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A fractal class of generalized Jackson interpolants

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In this paper, we establish a new formula that generalizes the Jackson trigonometric interpolation for a 2π-periodic function. This generalization is done by using various positive exponents in the basic nodal functions that gives a wide variety of bases during approximation. For a Hölder continuous periodic function, we compute the uniform interpolation error bound of the corresponding generalized Jackson interpolant and prove the convergence of the proposed interpolant. We also show that the mentioned approximation procedure is stable. In the last part, we consider a family of fractal interpolants associated with the generalized Jackson approximation functions under discussion.

KEYWORDS
curve fitting, fractals, Jackson interpolation, smoothing, trigonometric interpolation

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1 | INTRODUCTION

In the ordinary interpolation formula, a trigonometric sum of nth order is provided by the values of a function at $2n + 1$ evenly sampled points over the interval $[-\pi, \pi]$. We know that the convergence is not sure when the sampling points are indefinitely increased even for continuous functions. Jackson[1] proposed an interpolation formula which converges uniformly for every continuous function. This interpolation formula does not provide minimal errors for some periodic functions. Thus, we propose to modify Jackson trigonometric interpolation[1] by extending the class of the basic functions with different positive exponents. This is done by defining a family of nodal mappings linked to a uniform partition of the interval and assigning positive real exponents to each member of this family. In this way, we get a sequence of generalized Jackson interpolants based on the partition points of the interval. For a prescribed 2π-periodic Hölder continuous function, a bound for the uniform error with respect to its generalized Jackson interpolant is computed. In particular, if we choose the positive exponents in a suitable manner, the uniform error will be bounded, and the sequence of generalized Jackson interpolants will converge to the Hölder continuous function.

The content of this manuscript is described in the following order. In Section 2, we generalize the Jackson interpolation formula. It is shown that this formula gives lower errors for some periodic functions. Then, we deduce the uniform error bound for 2π-periodic Hölder continuous function. Finally, in the same section, convergence and stability results are proved for the new interpolant. We review some introductory results in Section 3. In Section 4, we consider a family of fractal interpolants associated with the generalized Jackson approximants. We derive an error bound for the fractal case as well, proving the convergence for suitable elections of a fractal parameter.

2 | NEW TYPE OF FOURIER INTERPOLANT

Jackson[1] used n-nodal basis functions and proposed the following interpolatory formula to a 2π-periodic function f: given n-points x_i in the interval $[-\pi, \pi]$ such that $x_{i+1} - x_i = 2\pi/n$ and $i \in \mathbb{N}_n$, where $\mathbb{N}_n$ is the first n natural numbers,

he defined

$$S_n(f)(x) = \frac{1}{n^2} \sum_{i=1}^{n} f(x_i) \left(\frac{\sin\left(n\frac{x_i-x}{2}\right)}{\sin\left(\frac{x_i-x}{2}\right)} \right)^2. \tag{1}$$

We extend the kernels (see, for instance, the works of Jackson[2] and Szabados[3]) with the help of positive exponents β_i as

$$Q_{n,i,\beta_i}(x) = \left| \frac{\sin\left(n\frac{x_i-x}{2}\right)}{\sin\left(\frac{x_i-x}{2}\right)} \right|^{\beta_i} \quad \text{for } x \neq x_i,$$

and

$$Q_{n,i,\beta_i}(x_i) = n^{\beta_i},$$

where $\beta_i > 0, i \in \mathbb{N}_n$. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ with $\beta_1 \geq \beta_2 \ldots \geq \beta_n$. Based on the above generalized kernels, we suggest the following approximation to f:

$$I_{n,\beta}(f)(x) = K_{n,\beta}(x) \sum_{i=1}^{n} f(x_i) Q_{n,i,\beta_i}(x), \tag{2}$$

and

$$K_{n,\beta}^{-1}(x) = \sum_{i=1}^{n} Q_{n,i,\beta_i}(x). \tag{3}$$

The selection of the exponential parameters in the above trigonometric expression (2) provides a greater flexibility for approximation of data sets, generating various types of periodic functions including the Jackson trigonometric interpolation formula. In particular, the use of variant exponents for different kernel functions allows greater flexibility to treat the data.

Several types of different trigonometric approximations can be found in other works.[4-8]

Proposition 1. *The proposed approximation formula (2) interpolates f at grid points.*

Proof. First, we will show that

$$Q_{n,i,\beta_i}(x_j) = n^{\beta_i}\delta_{ij}, \tag{4}$$

where $\delta_{ij} = 1$ if $i = j$, and zero otherwise. As per our construction, $Q_{n,i,\beta_i}(x_j) = n^{\beta_i}$ whenever $i = j$. For the case $i \neq j$, consider the natural number p such that $j = i + p$ (or $i = j + p$), then

$$x_j - x_i = \frac{2\pi p}{n},$$

and thus,

$$\sin\left(n\frac{x_i - x_j}{2}\right) = \sin(\pi p) = 0,$$

and hence, $Q_{n,i,\beta_i}(x_j) = 0$. Consequently,

$$K_{n,\beta}(x_j)^{-1} = Q_{n,j,\beta_j}(x_j) = n^{\beta_j},$$

and, for all $j = 1, 2, \ldots, n$,

$$I_{n,\beta}(f)(x_j) = K_{n,\beta}(x_j) \sum_{i=1}^{n} f(x_i) Q_{n,i,\beta_i}(x_j) = n^{-\beta_j} f(x_j) n^{\beta_j} = f(x_j).$$

$\square$

TABLE 1 The pointwise approximation errors of various periodic continuous function for the classical Jackson formula and our formula with different selections of n and β_i

	Value of 2π-periodic function	Jackson ($\beta_i = 2$)	$\beta_i = 2.5$	$\beta_i = 3$	$\beta_i = 4$
$n = 8$	$\sqrt{|\sin(2)|} + 2\sqrt{|\cos(2)|}$	0.2976	0.2850	0.2647	0.2110
$n = 10$	$2\sin^3(-1.5) + 3\cos^2(-1.5)$	1.0989	0.8293	0.6846	0.5135
$n = 10$	$\log\left(2 + \cos\left(\frac{\pi}{2}\right)\right)$	0.0267	0.0205	0.0167	0.0135
$n = 10$	$\min(\sin(1), \cos(1))$	0.2891	0.2268	0.1989	0.1882
$n = 12$	$\sqrt{\left|\sin\left(\frac{3\pi}{8}\right)\right|}$	0.0574	0.0436	0.0405	0.0264

Remark 1. It is worth to note that for natural exponents $\beta_i \in \mathbb{N}, i = 1, 2, \ldots, n$, the modulus is not required for the nodal basis Q_{n,i,β_i}, and for all $\beta_i = 2$, the proposed generalization (2) reduces to the original Jackson idea due to the following:

$$I_{n,2}(f)(x) = \frac{1}{n^2} \sum_{i=1}^{n} f(x_i) \left(\frac{\sin\left(n\frac{x_i-x}{2}\right)}{\sin\left(\frac{x_i-x}{2}\right)} \right)^2,$$ (5)

since

$$\sum_{i=1}^{n} \left(\frac{\sin\left(n\frac{x_i-x}{2}\right)}{\sin\left(\frac{x_i-x}{2}\right)} \right)^2 = n^2.$$ (6)

Now, we want to deduce the uniform error bound between a 2π-periodic Hölder continuous function, and its generalized interpolant. In our computation, we need the following propositions that are readily available in the work of Navascués et al.[9] Table 1 collects the errors committed in several trigonometric approximations for different exponents and values of n.

Proposition 2. *Let $\beta > 0$, and $y \in \mathbb{R}$. Then for all $m = 1, 2, \ldots$,*

$$\left| \frac{\sin my}{m \sin y} \right|^{\beta} \leq 1.$$ (7)

Proposition 3. *Let $y \in [0, \frac{\pi}{2}]$, then*

$$\sin y \geq \frac{2y}{\pi}.$$ (8)

Let $\text{Lip}_A d = \{f \in C(I) : |f(x) - f(y)| \leq A|x - y|^d, \forall x, y \in I\}$.

Theorem 1. *Let $f \in \text{Lip}_K d$, where $0 < d \leq 1$. Then, for $\beta_n > d + 1$*

$$\|f - I_{n,\beta}(f)\|_{\infty} \leq K\left(\frac{\pi}{n}\right)^d \left(\frac{\pi}{2}\right)^{\beta_1} \left(1 + 2^d + \frac{1}{\beta_n - (d+1)} + \frac{1}{\beta_n - 1}\right).$$

Proof. At first, we calculate the pointwise approximation error for the proposed method as

$$E_{n,\beta}(f)(x) := I_{n,\beta}(f)(x) - f(x) = K_{n,\beta}(x) \sum_{i=1}^{n} (f(x_i) - f(x)) Q_{n,i,\beta_i}(x).$$

Assume that $u_i \in [\frac{-\pi}{2}, \frac{\pi}{2}]$.[2] Substituting $x_i = x + 2u_i$, we have

$$|E_{n,\beta}(f)(x)| \leq K_{n,\beta}(x) \sum_{i=1}^{n} |f(x + 2u_i) - f(x)| \left| \frac{\sin(nu_i)}{\sin(u_i)} \right|^{\beta_i}.$$ (9)

The numerator and denominator of the right side of (9) are multiplied by n^{β_i+1}.

Let v_0 be the smallest of the numbers $|u_i|$, v_1 the second, and continuing similarly (see the work of Jackson[2(p454)]) using $f \in \text{Lip}_K d$, (9) reduces to

$$\left| E_{n,\beta}(f)(x) \right| \leq K_{n,\beta}(x) \sum_{i=0}^{n-1} n^{\beta_{i+1}} K 2^d v_i^d \left| \frac{\sin(nv_i)}{n\sin(v_i)} \right|^{\beta_{i+1}}, \tag{10}$$

where

$$\frac{\pi i}{2n} \leq v_i \leq \frac{\pi(i+1)}{2n} \leq \frac{\pi}{2}, \tag{11}$$

for $i \in \mathbb{N}_n \cup \{0\}$. From Proposition 2 for $i = 0, 1$, we obtain

$$\left| \frac{\sin(nv_i)}{n\sin(v_i)} \right|^{\beta_i} \leq 1. \tag{12}$$

It can be verified easily that, for remaining values ($i \geq 2$), after using the left inequality of (11), we get

$$n\sin(v_i) \geq n\frac{2v_i}{\pi} \geq i,$$

where we used Proposition 3 to obtain the first inequality. Therefore, for $i \geq 2$,

$$\left| \frac{\sin(nv_i)}{n\sin(v_i)} \right|^{\beta_i} \leq \left(\frac{1}{n\sin(v_i)} \right)^{\beta_i} \leq \left(\frac{1}{i} \right)^{\beta_i}. \tag{13}$$

Collecting the inequalities from (12) for $i = 0, 1$ and (13) for $i \geq 2$, we get

$$\left| E_{n,\beta}(f)(x) \right| \leq K 2^d K_{n,\beta}(x) n^{\beta_1} \left(v_0^d + v_1^d + \sum_{i=2}^{n-1} v_i^d \left(\frac{1}{i} \right)^{\beta_{i+1}} \right)$$

$$\leq K 2^d n^{\beta_1} K_{n,\beta}(x) \left(\frac{\pi^d}{2^d n^d} + \frac{(2\pi)^d}{2^d n^d} + \sum_{i=2}^{n-1} \pi^d \frac{(i+1)^d}{2^d n^d i^{\beta_{i+1}}} \right)$$

$$\leq K n^{\beta_1} K_{n,\beta}(x) \frac{\pi^d}{n^d} \left(1 + 2^d + \sum_{i=2}^{n-1} \frac{i^d + 1}{i^{\beta_n}} \right), \tag{14}$$

where, in the second step, we have applied (11) and, for the last step, the inequality $(i+1)^d \leq (i^d+1)$, where $0 < d \leq 1$ is applied. For the functions $\frac{1}{x^{\beta_n - d}}$ and $\frac{1}{x^{\beta_n}}$, we will use the lower Riemann sums in $[1, +\infty)$, where we have considered unit step. Thus, the last two summands in the above expression are evaluated as
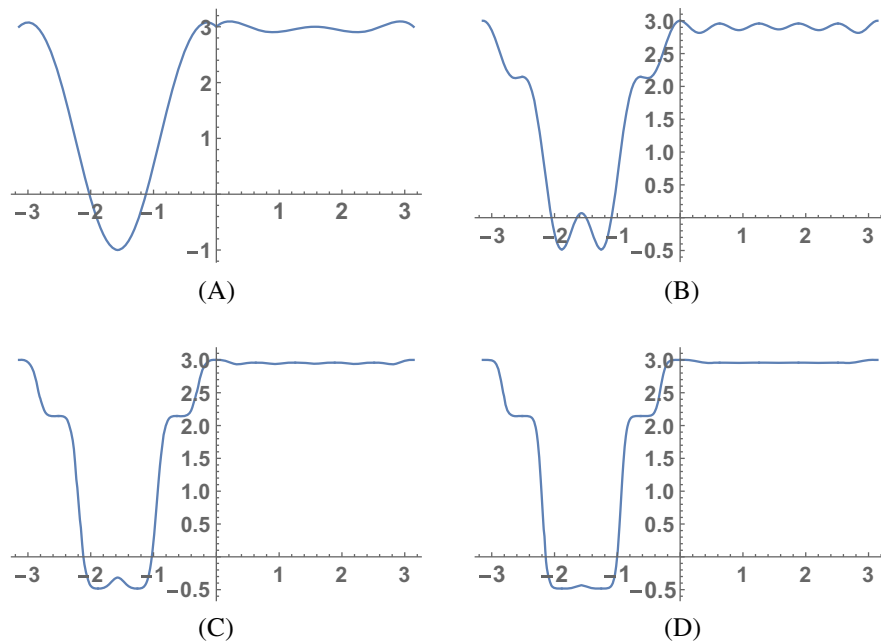
$$\sum_{i=2}^{n-1} \frac{1}{i^{\beta_n - d}} \leq \int_1^\infty \frac{dx}{x^{\beta_n - d}} = \frac{1}{\beta_n - (d+1)},$$

and

$$\sum_{i=2}^{n-1} \frac{1}{i^{\beta_n}} \leq \int_1^\infty \frac{dx}{x^{\beta_n}} = \frac{1}{\beta_n - 1}.$$

Adapting these bounds, we collect

$$\left| E_{n,\beta}(f)(x) \right| \leq K n^{\beta_1} K_{n,\beta}(x) \left(\frac{\pi}{n} \right)^d \left(1 + 2^d + \frac{1}{\beta_n - (d+1)} + \frac{1}{\beta_n - 1} \right), \tag{15}$$

FIGURE 1 Graph of interpolated function for different values of β_i. A, Original function; B, Jackson interpolated function; C, Generalized Jackson interpolated function for each $\beta_i = 3$; D, Generalized Jackson interpolated function for each $\beta_i = 4$

if $\beta_n > d + 1$. To estimate an upper bound for $K_{n,\beta}(x)$, we consider

$$K_{n,\beta}(x)^{-1} = \sum_{i=0}^{n-1} \left| \frac{\sin(nv_i)}{\sin(v_i)} \right|^{\beta_{i+1}} > \left| \frac{\sin(nv_0)}{\sin(v_0)} \right|^{\beta_1},$$

and

$$\sin nv_0 \geq \frac{2nv_0}{\pi}.$$

Consequently,

$$K_{n,\beta}(x)^{-1} > \left| \frac{2nv_0}{\pi v_0} \right|^{\beta_1} = \left(\frac{2n}{\pi} \right)^{\beta_1}, \tag{16}$$

which gives

$$n^{\beta_1} K_{n,\beta}(x) < \left(\frac{\pi}{2} \right)^{\beta_1}.$$

Applying (16) in (15), the uniform approximation error bound for the procedure is established as

$$\left\| f - I_{n,\beta}(f) \right\|_\infty \leq K \left(\frac{\pi}{n} \right)^d \left(\frac{\pi}{2} \right)^{\beta_1} \left(1 + 2^d + \frac{1}{\beta_n - (d+1)} + \frac{1}{\beta_n - 1} \right). \tag{17}$$

$\square$

The following example illustrates the proposed procedure.

Example: Let the function $f(x) = 2\sin^3 x + 3\cos^2 x + \sqrt{|\sin x|}$ be given over the interval $[-\pi, \pi]$. Figure 1A depicts the graph of the function f on the interval $[-\pi, \pi]$. Consider a partition of $[-\pi, \pi]$ with step length $\frac{\pi}{5}$. The corresponding interpolated function for Jackson interpolation is generated in Figure 1B. Using $n = 10$ and for each $\beta_i = 3$, the generalized interpolated function is generated in Figure 1C. Finally, with the same number of points of $[-\pi, \pi]$ for $\beta_i = 4$, the interpolated function is depicted in Figure 1D.

Corollary 1. *Let f be periodic and $f \in \text{Lip}_K d$. Then, for $\beta_n \geq r > (d+1)$, the interpolating function $I_{n,\beta}(f)$ is uniformly convergent to f whenever n approaches infinity. The order of convergence is $\mathcal{O}(n^{-d})$ and it can be noted that it is independent of β.*

In the concern for stability, first we review a result for the interpolant $I_m(f)$ on a partition $\{x_i^m\}_{i=1}^m$.[10]

Definition 1. The interpolation $I_m(f)$ is said to be stable if, for given $\varepsilon > 0$, there exists δ (depending on ε) > 0 so that $\max_{1 \leq i \leq m} |f(x_i^m)| \leq \delta$ implies $\|I_m(f)\|_\infty \leq \varepsilon$.

Definition 2. For the interpolation $I_m(f)$, the Lebesgue constant is defined as

$$\Lambda_m = \sup_{x \in I} \sum_{i=1}^{m} |\phi_i(x)|, \tag{18}$$

where I is the interval of definition and ϕ_i, $i = 1, \ldots, m$, are the associated nodal functions.

The next stability result can be recalled from the work of Hong.[10]

Theorem 2. *An interpolation is stable if and only if, for any Lebesgue constant sequence $\{\Lambda_m\}$, there exists a positive real number B satisfying $\Lambda_m \leq B$ for any natural number m.*

From the definition of nodal functions and using this result, we can culminate that $I_{n,\beta}$ is stable as the associated Lebesgue constants verify the following equality:

$$\Lambda_{n,\beta} = \sup_{x \in [-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{n} Q_{n,i,\beta_i}(x) = 1.$$

Hence, we obtain that the interpolation is stable.

In the work of Navascués et al,[11] we proposed a way of computation of optimal exponents of nodal functions. The procedure consists of taking half of the sampled data to perform the interpolation and using the rest as target points. The exponent is chosen such that the mean square error of the fitting process reaches minimum.

3 | IFS THEORY AND α-FRACTAL FUNCTIONS

In this section, we review the concept of iterated function system (IFS) and the construction of fractal functions which will be used in the sequel. Fractal interpolation function defined via IFS is a tool to approximate smooth and nonsmooth functions generating from real-world data. More details can be read from the works of Barnsley[12] and Navascués.[13,14]

Let $X \subset \mathbb{R}^n, n \in \mathbb{N}$ and (X, d_X) be a complete metric space with a metric d_X. Let $H(X) = \{A : A \neq \emptyset$, and A is compact in $X\}$. The Hausdorff distance between A and B in $H(X)$ is defined as $h(A, B) = \max\{d_X(A, B), d_X(B, A)\}$, where $d_X(A, B) = \sup_{x \in A} \inf_{y \in B} d_X(x, y)$. The space $(H(X), h)$ is called the space of fractals.[12] Let $\{w_i : X \to X; i \in \mathbb{N}_N\}$ be a collection of continuous maps on X. Then, $\{X; w_i, i \in \mathbb{N}_N\}$ is called an IFS on X. The above IFS is called hyperbolic if the maps $w_i, i \in \mathbb{N}_N$ are contractive, ie, $d_X(w_i(x), w_i(y)) < |\alpha_i| d_X(x, y), 0 \leq |\alpha_i| < 1$. For this hyperbolic IFS $\{X; w_i, i \in \mathbb{N}_N\}$, the set valued Hutchinson map $\mathcal{W} : H(X) \mapsto H(X)$ is defined as

$$\mathcal{W}(A) = \bigcup_{i=1}^{N} w_i(A).$$

It can be checked easily that $\mathcal{W}$ is a contraction and a contractive factor is $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, \ldots, N\}$. Thus, Banach fixed point theorem ensures the existence of a unique fixed point $G \in H(X)$ such that $G = \mathcal{W}(G)$. This G is said to be the attractor or the deterministic fractal of the corresponding IFS.

Now, we briefly describe α-fractal function stem from IFS. Consider the partition of $I = [a, b]$ as $\Delta = \{x_0, x_1, \ldots, x_N\}$ satisfying $a = x_0 < x_1 < \ldots < x_N = b$. Suppose the set of data points $\{(x_i, y_i), i \in \mathbb{N}_N \cup \{0\}\}$ be given. Let $I_i = [x_{i-1}, x_i]$. Consider N contractive homeomorphisms $L_i : I \to I_i$ such that

$$L_i(x_0) = x_{i-1}, \ L_i(x_N) = x_i. \tag{19}$$

Suppose $\mathcal{K} = I \times \mathbb{R}$ and $F_i : \mathcal{K} \to \mathbb{R}$ are N continuous mappings such that

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, \ |F_i(x, y) - F_i(x, y^*)| \leq |\alpha_i| |y - y^*|, \tag{20}$$

where $(x, y), (x, y^*) \in \mathcal{K}, |\alpha_i| < 1, i \in \mathbb{N}_N$. Consider $w_i : \mathcal{K} \to \mathcal{K}$ as $w_i(x, y) = (L_i(x), F_i(x, y)) \ \forall \ i \in \mathbb{N}_N$. Let us recall the principal theorem for the construction of FIFs.

Theorem 3. *Let $C(I) = \{f : I \mapsto \mathbb{R}, f \text{ is continuous}\}$ be associated with the uniform norm $\|g\|_\infty := \sup\{|g(x)| : x \in I\}$. Consider the subspace $C_{y_0, y_N}(I) := \{g \in C(I) : g(x_0) = y_0, g(x_N) = y_N\}$, which is a closed subspace of $C(I)$. Barnsley[12] proved the following:*

1. *The IFS $\{K; w_i, i = \mathbb{N}_N\}$ gives a unique attractor G which is the graph of a continuous real valued function $\tilde{f}$ and interpolates at the grid points.*
2. *The Read-Bajraktarević operator $T : C_{y_0, y_N}(I) \to C_{y_0, y_N}(I)$ defined by*

$$(Tg)(x) = F_i\left(L_i^{-1}(x), g \circ L_i^{-1}(x)\right), \ x \in I_i, \ i \in \mathbb{N}_N$$

determines $\tilde{f}$ as its fixed point.

The function $\tilde{f}$ obtained in Theorem 3 is said to be a FIF associated with $\{L_i(x), F_i(x, y)\}_{i=1}^N$ and it is a unique implicit function verifying

$$\tilde{f}(x) = F_i\left(L_i^{-1}(x), \tilde{f} \circ L_i^{-1}(x)\right) \ \forall x \in I_i, i \in \mathbb{N}_N. \tag{21}$$

Until now, most of the researchers studied FIF from the IFS

$$L_i(x) = a_i x + b_i, \ F_i(x, y) = \alpha_i y + q_i(x), \tag{22}$$

where $\alpha_i \in (-1, 1)$ is the IFS parameter, namely, vertical scale factor of the map w_i, $q_i : I \to \mathbb{R}$ are continuous functions obeying

$$q_i(x_0) = y_{i-1} - \alpha_i y_0, \quad q_i(x_N) = y_i - \alpha_i y_N$$

due to conditions (20). The vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in (-1, 1)^N$ is the corresponding vertical scaling factor.

Let $f \in C(I)$. Select a partition $\{a = x_0 < x_1 < \ldots < x_N = b\}$ of I, and let $q_i(x) = f \circ L_i(x) - \alpha_i b(x)$, $i \in \mathbb{N}_N$, where b is a continuous map satisfying $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$.

Definition 3. For $f \in C(I)$, base function b, scale vector α, the IFS (21)-(22) determines a continuous function f^α and this f^α is named as the α-fractal function corresponding to f.

According to (21) and (22), f^α verifies the fixed point equation

$$f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b) \circ L_i^{-1}(x), x \in I_i, i \in \mathbb{N}_N. \tag{23}$$

The uniform distance between f^α and f can be bounded as (see for instance the work of Navascués[13])

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty, \tag{24}$$

where $|\alpha|_\infty = \max\{|\alpha_i|; i \in \mathbb{N}_N\}$.

Remark 2. The map f^α is interpolatory.

Remark 3. It can be checked from the inequality (24) that, if $\alpha = 0$ or $f = b$, then $f^\alpha = f$.

4 | FRACTAL INTERPOLANTS OF JACKSON TYPE

In this section, to obtain the fractal analogue of the generalized Jackson interpolant, we will perturb the basis functions $Q_{n,i,\beta_i}(x)$ with suitable IFS parameters such as base functions $b_{n,i,\beta_i}(x)$, scale vector α, and partition of the interval as delineated in Section 3. We define the generalized fractal Jackson trigonometric interpolation as

$$I_{n,\beta}^\alpha(f)(x) = K_{n,\beta}(x) \sum_{i=1}^n f(x_i) Q_{n,i,\beta_i}^\alpha(x). \tag{25}$$

Using Remark 2 and (4), we obtain

$$Q_{n,i,\beta_i}^\alpha(x_j) = Q_{n,i,\beta_i}(x_j) = n^{\beta_i} \delta_{ij} \text{ for all } j = 1, 2, \ldots, n.$$

Thus, expression (25) interpolates at the grid points.

Theorem 4. *Let $f \in \text{Lip}_K d$, where $0 < d < 1$. Then, for $\beta_n > d + 1$,*

$$\left\| I^\alpha_{n,\beta}(f) - f \right\|_\infty \leq K \left(\frac{\pi}{n} \right)^d \left(\frac{\pi}{2} \right)^{\beta_1} \left(1 + 2^d + \frac{1}{\beta_n - (d+1)} + \frac{1}{\beta_n - 1} \right) + n \left(\frac{\pi}{2} \right)^{\beta_1} \|f\|_\infty \frac{|\alpha|_\infty}{1 - |\alpha|_\infty},$$

where α is a suitable IFS parameter used for the construction of the fractal perturbation of Q_{n,i,β_i} in the interval $[-\pi, \pi]$.

Proof. Consider the following triangle inequality:

$$\left\| I^\alpha_{n,\beta}(f) - f \right\|_\infty \leq \left\| I^\alpha_{n,\beta}(f) - I_{n,\beta}(f) \right\|_\infty + \left\| I_{n,\beta}(f) - f \right\|_\infty. \tag{26}$$

The bound for the second term was treated in Theorem 1. From (16), we have $K_{n,\beta}(x) < \left(\frac{\pi}{2n} \right)^{\beta_1}$, and this bound can be applied for the error in the first term of (26) since

$$
\begin{aligned}
\left| I^\alpha_{n,\beta}(f)(x) - I_{n,\beta}(f)(x) \right| &= \left| K_{n,\beta}(x) \sum_{i=1}^n f(x_i) \left(Q^\alpha_{n,i,\beta_i}(x) - Q_{n,i,\beta_i}(x) \right) \right| \\
&\leq K_{n,\beta}(x) \sum_{i=1}^n |f(x_i)| \left| Q^\alpha_{n,i,\beta_i}(x) - Q_{n,i,\beta_i}(x) \right| \\
&\leq K_{n,\beta}(x) \|f\|_\infty \sum_{i=1}^n \left| Q^\alpha_{n,i,\beta_i}(x) - Q_{n,i,\beta_i}(x) \right| \\
&\leq \left(\frac{\pi}{2n} \right)^{\beta_1} n \|f\|_\infty \max_{1 \leq i \leq n} \left\| Q^\alpha_{n,i,\beta_i} - Q_{n,i,\beta_i} \right\|_\infty \\
&\leq \left(\frac{\pi}{2n} \right)^{\beta_1} n \|f\|_\infty \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \max_{1 \leq i \leq n} \left\| Q_{n,i,\beta_i} - b_{n,i,\beta_i} \right\|_\infty,
\end{aligned}
\tag{27}
$$

where we employed (24) for the last step. b_{n,i,β_i} are the corresponding base functions required to construct the α-fractal functions Q^α_{n,i,β_i}. These functions may be selected such that

$$\| Q_{n,i,\beta_i} - b_{n,i,\beta_i} \|_\infty \leq \| Q_{n,i,\beta_i} \|_\infty.$$

Hence,

$$\max_{1 \leq i \leq n} \| Q_{n,i,\beta_i} - b_{n,i,\beta_i} \|_\infty \leq \max_{1 \leq i \leq n} \| Q_{n,i,\beta_i} \|_\infty = n^{\beta_i} \leq n^{\beta_1}.$$

Finally, collecting the preceding bound, (17) and (27) altogether in (26), we obtain the required error evaluation. $\square$

Remark 4. It can be noted that, in the process of nonsmooth generalization, to obtain more flexibility, one can define the fractal version of generalized interpolant using different scale vector α^i for each nodal function Q_{n,i,β_i} separately.

Figure 2 represents the graph of a fractal interpolant of the function $f(x) = 2\sin^3 x + 3\cos^2 x + \sqrt{|\sin x|}$, on the interval $[-\pi, \pi]$. The number of subintervals is 10, the scale vector is $(0.14, -0.27, 0.1, -0.24, 0.05, -0.08, 0.1, -0.13, 0, 0.08)$ and all the exponents β_i are equal to 2.
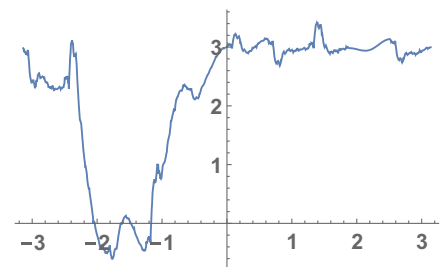
FIGURE 2 Graph of a fractal interpolant

5 | CONVERGENCE AND STABILITY OF THE FRACTAL CASE

The convergence and stability of generalized Jackson interpolation process were checked in Section 2. The convergence and stability of the generalized fractal Jackson trigonometric interpolant (25) are proven in the following.

Theorem 5. *Let f be 2π periodic and $f \in \mathrm{Lip}_K d$ such that $0 < d \leq 1$. If $\beta_n \geq r > (d+1)$, and if we select the sequence of IFS parameters α^n as $\alpha^n = \mathcal{O}\left(n^{-(d+1)}\right)$, then the interpolating function $I_{n,\beta}^{\alpha^n}(f)$ is uniformly convergent to f whenever n approaches infinity. The order of convergence is $\mathcal{O}(n^{-d})$ and, consequently, it is independent of β.*

Now, adapting similar computation that we used at the end of Theorem 4 (take $f = 1$),

$$
\begin{aligned}
\Lambda_{n,\beta}^{\alpha} &= \sup_{x \in [-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{n} \left| Q_{n,i,\beta_i}^{\alpha}(x) - Q_{n,i,\beta_i}(x) + Q_{n,i,\beta_i}(x) \right| \\
&\leq \sup_{x \in [-\pi,\pi]} K_{n,\beta}(x) \left(\sum_{i=1}^{n} \left| Q_{n,i,\beta_i}^{\alpha}(x) - Q_{n,i,\beta_i}(x) \right| \right) + \sup_{x \in [-\pi,\pi]} K_{n,\beta}(x) \sum_{i=1}^{n} \left| Q_{n,i,\beta_i}(x) \right| \\
&\leq \sup_{x \in [-\pi,\pi]} K_{n,\beta}(x) \left(\sum_{i=1}^{n} \left| Q_{n,i,\beta_i}^{\alpha}(x) - Q_{n,i,\beta_i}(x) \right| \right) + \Lambda_{n,\beta} \\
&\leq 1 + \sup_{x \in [-\pi,\pi]} n^{\beta_1} K_{n,\beta}(x) n \frac{|\alpha|_{\infty}}{1 - |\alpha|_{\infty}} \\
&\leq 1 + \left(\frac{\pi}{2} \right)^{\beta_1} n \frac{|\alpha|_{\infty}}{1 - |\alpha|_{\infty}}.
\end{aligned}
$$

As a result, one can check that if we choose the sequence of IFS parameter α^n as $\alpha^n = \mathcal{O}(n^{-1})$, then the interpolation $I_{n,\beta}^{\alpha^n}(f)$ is stable for any $\beta_n \geq r > (d+1)$.

Let us study the fractal nodal basis

$$
k_{n,i,\beta_i}^{\alpha}(x) = K_{n,\beta}(x) Q_{n,i,\beta_i}^{\alpha}(x). \tag{28}
$$

These functions are continuous as the denominator $K_{n,\beta}^{-1}(x)$ is not null for every $x \in I$ due to (16). Let us look at the space spanned by these mappings

$$
S_{n,\beta}^{\alpha} = \mathrm{span} \left(k_{n,i,\beta_i}^{\alpha} \right)_{i=1}^{n}.
$$

For a partition of the interval such that

$$
x_{i+1} - x_i = \frac{2\pi}{n},
$$

let us define the bilinear form in $C[-\pi, \pi]$

$$
(f^{\alpha}, g^{\alpha}) = \sum_{i=1}^{n} f^{\alpha}(x_i) g^{\alpha}(x_i) = \sum_{i=1}^{n} f(x_i) g(x_i).
$$

The mappings $(k_{n,i,\beta_i}^{\alpha})_{i=1}^{n}$ are orthonormal with respect to this product and hence independent. Therefore,

$$
\dim \left(S_{n,\beta}^{\alpha} \right) = n.
$$

Let us consider the functionals of point evaluation

$$
\mathcal{L}_{n,i}(f) = f(x_i).
$$

The systems $\{\mathcal{L}_{n,i}\}$ and $\{k_{n,i,\beta_i}^{\alpha}\}$ are biorthonormal since

$$
\mathcal{L}_{n,i} \left(k_{n,j,\beta_i}^{\alpha} \right) = k_{n,j,\beta_i}^{\alpha}(x_i) = K_{n,\beta}(x_i) Q_{n,j,\beta_i}^{\alpha}(x_i) = \delta_{ij}.
$$

The next result is obvious from the definition of the interpolation $I_{n,\beta}^{\alpha}(f)$.

Lemma 1. *If $f, g \in C[-\pi, \pi]$ agree at the nodes*

$$f(x_i) = g(x_i),$$

for $i = 1, 2, \ldots, n$, then $I^\alpha_{n,\beta}(f) = I^\alpha_{n,\beta}(g)$.

Proposition 4. *The operator $I^\alpha_{n,\beta} : C[-\pi, \pi] \mapsto C[-\pi, \pi]$ is a projection, that is to say,*

$$I^\alpha_{n,\beta} \circ I^\alpha_{n,\beta} = I^\alpha_{n,\beta}.$$

Proof. For any $f \in C[-\pi, \pi]$, f agrees with $I^\alpha_{n,\beta}(f)$ at the nodes. Thus, applying Lemma 1,

$$I^\alpha_{n,\beta}(f) = I^\alpha_{n,\beta}\left(I^\alpha_{n,\beta}(f)\right),$$

and the result is achieved. □

Proposition 5. *The function $g \in C[-\pi, \pi]$ is a fixed point of $I^\alpha_{n,\beta}$ if and only if $g \in S^\alpha_{n,\beta}$.*

Proof. If $g \in S^\alpha_{n,\beta}$, then

$$g = \sum_{i=1}^{n} \lambda_i k^\alpha_{n,i,\beta_i}.$$

Due to the orthogonality of k^α_{n,i,β_i},

$$g(x_j) = \lambda_j,$$

for any $j \in \mathbb{N}_n$, and then $g = I^\alpha_{n,\beta}(g)$, in view of the definition of $I^\alpha_{n,\beta}$. The other implication is a direct consideration from the definition of the interpolation. □

The Lebesgue constant of the partition is in this case

$$\Lambda^\alpha_{n,\beta} = \sup_{x \in [-\pi, \pi]} \sum_{i=1}^{n} \left| k^\alpha_{n,i,\beta_i}(x) \right|.$$

The norm of the nth interpolation can be acquired considering that, due to the definition (25),

$$\left\| I^\alpha_{n,\beta}(f) \right\|_\infty \leq \|f\|_\infty \sup_{x \in [-\pi, \pi]} \sum_{i=1}^{n} \left| k^\alpha_{n,i,\beta_i}(x) \right| = \Lambda^\alpha_{n,\beta} \|f\|_\infty, \tag{29}$$

and thus,

$$\left\| I^\alpha_{n,\beta} \right\| \leq \Lambda^\alpha_{n,\beta}.$$

For $f = 1$, we obtain the equality.

The Lebesgue constant can be viewed as a condition number with respect to the uniform norm of the interpolation, relative to changes in function values, since due to (29),

$$\left\| I^\alpha_{n,\beta}(f) - I^\alpha_{n,\beta}(\tilde{f}) \right\|_\infty \leq \Lambda^\alpha_{n,\beta} \|f - \tilde{f}\|_\infty,$$

where f and $\tilde{f}$ are the original and any perturbed function of f, respectively.

This number measures as well the separation of the interpolant with respect to the closest function in $S_{n,\beta}^\alpha$. If we denote as $f_{n,\beta}^*$ the closest function to f in this space, and $d_{n,\beta}^*(f)$ as the minimum distance from f to $S_{n,\beta}^\alpha$,

$$\left\| f - I_{n,\beta}^\alpha(f) \right\|_\infty \leq \left\| f - f_{n,\beta}^* \right\|_\infty + \left\| f_{n,\beta}^* - I_{n,\beta}^\alpha(f) \right\|_\infty \leq \left(1 + \Lambda_{n,\beta}^\alpha \right) \left\| f - f_{n,\beta}^* \right\|_\infty = \left(1 + \Lambda_{n,\beta}^\alpha \right) d_{n,\beta}^*(f),$$

since $f_{n,\beta}^* \in S_{n,\beta}^\alpha$, and thus, by Proposition 5, $f_{n,\beta}^* = I_{n,\beta}^\alpha(f_{n,\beta}^*)$. Following the same theorem and the properties of the modulus of continuity, we obtain a theorem of Jackson type for the spaces $S_{n,\beta}^\alpha$ in the case $\alpha = 0$.

Proposition 6. *For $\beta_n > d + 1$, the distance of $f \in C[-\pi, \pi]$ to the space $S_{n,\beta}^0$ verifies the following inequality:*

$$d_{n,\beta}^0(f) \leq K_{\beta,d} \left(\frac{\pi}{n} \right)^d,$$

where $K_{\beta,d}$ is determined by the values of β and f.

Proof. The result is a consequence of Theorem 1 since

$$d_{n,\beta}^0(f) \leq \left\| f - I_{n,\beta}(f) \right\|_\infty.$$

$\square$

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