (37) we had discussed before. As in the earlier example, the adiabatic and the effective speeds of sound associated with the perturbations reduce to

$$c_A^2 = c_s^2 = \left(\frac{1}{2\alpha - 1} \right) = w. \quad (61)$$

Note that the two Lagrangian densities that we have considered in this sub-section until now can be related to each other through the relations (36).

The third type of scalar fields that we shall consider are those that are described by the Lagrangian density

$$\mathcal{L}(X, \phi) = \mathcal{B} \exp [X^\alpha V(\phi)], \quad (62)$$

where \mathcal{B} is a negative constant, and α is a real number. Clearly, as the previous example, this Lagrangian density too will satisfy the condition (34), courtesy of the relations (36). We obtain the homogeneous energy density and pressure associated with the above Lagrangian density to be

$$\rho = \mathcal{B} [2\alpha X^\alpha V(\phi) - 1] \exp [X^\alpha V(\phi)] \quad \text{and} \quad p = \mathcal{B} \exp [X^\alpha V(\phi)]. \quad (63)$$

Also, we find that

$$c_A^2 = c_s^2 = \left(\frac{1}{2\alpha (1 + V(\phi) X^\alpha) - 1} \right), \quad (64)$$

so that we can write

$$c_s^2 = \left(\frac{w}{2\alpha w + 1} \right). \quad (65)$$

Note that, if $\alpha > (3/2)$, then c_s^2 is a positive definite quantity, and w can be less than $-(1/3)$.

Though the above three examples behave as though they explicitly depend on the scalar field, we find that they can be turned into pure kinetic models by a suitable redefinition of the field. For instance, the transformation

$$\chi(\phi) = \int d\phi \left[\frac{V(\phi)}{V_0} \right]^{(1/2\alpha)}, \quad (66)$$

reduces the Lagrangian density (58) to the example (37). Evidently, such a transformation will also convert the Lagrangian density (62) into a purely kinetic form. Moreover, we find that the following transformation

$$\chi(\phi) = \int d\phi \exp - [V(\phi)/2\alpha], \quad (67)$$

removes the explicit dependence on the scalar field in the first example above [cf. Eqs. (54) and (55)]. In other words, classically, the three Lagrangian densities that we have presented in this sub-section are essentially pure kinetic models that masquerade as though they depend on the scalar field. The fact that field redefinitions can relate different, but dynamically equivalent, Lagrangian densities for scalar fields has been noticed earlier (in this context, see, in particular, Ref. [11]). However, we should add that it is not clear whether such a behavior will be preserved when the quantum nature of the scalar fields are taken into account, as it is required, say, in the context of inflation.

In this section, we had presented a few specific examples of scalar fields that satisfy the condition (34). It should be noted that the most general solution to this equation can be expressed in terms of the product $[X f(\phi)]$, where $f(\phi)$ is an arbitrary function of the scalar field (for details, see Ref. [9]).

V. AN INTERESTING IMPLICATION FOR INFLATION

We find that the perfect scalar fields that satisfy the condition (34) behave in a particular fashion when they are treated as the inflaton. We shall now turn to discuss this property.

It is well known that, in the slow roll inflationary scenario involving a single scalar field, the amplitude of the curvature perturbation approaches a constant value soon after the modes leave the Hubble radius (see, any of the standard texts [1] or reviews [2]). However, this is not true when there are one or more brief periods of deviation from slow roll inflation¹. In such cases, the amplitude of the curvature perturbation over a suitable range of modes (which leave the Hubble radius just before the deviation from slow roll) can be enhanced or suppressed at super Hubble scales when compared to their value at Hubble exit. It can be shown that this behavior essentially arises due to the fact that, in such scenarios, the intrinsic entropy perturbation (i.e. the non-adiabatic pressure perturbation) grows rapidly around the period of fast roll at super Hubble scales [3].

Since the non-adiabatic pressure perturbation vanishes identically, the phenomenon described above can not occur in models which satisfy the condition (34). It is then possible that the demand that the non-adiabatic pressure perturbation vanishes exactly is so restrictive that the perfect scalar fields will not allow a brief period of departure from slow roll inflation. This expectation indeed turns out to be true and, in what follows, we shall outline its proof.

Recall that the first slow roll parameter ϵ is defined as [1, 2]

$$\epsilon = - \left(\frac{\dot{H}}{H^2} \right) = \left(\frac{3(1+w)}{2} \right), \quad (68)$$

and inflation corresponds to the epoch wherein $\epsilon < 1$ or, equivalently, $w < -(1/3)$. The time derivative of the quantity ϵ is given by

$$\dot{\epsilon} = \left(\frac{9H}{2} \right) (1+w) (w - c_A^2) = \left(\frac{9H}{2} \right) (1+w) (w - c_s^2), \quad (69)$$

¹ Actually, it could even be a short period of departure from inflation. But, the deviation from slow roll *has* to be brief, since a prolonged deviation may not allow inflation to restart.

where, in arriving at the last equality, we have set $c_A^2 = c_s^2$, as it is in the case of perfect scalar fields. The scalar power spectrum is determined by the amplitude of the curvature perturbations when the modes are well outside the Hubble radius during the inflationary epoch [1, 2]. Also, these curvature perturbations are evolved from the sub Hubble to the super Hubble scales by imposing standard initial conditions (viz. the Bunch-Davies conditions) when the modes are well inside the Hubble radius. It is the quantity c_s that describes the speed of propagation of the curvature perturbations. Hence, c_s has to be a real quantity, if it has to be ensured that the curvature perturbations do not grow exponentially at sub Hubble scales, so that the Bunch-Davies initial conditions can be imposed on them (see Ref. [7]; in this context, also see Ref. [17]). So, during inflation, in addition to $w < -(1/3)$, we require that $c_s^2 > 0$. It is then evident from the last equality in the expression (69) above that

$$\dot{\epsilon} < 0. \quad (70)$$

In other words, ϵ will *always* be a monotonically *decreasing* function of time when perfect scalar fields drive inflation. Therefore, clearly, once slow roll inflation has commenced, these models will not allow a period of deviation from slow roll inflation. In fact, terminating inflation becomes a problem in such cases, and one will have to invoke an additional scalar field to exit inflation. We should point out that, amongst the various examples of perfect scalar fields that we have discussed, only the models described in Eqs.(40), (45) and (62) satisfy both the conditions $w < -(1/3)$ and $c_s^2 > 0$ to act as a inflaton.

It has been recognized that certain features in the primordial spectrum lead to a better fit to the cosmic microwave background data than the conventional, featureless, power law, primordial scalar perturbation spectrum (in this context, see, for example, Refs. [18], and the long list of references therein). Interestingly, a short epoch of deviation from slow roll inflation is essential for generating features in the inflationary, scalar perturbation spectrum. In particular, it seems mandatory that the first slow roll parameter ϵ has to rise and fall if features are to be produced in the scalar power spectrum². However, as we discussed above, it is not possible to achieve such a behavior for ϵ in perfect scalar field models. This, in turn, implies that scalar fields with a vanishing non-adiabatic pressure perturbation, if they are treated as the inflaton, can not lead to features in the scalar perturbation spectrum!

VI. CONCLUDING REMARKS

In this paper, we have obtained a condition on the Lagrangian density of an arbitrary scalar field under which the intrinsic, non-adiabatic pressure perturbation associated with the scalar field vanishes identically, at the first order in the perturbations. Motivated by the analogy with fluids, we have termed these fields as perfect scalar fields. Scalar field models that depend only on the kinetic energy are known to contain no non-adiabatic pressure perturbation. In addition to discussing a couple of such examples, we have presented a few models which seemingly depend on the scalar field explicitly, and possess no non-adiabatic pressure perturbation. But, we find that all such models that we were able to construct could be reduced to purely kinetic models by a redefinition of the field. In fact, it can be proved that all Lagrangian densities that satisfy the condition (34) can be reduced to a purely kinetic form (in this context, see the discussion following Eq. (2.22) in Ref. [9]). Interestingly, we have also shown that, if perfect scalar fields are used to drive inflation, then they can not lead to features in the scalar power spectrum.

Note that, throughout the paper, we had restricted our attention to the first order in the perturbations. We should point out that, since all the scalar fields that satisfy the condition (34) can be reduced to pure kinetic models, such perfect scalar fields will not contain any non-adiabatic pressure perturbation at the higher orders in the perturbations either (in this context, see Refs. [20]). In other words, these scalar fields are truly perfect.

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² For example, it is known that the spectrum can remain featureless even if the second slow roll parameter is large for a short duration of time (see, for instance, Refs. [19]).

attention to a few relevant references.

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